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Atypical representations of the Lie superalgebra sl(1,n) for n greater than 1

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1. Preliminaries

The Lie superalgebra (LSA) $\mathrm{sl}(1,n)$ with $n>1$ is a concrete form of the classical LSA $A(0,n-1)$ (in the classification of Kac ([K78]); cf. also [B92]). We can realize $\mathrm{sl}(1,n)$ by $(n+1)\times(n+1)$-matrices:

$$G=\mathrm{sl}(1,n)=\left\{\left\|\begin{matrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n0} & a_{n1} & \cdots & a_{nn}\end{matrix}\right\| ;\ a_{ij}\in\mathcal{C},\ a_{00}-(a_{11}+\cdots+a_{nn})=0\right\}$$

(identical representation). The even part G_0 is characterized by the condition $a_{0i}=a_{i0}=0$ for $i=1,\dots,n$. G_0 is isomorphic to $gl(n)$.

We choose the Cartan subalgebra H as the set of diagonal matrices of G and G_0 resp.

Let e_{ij} be denote the $(n+1)\times(n+1)$-matrix with 1 at the place (i,j) and 0 everywhere $(i,j=0,\dots,n)$. Then we get by $h_1=e_{00}+e_{11}$, $h_i=e_{i-1,i-1}-e_{ii}$ $(i=2,\dots,n)$ a basis of H. The sets

$$\Delta^+=\{e_{ij};i<j,i,j=0,\dots,n\},\Delta_1^+=\{e_{0j};j=1,\dots,n\}$$

are the positive and the odd positive root vectors (we define $\Delta_0^+,\Delta^-,\Delta_1^-,\Delta_0^-$ in an analogous way). $\{h_i,e_i=e_{i-1,i},f_i=e_{i,i-1};i=1,\dots,n\}$ build a Chevalley basis of $\mathrm{sl}(1,n)$.

The representation theory of the classical LSAs developed by Kac gives some general results which we formulate for $G=\mathrm{sl}(1,n)$ in the following.

Let Λ be a linear functional over H. It is possible to construct for every such Λ a G-module $\overline{W}(\Lambda)$ by a standard induction process. $\overline{W}(\Lambda)$ contains a proper maximal G-submodule $\overline{I}(\Lambda)$ which can be the null space. We denote $\overline{W}(\Lambda)/\overline{I}(\Lambda)$ by $W(\Lambda)$: $W(\Lambda)$ is the irreducible G-module with the highest weight Λ.

Proposition 1.1.
a) $W(\Lambda_1)\cong W(\Lambda_2) \iff \Lambda_1=\Lambda_2$ $(\Lambda_1,\Lambda_2\in H^*)$.
b) Every finite dimensional irreducible G-module is equivalent to a module $W(\Lambda)$.
c) $W(\Lambda)$ *is finite dimensional* $\iff$ $a_i=\Lambda(h_i)\in\mathcal{Z}_+$, $i=2,\dots,n$; $a_1=\Lambda(h_1)$ *can be an arbitrary complex number.*

In the following we look only to finite dimensional representations and G-modules resp. We use both the language of representations and the language of modules.

The n-tuple $(a_1, a_2, \ldots, a_n)$ introduced in c) determines the linear functional Λ completely, because the h_i are a basis of H. So we can replace Λ by $(a_1, a_2, \ldots, a_n)$(also denote as highest weight), where $a_1 \in \mathcal{C}$, $a_i \in \mathcal{Z}_+$ for $i = 2, \ldots, n$; in many cases the signature $[m]_{n+1} = [m_1, \ldots, m_n]_{n+1}$ defined by $m_1 = a_1, m_i = a_1 - (a_2 + \cdots + a_i)$ for $i = 2, \ldots, n$ is used to denote the representation determined by Λ. This we will do now. The m_i are complex numbers having all the them imaginary part; furthermore $m_i - m_{i+1} = a_{i+1} \in \mathcal{Z}_+$ $(i = 1, \ldots, n-1)$ such that $m_1 \geq m_2 \geq \ldots \geq m_n$ if $m_1 = a_1 \in \mathcal{Z}_+$. We assume that this condition is always fulfilled in the following because for the atypical case below considered we have this condition.

The index $n+1$ at the right bracket of the signature we take to differ the representation of the LSA $\mathrm{sl}(1,n)$ from the representation of the even part $gl(n)$ with the same signature known from the classical theory.

The induced G-module $\overline{W}([m]_{n+1})$ can be irreducible; but if it is reducible then it is not completely reducible. So we have two possible cases which we describe by the following definition:

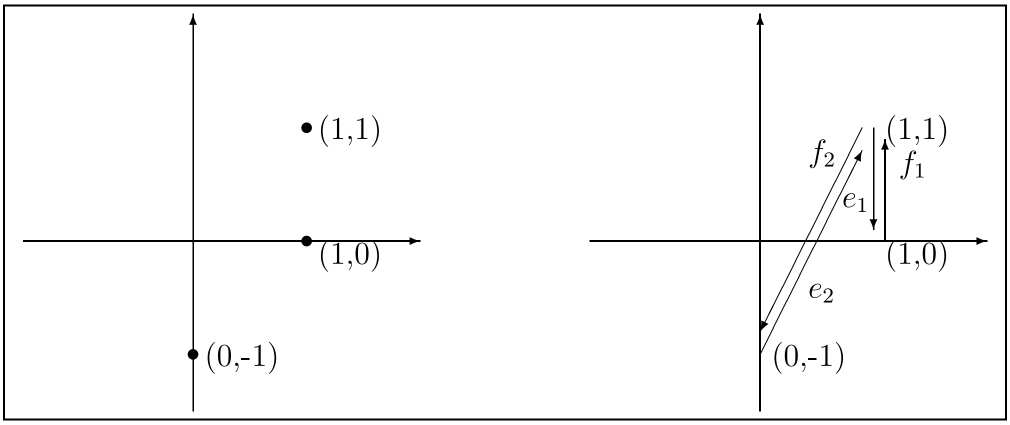
The module $W([m]_{n+1})$ (the representation $[m]_{n+1}$) is typical if the maximal invariant subspace $\overline{I}([m]_{n+1}) = \{0\}$ or $W([m]_{n+1}) = \overline{W}([m]_{n+1})$; $W([m]_{n+1})$ is atypical or nontypical if $\overline{I}([m]_{n+1}) \neq \{0\}$ or $W([m]_{n+1}) \neq \overline{W}([m]_{n+1})$.

So, in the atypical case we have to distinguish between $W([m]_{n+1})$ and $\overline{W}([m]_{n+1})$. Both modules have the same highest weight and the same signature. For this we will denote the atypical irreducible representation (given by $W([m]_{n+1})$) with $[m]^{irr}_{n+1} = [m_1, \ldots, m_n]^{irr}_{n+1}$ in difference to the indecomposable representation (given by $\overline{W}([m_{n+1}])$). The last one and the typical representation are denoted by $[m]_{n+1} = [m_1, \ldots, m_n]_{n+1}$.

A simple criterion for the atypical case is the existence of a k with $m_k = k-1$. It is easy to see that only one k with this property can exist in a signature $[m]_{n+1}$.

2. Weight diagrams for sl(1,2)

In [P87] Palev describes the action of the generators of $\mathrm{sl}(1,n)$ of a representation $[m]_{n+1}$ in a Gelfand-Zetlin-basis. The vectors in such a basis are weight vectors and can be indicated by the n-tuples of eigenvalues of the operators $h_1, \ldots, h_n$ (we denote the images of the elements of $\mathrm{sl}(1,n)$ by a representation with the same symbol as by the identical representation). The $h_1, \ldots, h_n$ act diagonally in this basis. The n-tuples of eigenvalues from $h_1, \ldots, h_n$ build the weight diagram for $[m]_{n+1}$. For instances, we have at the identical representation $[1,1]^{irr}_3$ of $sl(1,2)$ the weight diagram (the left diagram in the following picture)



corresponding to the pairs of eigenvalues of

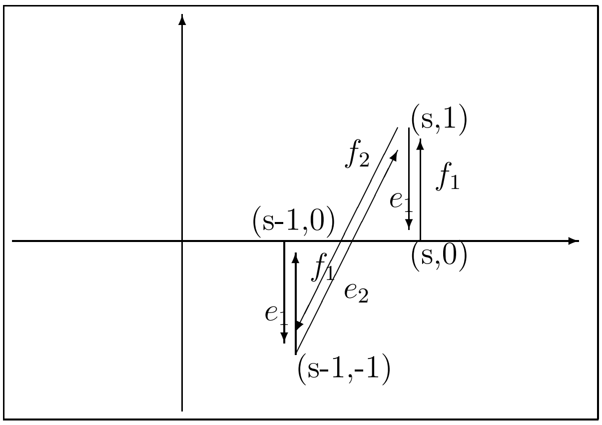
$$h_1 = \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right\|, \quad h_2 = \left\| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right\|.$$

$(1,0)$ is the highest weight.

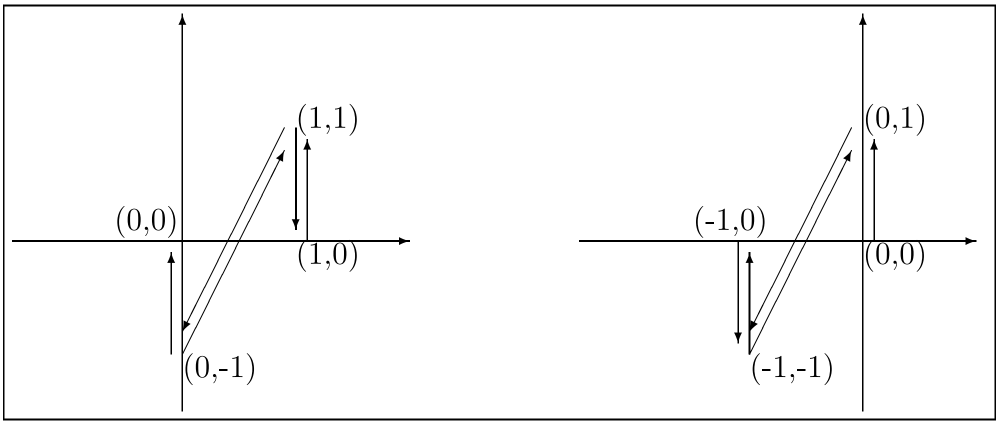
To describe the action of $\mathrm{sl}(1,n)$ it is enough to describe the action of the Chevalley basis $\{e_i, f_i, h_i\}$. If (j_1, j_2) is a pair of eigenvalues then the symbol $|j_1, j_2>$ denotes a corresponding basis vector. We draw in the following an arrow from (j_1, j_2) to (k_1, k_2) with the name g if $g(|j_1, j_2>) = \alpha |k_1, k_2>$ for some $\alpha \in \mathcal{C}, \alpha \neq 0$. So in the case $[1,1]_3^{irr}$ we get the right diagram (above drawn). The action of the h_i is known by construction of the weight diagram. All operators of the representation we can compose by the drawn actions.

The identical representation is an atypical representation ($m_2 = 1$). We will consider the more general case $[s,s]_3, s \in \mathcal{Z}$. The representations $[s,s]_3$ have the smallest dimension (namely 4) which is possible if we look to the representations $[m_1, m_2]_3$ (cf. 3).

In [P87] we find explicit formulas of the action of the Chevalley basis and hence we can compute the eigenvalues of the h_i. The mentioned formulas describe the representations $[m]_{n+1}$ but not the cases $[m]_{n+1}^{irr}$. So the weight diagram of $[s,s]_3$ has the form:



If $s = 1$ or $s = 0$ we have the atypical case. What is the speciality there? In both cases the action of one of the odd operators $f_1 = e_{10}, e_1 = e_{01}$ will be zero at some places where it does not vanish for $s \neq 1$ and $s \neq 0$ resp. So we get the following diagrams (left: $[1,1]_3$, right: $[0,0]_3$):



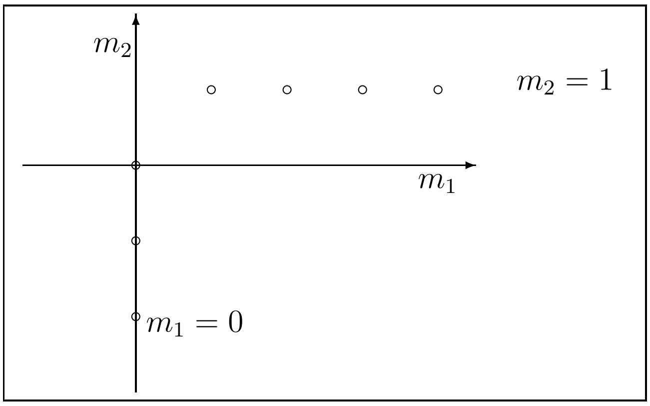
We see that for $s = 1$ the subspace $\overline{I} = \{0\}$, for $s = 0$ the subspace $\overline{I} = \text{span}\{|0, 1>, |-1, -1>, |-1, 0>\}$ is invariant. But the representations $[1, 1]_3$ and $[0, 0]_3$ are indecomposable because the complement of $\overline{I}$ is not invariant. Furthermore the subspace $\overline{I}$ and the factor space $\overline{W}/\overline{I}$ are spaces of irreducible representations: In $[1, 1]_3$ we find $[1, 1]_3^{irr}$ and $[0, 0]_3^{irr}$ and in $[0, 0]_3$ we find $[0, 0]_3^{irr}$ and $[0, -1]_3^{irr}$. We see that $[0, 0]_3^{irr}$ is contained in $[1, 1]_3$ by $\overline{I}$, in $[0, 0]_3$ by $\overline{W}/\overline{I}$. It is possible to generalize this property for all indecomposable representations of $sl(1, 2)$ ($\text{sl}(1, n)$; cf. 3): All atypical irreducible representations we find pairwise in the indecomposable representations $[m_1, 1]$ and $[0, m_2]$. These pairs are:

$m_1 > 1,\ m_2 = 1$: $[m_1, 1]_3^{irr}$, $[m_1 - 1, 1]_3^{irr}$

$m_1 = 1,\ m_2 = 1$: $[1, 1]_3^{irr}$ and $[0, 0]_3^{irr}$

$m_1 = 0,\ m_2 \leq 0$: $[0, m_2]_3^{irr}$ and $[0, m_2 - 1]_3^{irr}$.

If we draw the $[m_1, m_2]$ with real numbers m_1, m_2 as 2-tuples in a diagram we see the atypical representations along two rays:



The pairs above mentioned are neighbouring. The rays are connected by the pair $[1, 1]_3^{irr}, [0, 0]_3^{irr}$.

3. Connections between atypical representations of sl(1,n)

We will generalize the results of 2. For this first we descibe the structure of an induced finite dimensional module $\overline{W}([m]_{n+1})$ following Palev([P87]). The restriction to the even part $\text{sl}(1, n)_0 \cong gl(n)$ is the direct sum of finite dimensional irreducible $gl(n)$-modules. Every such submodule is a module with highest weight; the corresponding signatures are given in an one-by-one condition by aid of socalled Θ-tuples $(\Theta_1, \ldots, \Theta_n)$ where $\Theta_i \in \{0, 1\}$ for all $i = 1, \ldots, n$. We set

$$m(\Theta) = (m_1 + \Theta_1 - \sum_{j=1}^{n} \Theta_j, \ldots, m_n + \Theta_n - \sum_{j=1}^{n} \Theta_j).$$

If $(\Theta_1, \ldots, \Theta_n)$ ranges through all Θ-tuples then $[m(\Theta)]_{n+1}$ ranges trough the signatures of the irreducible $gl(n)$-submodules of $\overline{W}([m]_{n+1})$. We denote these submodules with $V(m(\Theta))$. In the atypical case ($m_k = k - 1$ for some k) the maximal invariant submodule $\overline{I}$ is the direct sum of all $V(m(\Theta))$ where $\Theta_k = 1$:

$$\overline{I} \cong \sum_{\Theta, \Theta_k = 1} V(m(\Theta)) \tag{1}$$

$$\overline{W}/\overline{I} \cong \sum_{\Theta, \Theta_k = 0} V(m(\Theta)) \tag{2}$$

In general, the signature $[m']_{n+1}^{irr} = [(m-1)^k]_{n+1}^{irr}$ of the atypical $\mathrm{sl}(1, n)$-module $\overline{I}$ is given by $\Theta_k = 1, \Theta_j = 0$ everywhere; the signature of $\overline{W}/\overline{I}$ is described by $\Theta_i = 0$ everywhere. Both irreducible representations are paired in $[m]_{n+1}$. But we have to look to exceptional cases for $k > 1$: If for some $\kappa > 0$ is valid:

$$m_{k-\kappa-1} > m_{k-\kappa} = \cdots = m_k = k - 1$$

then the signature $[m']_{n+1}^{irr}$ is given by $\Theta_{k-\kappa} = \cdots = \Theta_k = 1$ and $\Theta_j = 0$ everywhere. Here atypical representations are connected which belong to $m_k = k - 1$ and $m_{k-\kappa} = k - \kappa - 1$. For $\kappa = 0$ we go back to the general case.

These facts we can see from the formulas in [P87]:The images of the vectors $|m>$, $|m'>$ are zero by the positive root vectors.

Proposition 3.1. *Atypical finite dimensional irreducible $sl(1, n)$-modules are contained pairwise in the induced finite dimensional $sl(1, n)$-modules with signature $[m]_{n+1} = [m_1, \ldots, m_k = k - 1, \ldots, m_n]_{n+1}$. The maximal invariant submodule $\overline{I}$ is given by*

$$[(m-1)^k]_{n+1}^{irr} = [m_1 - 1, \ldots, m_{k-1} - 1, m_k, m_{k+1} - 1, \ldots, m_n - 1]_{n+1}^{irr}$$

the factormodule $\overline{W}/\overline{I}$ by

$$[m]_{n+1}^{irr} = [m_1, \ldots, m_k, \ldots, m_n]_{n+1}^{irr}.$$

Particular, we have the condition

$$\dim([(m-1)^k]_{n+1}^{irr}) + \dim([m]_{n+1}^{irr}) = 2^n \dim([m]_n) = 2^n \prod_{\substack{i<j \\ i,j=1,\ldots,n}} \frac{m_i - m_j + j - i}{j - i}$$

In the exceptional cases are fullfilled also these conditions; we have to change only $[(m-1)^k]_{n+1}^{irr}$ in an way above mentioned.

The proposition follows from the theory developed in [P87] and [P88]. The right side of the dimension formula we get from the classical theory.

4. The polynomial solution of the equation $g(x_1, \ldots, x_n) + g(x_1 - 1, \ldots, x_n - 1) = 2^{n+1} x_1 \cdots x_n$

The computation of the dimension of an atypical representation will be reduced in the following to the solution of an equation of the form

$$g(x_1, \ldots, x_n) + g(x_1 - 1, \ldots, x_n - 1) = 2^{n+1} x_1 \cdots x_n. \tag{3}$$

For the solution of this difference equation we use some ideas from the theory of difference equations in one variables (cf. [G58]) and some own ideas. Let be $\sigma_l(x_1, \ldots, x_n)$ denote the l-th elementary symmetric function in n variables:

$$\sigma_l(x_1, \ldots, x_n) = x_1 \cdots x_l + x_1 \cdots x_{l-1} x_{l+1} + \cdots + x_{n-l+1} \cdots x_n.$$

The solution of (3) is a linear combination of the σ_i with coefficients H_l at σ_{n-l}; the H_l have some remarkable properties.

Computations with a computer give the following values:

l	0	1	2	3	4	5	6	7	8	9	10	11
H_l	1	1/2	0	-1/4	0	1/2	0	-17/8	0	31/2	0	-691/4

It seems that $H_l = 0$ if l is even; the H_l could be similar to the Bernoulli numbers B_l. Furthermore, for the given numbers in the table we have

$$H_l = \frac{2(2^{l+1} - 1)}{l + 1} B_{l+1}. \tag{4}$$

So, we come to the idea that a generating function belongs to the H_l. It is possible to prove the following lemma ([S93]):

Lemma 4.1.

$$2\frac{e^t}{e^t + 1} = \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}.$$

Now we can show the relation (4) between H_l and the Bernoulli numbers B_{l+1}. By [G58] $B_1 = -1/2$ holds, and for small t:

$$\frac{t}{e^t - 1} = \sum_{l=0}^{\infty} B_l \frac{t^l}{l!}. \tag{5}$$

Now we get

$$\begin{aligned}
2\frac{e^t}{e^t + 1} &= 2\frac{e^t(e^t - 1)}{(e^t + 1)(e^t - 1)} = \frac{2}{t}\left(t - \frac{t}{e^t - 1} + \frac{2t}{e^{2t} - 1}\right) \\
&= 2 - \frac{2}{t}\sum_{l=0}^{\infty} B_l \frac{t^l}{l!} + \frac{2}{t}\sum_{l=0}^{\infty} B_l \frac{(2t)^l}{l!} = 2 - \frac{2}{t}\left(\sum_{l=2}^{\infty}(B_l - 2^l B_l)\frac{t^l}{l!}\right) - 2B_1 + 4B_1 \\
&= 1 + \sum_{l=1}^{\infty} 2(2^{l+1} - 1) B_{l+1} \frac{t^l}{(l + 1)!} = \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}
\end{aligned} \tag{6}$$

and

$$H_0 = 1, H_l = \frac{2(2^{l+1} - 1)}{l + 1} B_{l+1} \tag{7}$$

for $l = 1, 2, \ldots$

Similar to the Bernoulli numbers the H_l satisfy some recurrence relations. For instances we obtain:

a)From

$$2e^t = (e^t + 1) \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}$$

and development in power series we get by comparison of coefficients $2H_0 = 2, 2H_1 + H_0 = 2, 2H_2 + 2H_1 + H_0 = 2$ etc. and in general

$$\sum_{l=0}^{k} \binom{k}{l} H_l = 2 - H_k$$

. b)From

$$2 = (e^{-t} + 1) \sum_{l=0}^{\infty} H_l \frac{t^l}{l!}$$

and development in potency series we get

$$\sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} H_l = H_k.$$

Furthermore, it is easy to express the H_l by aid of the important values $B_l(1/2)$ and $B_l(1)$ of the Bernoulli polynomials $B_l(x)$.

Proposition 4.2. *The unique polynomial solution of (3) is*

$$g(x_1, \ldots, x_n) = 2^n \sum_{l=0}^{n} H_l \sigma_{n-l}(x_1, \ldots, x_n).$$

Proof. By the foregoing we have to prove the uniqueness only. For this we look to the equation

$$g(x_1, \ldots, x_n) + g(x_1 - 1, \ldots, x_n - 1) = 0 \tag{8}$$

and show that the only polynomial solution is the null polynomial. From this fact it is easy to get the uniqueness of the solution of (8) with an arbitrary polynomial on the right side.

Let $cx_1^{t_1} \cdots x_n^{t_n}$ be a summand in g with maximal degree $t_1 + \cdots + t_n$ and

$$g(x_1, \ldots, x_n) = cx_1^{t_1} \cdots x_n^{t_n} + d(x_1, \ldots, x_n).$$

On the left side of (8) we get

$$\begin{aligned} cx_1^{t_1} \cdots x_n^{t_n} + c(x_1 - 1)^{t_1} \cdots (x_n - 1)^{t_n} + d(x_1, \ldots, x_n) + d(x_1 - 1, \ldots, x_n - 1) &= \\ 2cx_1^{t_1} \cdots x_n^{t_n} + h(x_1, \ldots, x_n) & \end{aligned}$$

where h contains only summands of the degree less or equal to $t_1 + \cdots + t_n$ and the summands with this maximal degree are linear independent from the first summand. Then c must be zero and g has no summand of maximal degree. ∎

5. The dimension of a finite dimensional irreducible representation of sl(1,n)

We use the denotations of 1.-3.

The dimension D of $\overline{W}([m]_{n+1})$ is determined by the finite dimensional irreducible $gl(n)$-module with the signature $[m]_n$. The dimension of $[m]_n$ is given by the expression (cf. 3)

$$\dim([m]_n) = \prod_{\substack{i<j \\ i,j=1,\dots n}} \frac{m_i - m_j + j - i}{j-i} = \prod_{\substack{i<j \\ i,j=1,\dots,n}} \frac{r_{ij}}{j-i} \tag{9}$$

where $r_{ij} = m_i - m_j + j - i$ (for the following we extend the definition of r_{ij} on the case $i > j$ also) and

$$D = 2^n \dim([m]_n).$$

So we can consider the dimension of a typical finite dimensional irreducible $\mathrm{sl}(1,n)$-module as known.

On the other side by the sum decomposition given by (1) and (2) it is possible to describe D in the following way: Let for a given Θ-tuple $(\Theta_1,\dots,\Theta_n)$ be $r_{st}(\Theta) = m_{s,n+1} - m_{t,n+1} + t - s + \Theta_s - \Theta_t$. Then

$$D = \sum_{\Theta} \prod_{\substack{s<t \\ s,t=1,\dots,n}} \frac{r_{st}(\Theta)}{t-s} \tag{10}$$

where the summation goes over all possible $\Theta-$tuple. If $\overline{W}([m]_{n+1})$ is indecomposable and $m_k = k-1$ then the dimensions
$D_1 = \dim \overline{I}$ and $D_2 = \dim \overline{W}([m]_{n+1})/\overline{I}$
are given by the same formula (10) with the difference that the summation goes over all possible Θ-tuple with $\Theta_k = 1$ and $\Theta_k = 0$ resp. (cf. (1),(2)). This gives a possibility for the numerical calculation of the dimension of an atypical finite dimensional irreducible $\mathrm{sl}(1,n)$-module. In the following we will prove an explicit formula for such dimensions. We assume first that for $k > 1$ we have $m_{k-1} > m_k = k-1$.

We look to the maximal invariant subspace $\overline{I}$ of $\overline{W}([m]_{n+1})$. $\overline{I}$ itself is an atypical finite dimensional irreducible $\mathrm{sl}(1,n)$-module with the signature

$$[m']_{n+1} = [m_1 - 1, \dots, m_k, \dots, m_n - 1]_{n+1}$$

and the same k as in $\overline{W}([m]_{n+1})/\overline{I}$ in the general case. Hence we can compute D_1 by the same formula as D_2 replacing m_i by m'_i and r_{st} by r'_{st}, where $r'_{sk} = r_{sk} - 1$ for $s < k$, $r'_{kt} = r_{kt} + 1$ for $t > k$ and $r'_{st} = r_{st}$ in all other cases. On the other side from the analogous formula to (10) we see that D_2 is a polynomial $f(r_{12},\dots,r_{n-1,n})$. So we have to look to a polynomial f which satisfied the functional equation

$$f(r_{12},\dots,r_{n-1,n}) + f(r'_{12},\dots,r'_{n-1,n}) = 2^n \prod_{\substack{s<t \\ s,t=1,\dots,n}} \frac{r_{st}}{t-s}. \tag{11}$$

Because $r_{st} = r_{sk} + r_{kt}$ for arbitrary s,t,k we look to the polynomial f as a polynomial in $n-1$ variables only. We choose $x_1 = r_{1k}, \dots, x_{k-1} = r_{k-1,k}, x_{k+1} =$

$r_{k,k+1}, \dots, x_n = r_{k,n}$ as such variables which are independent. On the right side of (11) we separate the factor

$$2^n \prod_{s=1,\dots,k-1} r_{sk} \prod_{t=k+1,\dots,n} r_{kt} = 2^n \prod_{\substack{i=1,\dots,n \\ i \neq k}} x_i$$

from the rest R_{kn} and consider the equation

$$f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) + f_k(x_1 - 1, \dots, x_{k-1} - 1, x_{k+1} + 1, \dots, x_n + 1) = \\ = 2^n \prod_{\substack{i=1,\dots,n \\ i \neq k}} x_i \tag{12}$$

and for $k = n$

$$f_n(x_1, \dots, x_{n-1}) + f_n(x_1 - 1, \dots, x_{n-1} - 1) = 2^n \prod_{i=1,\dots,n-1} x_i \tag{13}$$

It is easy to see that $R_{kn} f_k(r_{1k}, \dots, r_{k-1,k}, r_{k,k+1}, \dots, r_{kn})$ is a solution of (11) where f_k solves (12). Furthermore, the solutions of (11) and (12) resp. are unique by the same arguments as in the proof of the proposition of 4.

Now we compare (12) for an arbitrary k with the case (13). If we replace r_{kt} by $-r_{kt} = r_{tk}$, we see by (10) that

$$f_k(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n) = (-1)^{n-k} f_n(x_1, \dots, x_{k-1}, -x_{k+1}, \dots, -x_n).$$

So it is enough to solve (13) obtaining the solution of (11). But (13) is solved by the proposition of 4.

Theorem 5.1. *Let $Y_k([m]_{n+1}) = \overline{W}([m]_{n+1})/\overline{I}$ be an atypical finite dimensional irreducible sl(n)-module with signature $[m_1, \dots, m_k = k-1, \dots, m_n]$. For all possible s,t,k we set:*

$$r_{st} = m_s - m_t + t - s,$$

$$R_{kn} = \prod_{\substack{s<t \\ s,t=1,\dots,n}} \frac{1}{t-s} \prod_{\substack{i<j, i\neq k, j \neq k \\ i,j=1,\dots,n}} r_{ij}.$$

The dimension of $Y_k([m]_{n+1})$ we get by the following formula:

$$\dim(Y_k([m]_{n+1})) =$$

$$2^{n-1} R_{kn} \sum_{l=0}^{n-1} (-1)^{n-k} H_l \sigma_{(n-1)-l}(r_{1k}, \dots, r_{k-1,k}, -r_{k,k+1}, \dots, -r_{kn})$$

where the $\sigma_{(n-1)-l}$ are the elementary symmetric functions of the first kind and the H_l are the coefficients determined in 4. More exact, because $m_k = k-1$ we have

$$r_{sk} = m_s - s + 1, -r_{kt} = m_t - t + 1$$

such that

$$\dim(Y_k([m]_{n+1}) =$$

$$2^{n-1} R_{kn} \sum_{l=0}^{n-1} (-1)^{n-k} H_l \sigma_{(n-1)-l}(m_1, m_2 - 1, \dots, m_{k-1} - (k-2), m_{k+1} - k, \dots, m_n - (n-1))$$

The theorem is correct also in the exceptional case: There for a natural number $\kappa > 0$ we have $m_{k-\kappa-1} > m_{k-\kappa} = \cdots = m_k = k-1$. Only this case we have to prove. The proof consists in a series of calculations and in producing a connection to the general case ([S93]). As examples we list the explicit expressions for the dimensions of the atypical irreducible $sl(1,3)$ modules:

$$\begin{aligned}
&\dim Y_3([m_1, m_2, 2])\\
&= 4\cdot\tfrac{1}{2}(m_1-m_2+1)\Big(H_0\sigma_2(m_1,m_2-1)+H_1\sigma_1(m_1,m_2-1)+H_2\Big)\\
&= (m_1-m_3+2)(-2m_1m_3+3m_1-m_3+2),\\
&\dim Y_2([m_1, 1, m_3])\\
&= 4\cdot\tfrac{1}{2}\cdot(-1)(m_1-m_3+2)\Big(H_0\sigma_2(m_1,m_3-2)+H_1\sigma_1(m_1,m_3-2)+H_2\Big)\\
&= (m_1-m_3+2)(-2m_1m_3+3m_1-m_3+2),\\
&\dim Y_1([0, m_2, m_3])\\
&= 4\cdot\tfrac{1}{2}(m_2-m_3+1)\Big(H_0\sigma_2(m_2-1,m_3-2)+H_1\sigma_1(m_2-1,m_3-2)+H_2\Big)\\
&= (m_2-m_3+1)(2m_2m_3-3m_2-m_3+1).
\end{aligned}$$

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