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A class of L^p convolutors on harmonic extensions of H-type groups

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Abstract. Let S be a harmonic extension of an H-type group and call δ its modular function. We find sufficient conditions on the spherical transform of a radial distribution k on S, so that $\delta^{-z}k$ is a left [resp. a right] convolutor of $L^p(S)$, $1 < p < \infty$, for every complex z with $0 \leq \mathrm{Re}z \leq 1$ [resp. $-1 + 1/p \leq \mathrm{Re}z \leq 1/p$].

1. Introduction

Given an H-type group N, let $S = NA$ be the solvable extension of N obtained by letting $A = \mathbb{R}^+$ act on N by homogeneous dilations; S is nonunimodular, hence, as a Riemannian manifold, it is of exponential growth.

Geometric properties of these groups were studied by several authors; among these E. Damek [8,9], who proved that in the general case there is no group K acting transitively on geodesic spheres, A. Korányi [19], who proved that the nilpotent algebra in the Iwasawa decomposition of a rank one semisimple Lie algebra is H-type, and M. Cowling, A. H. Dooley, Korányi and F. Ricci [5,6], who proved that these manifolds, whose structure is very similar to that of rank one symmetric spaces, are in fact symmetric only in a few cases. Moreover Damek and Ricci [10] showed that these extensions provide examples of nonsymmetric harmonic manifolds, so disproving the Lichnerowicz conjecture [22].

In the very last years, Damek and Ricci [11], Ricci [20] began the study of spherical analysis, i.e. the analysis of spherical functions and distributions on these groups. They proved that, despite the lack of a compact group acting transitively on geodesic spheres, one can define a transform, mapping radial distributions on S to even distributions on $\mathbb{R}$, analogous to the spherical transform on rank one symmetric spaces.

A problem which has been studied in spherical analysis on a symmetric space G/K is the L^p spherical multiplier problem: give sufficient conditions on

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the spherical transform of a K–invariant distribution k on the symmetric space G/K so that the left-invariant operator $f \mapsto f * k$ is bounded on $L^p(G/K)$.

This problem has attracted some attention recently; we only mention a few results in the case of symmetric spaces of the noncompact type and we refer to [7] for further references.

The investigation started with the work of J. L. Clerc and E. M. Stein [4] for multipliers of the Laplace–Beltrami operator (i.e. K–invariant right convolutors) in the complex case; their work was carried on by R. J. Stanton and P. A. Tomas [21] in the rank one case; they proved that if m extends to a bounded holomorphic function on the strip $\mathcal{T} = \{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| < \rho\}$ and it satisfies Mihlin conditions of a suitable order, then m is an L^p multiplier; moreover an L^p multiplier needs to be holomorphic on the smaller strip $\{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| < |1-2/p|\rho\}$.

This result was improved by J. Ph. Anker [1] on symmetric spaces and by S. Giulini, G. Mauceri and S. Meda [14] on rank one symmetric spaces, in the complex case and on free groups. The first author proved that it is enough to check that a Hörmander condition holds on the upper edge of the strip $\mathcal{T}$. The latter considered more general kernels, which are not necessarily integrable at infinity; in the rank one case (of dimension n), given an even, holomorphic function m defined on the strip of the complex plane $\mathcal{T}$ and defining the function ω on $\mathbb{C}$ via $\omega(\lambda) = (\lambda^2+4\rho^2)^{(n-1)/4}$, they proved that if $(\omega m)(\cdot+i\rho)$ is a Euclidean Fourier multiplier of $L^p(\mathbb{R})$, then m is a multiplier of L^p (i.e. the inverse spherical transform of m is a right convolutor of L^p).

A nonholomorphic functional calculus was found by Cowling, Giulini, A. Hulanicki and Mauceri [7] for multipliers of a distinguished right-invariant laplacian Δ on symmetric spaces, which is closely related to the Laplace–Beltrami operator L. A similar result can be obtained also in the case of the harmonic groups we are considering [2]; we remark that the kernel κ of the operator $m(\Delta)$ is related to the radial kernel k of the operator $m(L)$ via the formula $\kappa = \delta^{-1/2}k$, where δ is the modular function of the group S. Moreover, since L is a left-invariant operator and Δ is a right-invariant operator, we have $m(L)f = f * k$ and $m(\Delta)f = \kappa * f$ for every function f on the group S.

These results suggest the investigation of the following problem: given a radial distribution k on S and z in $\mathbb{C}$, find sufficient conditions on the spherical transform m of k, so that $\delta^{-z}k$ is a left or a right convolutor of $L^p(S)$ (with respect to left Haar measure).

Suppose that the function m is holomorphic on the strip

$$\mathcal{T} = \{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| < Q/2\},$$

where Q is the homogeneous dimension of N, and define the function $\omega : \mathcal{T} \to \mathbb{C}$ by the rule

$$\omega(\lambda) = (\lambda^2 + Q^2)^{(n-1)/4}.$$

Assume that ωm is bounded (and holomorphic) on $\mathcal{T}$, so that, by the Fatou Theorem, ωm has nontangential limit on the upper edge of $\mathcal{T}$, i.e., the line $\lambda + iQ/2$, $\lambda \in \mathbb{R}$; denote by $(\omega m)_{Q/2}$ this nontangential limit.

Our result is the following:

Theorem 1.1. *Let S be a harmonic extension of an H-type group and suppose that m is an even, holomorphic function on the strip $\mathcal{T}$ such that $(\omega m)_{Q/2}$ is a Euclidean Fourier multiplier of $\mathbb{R}$ for some p, $1 < p < \infty$. Then $\delta^{-z}k$ is a left convolutor [resp. a right convolutor] of $L^p(S)$ for every complex z, $0 \leq Re(z) \leq 1$ [resp. $-1/p' \leq Re(z) \leq 1/p$] with norm bounded by $\|(\omega m)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}$.*

As we are requiring the function m to be holomorphic, the result in [2] does not follow from Theorem 1.1; however Theorem 1.1 extends the result in [14] regarding rank one symmetric spaces to the harmonic groups S we are considering.

As proved in [14] a theorem imposing Hörmander conditions follows from ours and the condition we impose on m is optimal for k to be a right convolutor of $L^p(S)$, when S is symmetric.

The methods we employ are closely related to those in [14] and are based on precise information about spherical functions and a transference result of C. Herz [16], which lets us link left radial convolutors on S and convolutors on $\mathbb{R}$ in a simple way. Moreover we notice that in the same hypotheses k is also a radial left convolutor of $L^{p'}(S)$, so that $\delta^{-1}k$ is a left convolutor of $L^p(S)$ and our result follows by an interpolation argument.

Our paper is organized as follows: in Section 2 we introduce some notation and recall the basic facts about H-type groups and Figà-Talamanca–Herz algebras.

In Section 3 we obtain asymptotic estimates of spherical functions both when the distance from the identity is small, applying the same techniques of Stanton and Tomas [21], and when the distance from the identity is large, extending the results of Harish-Chandra and R. Gangolli [13].

Finally, in Section 4 we prove Theorem 1.1.

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2. Preliminaries

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra equipped with an inner product; $\mathfrak{n}$ has a nontrivial centre $\mathfrak{z}$ and we denote by $\mathfrak{v}$ its orthogonal complement with respect to the inner product.

According to A. Kaplan [17], $\mathfrak{n}$ is said to be an H-type Lie algebra if for every unitary Z in $\mathfrak{z}$ the map $J_Z : \mathfrak{v} \longrightarrow \mathfrak{v}$, defined by the relation

$$\langle J_Z X, Y\rangle = \langle Z, [X, Y]\rangle,$$

is orthogonal.

An H-type group N is a connected, simply connected Lie group whose Lie algebra $\mathfrak{n}$ is H-type. Let S be a one dimensional extension of the group N obtained by making $A = \mathbb{R}^+$ act on N by homogeneous dilations; let H denote the vector of $\mathfrak{a} \simeq \mathbb{R}$, the Lie algebra of A, acting on $\mathfrak{n}$ with eigenvalues $1/2$ and 1; we can extend the original inner product on $\mathfrak{n}$ to the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of the group S by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be unitary.

The map

$$\begin{array}{ccc} \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}^+ & \longrightarrow & S \\ (X, Z, a) & \mapsto & \exp_S(X + Z)\exp_S(\log aH) \end{array}$$

is a global chart and using these coordinates the group law can be written as

$$(X,Z,a)(X',Z',a') = \Big(X + a^{1/2}X', Z + aZ' + \frac{1}{2}a^{1/2}[X,X'], aa'\Big). \qquad (1)$$

If $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$ are the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ respectively, then $n = m_{\mathfrak{v}} + m_{\mathfrak{z}} + 1$ and $Q = m_{\mathfrak{v}}/2 + m_{\mathfrak{z}}$ are the dimension of S and the homogeneous dimension of N. From formula (1), it is easy to check that the left Haar measure is $a^{-Q-1}dXdZda$, and the right Haar measure is $a^{-1}dXdZda$, so that the modular function is $\delta(X,Z,a) = a^{-Q}$.

Cowling, Dooley, Korányi and Ricci [6] proved that a model for the action of the group S is the unit ball $\mathcal{B} = \{(X,Z,t) \in \mathfrak{s} : \|X\|^2 + \|Z\|^2 + |t|^2 < 1\}$ and we can identify the group S with the ball $\mathcal{B}$ via the map

$$\begin{aligned}\tau(X,Z,a) &= \frac{1}{\left(1+a+\frac{1}{4}\|X\|^2\right)^2 + \|Z\|^2}\Big(\Big(1 + a + \frac{1}{4}\|X\|^2 - J_Z\Big)X, 2Z, \\ &\qquad\qquad -1 + \Big(a + \frac{1}{4}\|X\|^2\Big)^2 + \|Z\|^2\Big).\end{aligned}$$

In this model the geodesics through the identity are the radii in $\mathcal{B}$ and geodesic spheres are the spheres in the norm of $\mathfrak{s}$.

We say that a function ϕ on S is radial if it depends only on the distance from the identity. A radial function ϕ is said to be spherical if

(*i*) $\phi(e) = 1$;

(*ii*) ϕ is an eigenfunction of the Laplace–Beltrami operator L.

Let r be the distance of the point (X,Z,a) from the identity and define the function $A : \mathbb{R}^+ \longrightarrow \mathbb{R}$ by

$$A(r) = 2^{m_{\mathfrak{v}}+2m_{\mathfrak{z}}}\,(\sinh(r/2))^{m_{\mathfrak{v}}+m_{\mathfrak{z}}}\,(\cosh(r/2))^{m_{\mathfrak{z}}};$$

then the left Haar measure dx of the group S may be normalized so that

$$dx = A(r)\,dr\,d\sigma(\omega),$$

where $d\sigma(\omega)$ is the surface measure on the unit sphere $\Omega = \partial\mathcal{B}$.

For every function f in $C_c^\infty(S)$ define the radial function πf in $C_c^\infty(S)$ by averaging the values of f over geodesic spheres; the operator π defined in this way is continuous on $C_c^\infty(S)$, so it extends to distributions. We say that a distributon T on S is radial if $\pi T = T$; every distribution T can be decomposed uniquely into the sum $T_1 + T_2$, with T_1 radial and $\pi T_2 = 0$.

As proved in [11], all spherical functions are of the form

$$\phi_\lambda = \pi(\delta^{i\lambda/Q-1/2}), \qquad \lambda \in \mathbb{C};$$

the corresponding eigenvalue is $-(\lambda^2 + Q^2/4)$ and $\phi_\lambda = \phi_{-\lambda}$.

If f is a radial function, its spherical transform is defined by

$$\widetilde{f}(\lambda) = \int_S f(x)\,\phi_\lambda(x)\,dx$$

for all values of λ for which the integral converges.

Ricci [20] proved an inversion formula for this spherical transform: the rôle of Harish-Chandra $\mathbf{c}$–function in the case of rank one symmetric spaces is played here by the function

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)}{\Gamma\left(\frac{Q+2i\lambda}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m_{\mathfrak{v}}+4i\lambda+2}{4}\right)}, \qquad \forall\lambda \in \mathbb{R}.$$

The Plancherel measure is given by $d\mu(\lambda) = |\mathbf{c}(\lambda)|^{-2} d\lambda$ on $\mathbb{R}$ so that, for radial functions f in $C_c(S)$, the inversion formula reads

$$f(x) = c_S \int_{\mathbb{R}} \widetilde{f}(\lambda)\, \phi_\lambda(x)\, d\mu(\lambda),$$

where the constant c_S depends only on $m_{\mathfrak{v}}$ and $m_{\mathfrak{z}}$.

We recall some facts about Figà-Talamanca–Herz algebras on locally compact groups G that will be crucial in our proof (see [12,16]). L^p spaces will always be considered with respect to left Haar measure.

Let p be in $(1,\infty)$, and denote by p' its conjugate index. The space $LCv_p(G)$ of left convolutors of $L^p(G)$ is defined as the space of all bounded linear operators on $L^p(G)$ which commute with right translations.

For every k in $L^1(G)$, let $\lambda_p(k)$ denote the operator $f \mapsto k * f$ on $L^p(G)$; clearly $\lambda_p(L^1(G))$ is contained in $LCv_p(G)$, and its ultraweak closure in $LCv_p(G)$ is the space of left p-pseudomeasures $PM_p(G)$; when $p = 2$ or G is amenable, we have $PM_p(G) = LCv_p(G)$.

$PM_p(G)$ is the Banach dual of a space of continuous functions on G vanishing at infinity; more precisely, the Figà-Talamanca–Herz algebra $A_p(G)$ is defined as the space of all functions φ in $C_o(G)$ that can be expressed as a sum $\sum_i g_i * \check{f}_i$, with $f_i \in L^p(G)$, $g_i \in L^{p'}(G)$ and $\check{f}_i(x) = f_i(x^{-1})$.

The norm of a function φ in $A_p(G)$ is

$$\|\varphi\|_{A_p(G)} = \inf\left\{\sum_i \|g_i\|_{L^{p'}(G)} \|f_i\|_{L^p(G)} : \varphi = \sum_i g_i * \check{f}_i,\ f_i \in L^p(G),\ g_i \in L^{p'}(G)\right\}.$$

The dual $A_p(G)^*$ is isometrically identified with $PM_p(G)$ via the pairing

$$\langle T, \varphi\rangle = \sum_i \int_G T f_i\, g_i\, dx \qquad T \in PM_p(G),\ \varphi \in A_p(G),$$

for any decomposition of φ as above.

The Herz restriction Theorem [16] asserts that if H is a closed subgroup of G, then the restriction to H of functions on G is a continuous operator from $A_p(G)$ to $A_p(H)$.

3. Asymptotic expansions for the spherical functions

As the action of the Laplace–Beltrami operator L on a smooth radial function f is

$$L_{rad} f = \frac{\partial^2}{\partial r^2} f + \left(\frac{m_{\mathfrak{v}}}{2} \coth \frac{r}{2} + m_{\mathfrak{z}} \coth r\right) \frac{\partial}{\partial r} f \qquad (2)$$

(see [11]), spherical functions may be expressed as hypergeometric functions via the formula

$$\phi_\lambda(r) = {}_2F_1\left(\frac{1}{2}(Q-2i\lambda), \frac{1}{2}(Q+2i\lambda), \frac{n}{2}, -\sinh^2\frac{r}{2}\right), \tag{3}$$

or as Jacobi functions (see [18]) $\phi_\lambda^{(\alpha,\beta)}$ of parameters $\alpha = (m_{\mathfrak{v}} + m_{\mathfrak{z}} - 1)/2$ and $\beta = (m_{\mathfrak{z}} - 1)/2$ via

$$\phi_\lambda(r) = \phi_{2\lambda}^{(\alpha,\beta)}(r/2).$$

Our goal is to obtain an expansion of spherical functions ϕ_λ in terms of Bessel functions J_μ as in [21].

It is useful to define the number

$$c_0 = 2^{m_{\mathfrak{z}}}\,\pi^{-1/2}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}$$

and the functions on $\mathbb{C}$

$$\mathcal{J}_\mu(z) = 2^\mu\,\pi^{1/2}\,\Gamma\left(\mu+\frac{1}{2}\right)\frac{J_\mu(z)}{z^\mu},$$

for every $\mu \geq 0$.

In the next theorem we examine the local behaviour of spherical functions and obtain an asymptotic expansion.

Theorem 3.1. *There exist R_0, $2 < R_0 < 2R_1$, such that for any r, $0 \leq r \leq R_0$ and any integer $N \geq 0$, we have*

$$\phi_\lambda(r) = c_0\left(\frac{r^{n-1}}{A(r)}\right)^{1/2}\sum_{\ell=0}^{N} a_\ell(r)\mathcal{J}_{\frac{n-2}{2}+\ell}(\lambda r)\,r^{2\ell} + E_{N+1}(\lambda,r)$$

where

$$a_0 \equiv 1 \qquad |a_\ell(r)| \leq C\,(4R_1)^{-\ell}$$

and the error term has the following behaviour

$$|E_{N+1}(\lambda,r)| \leq c_N \begin{cases} r^{2(N+1)} & \text{if } |\lambda r| \leq 1 \\ r^{2(N+1)}|\lambda r|^{-((n-1)/2+N+1)} & \text{if } |\lambda r| > 1. \end{cases}$$

Moreover, for every r, $0 \leq r < 2$, the series

$$\phi_\lambda(r) = c_0\left(\frac{r^{n-1}}{A(r)}\right)^{1/2}\sum_{\ell=0}^{\infty} a_\ell(r)\mathcal{J}_{\frac{n-2}{2}+\ell}(\lambda r)\,r^{2\ell}$$

is absolutely convergent.

We omit the proof of this theorem, which is a straightforward adaptation from [21], Theorem 2.1.

Now we employ Harish-Chandra method (see [15], p. 427), to study the asymptotic behaviour of spherical functions ϕ_λ when the distance from the identity is large.

We can write the radial part L_{rad} of the Laplace–Beltrami operator (2) as

$$\frac{\partial^2}{\partial r^2} + Q\frac{\partial}{\partial r} + \frac{m_{\mathfrak{v}}}{2}\left(\coth\frac{r}{2} - 1\right)\frac{\partial}{\partial r} + m_{\mathfrak{z}}\left(\coth r - 1\right)\frac{\partial}{\partial r}$$

and regard the last two terms as perturbations for large r.

So we try to find radial solutions to the equation $Lu = -(\lambda^2 + Q^2/4)u$ of the form

$$\sum \Gamma_\mu(\lambda)\, e^{(i\lambda - Q/2-\mu)r}.$$

Substituting this expression into the equation and equating coefficients to the powers $e^{(i\lambda-Q/2-\mu)r}$, we find the following recursion formula for the coefficient Γ_μ:

$$\begin{aligned}\Gamma_\mu(\lambda)\,(\mu^2 - 2i\mu\lambda) \quad &= \quad m_{\mathfrak{v}} \sum_{j=1}^{\mu} \Gamma_{\mu-j}(\lambda)\,(Q/2 + \mu - j - i\lambda) \\ &\qquad + 2m_{\mathfrak{z}} \sum_{j=1}^{[\mu/2]} \Gamma_{\mu-2j}(\lambda)\,(Q/2+\mu-2j-i\lambda),\end{aligned} \tag{4}$$

where $[\mu/2]$ is the greatest integer $\leq \mu/2$.
Let $\Gamma_0 \equiv 1$; then $\Gamma_\mu(\lambda)$ is defined by formula (4) for every $\mu > 0$.

As in [15], p. 428, Lemma 5.3, we can prove that for λ in $\mathbb{C} \setminus \frac{1}{2}i\mathbb{Z}$, for any fixed $R > 0$, $|\Gamma_\mu(\lambda)| \leq C_{\lambda,R}\, e^{\mu R}$ so that the series $\sum \Gamma_\mu(\lambda) e^{(i\lambda-Q/2-\mu)r}$ is absolutely and uniformly convergent outside a fixed ball and the term by term differentiation is justified; moreover we can work out the analogue of Harish-Chandra well-known formula

$$\phi_\lambda(r) = \mathbf{c}(\lambda) \sum \Gamma_\mu(\lambda) e^{(i\lambda-Q/2-\mu)r} + \mathbf{c}(-\lambda) \sum \Gamma_\mu(-\lambda) e^{(-i\lambda-Q/2-\mu)r} \tag{5}$$

for λ in $\mathbb{C} \setminus \frac{1}{2}i\mathbb{Z}$.

As in the symmetric case, we can compute the function $\mathbf{c}$ by evaluating

$$\lim_{r\to+\infty} e^{(-i\lambda+Q/2)r}\phi_\lambda(r)$$

when $-m_{\mathfrak{v}}/4 - 1 < \mathrm{Im}(\lambda) < 0$ and $\lambda \in \mathbb{C}\setminus\frac{1}{2}i\mathbb{Z}$ in two different ways; remembering formula (5), this limit equals $\mathbf{c}(\lambda)$, while remembering formula (3), this limit equals

$$\frac{2^{Q-2i\lambda}\Gamma(2i\lambda)}{\Gamma\left(\frac{Q+2i\lambda}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m_{\mathfrak{v}}+4i\lambda+2}{4}\right)}.$$

By analytic continuation, we can conclude that

$$\mathbf{c}(\lambda) = \frac{2^{Q-2i\lambda}\Gamma(2i\lambda)}{\Gamma\left(\frac{Q+2i\lambda}{2}\right)} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{m_{\mathfrak{v}}+4i\lambda+2}{4}\right)} \quad \forall \lambda \in \mathbb{C} \setminus \left\{0,\, \frac{i}{2},\, i,\, \frac{3}{2}i, \ldots\right\}. \tag{6}$$

Let D be the region of the complex plane consisting of the points λ in $\mathbb{C}$ such that $|\mathrm{Im}(\lambda)| \leq |\mathrm{Re}(\lambda)|$ or $\mathrm{Im}(\lambda) \geq 0$; on D we determine a better control of the coefficients $\Gamma_\mu(\lambda)$ by adapting the method of Giulini, Mauceri and Meda [14], Lemma 3.3.

Theorem 3.2. *There exists a constant d such that*

$$\sup_{\lambda \in D} |\Gamma_\mu(\lambda)| \leq C\,(1+\mu)^d, \qquad \forall \mu \in \mathbb{N}.$$

Proof. The theorem is obviously true for $\mu = 0$ with $d = 0$; so we may assume that $\mu > 0$.

Let D_0 be the region $\{\lambda \in \mathbb{C} : |\mathrm{Im}(\lambda)| \leq |\mathrm{Re}(\lambda)|\}$; then D can be written as the union of the regions $D_0 + i\theta$, $\theta \geq 0$. We will prove the theorem by showing that, for every fixed $\theta \geq 0$, there exist constants C and d, independent from θ, such that

$$\sup_{\lambda \in D_0} |\Gamma_\mu(\lambda + i\theta)| \leq C\,\mu^d$$

for every $\mu \geq 1$.

We notice that for every a, b in $\mathbb{R}^+$

$$\sup_{z \in D_0} \left| \frac{a - iz}{b - iz} \right| \leq \sqrt{2}\max(1, a/b),$$

so that, remembering formula (4),

$$\sup_{\lambda \in D_0} |\Gamma_1(\lambda + i\theta)| \leq \frac{\sqrt{2}\,m_{\mathfrak{v}}}{2} \max\left(\frac{2\theta + Q}{2\theta + 1}, 1\right) \leq \frac{\sqrt{2}\,Q\,m_{\mathfrak{v}}}{2}.$$

Let $\bar{c}$ be the constant $\sqrt{2}Qm_{\mathfrak{v}}/2$. We proceed by induction on μ. Suppose that the estimate

$$\sup_{\lambda \in D_0} |\Gamma_\nu(\lambda + i\theta)| \leq \bar{c}\,\nu^d$$

holds for every $\nu \leq \mu - 1$; we will show that

$$\sup_{\lambda \in D_0} |\Gamma_\mu(\lambda + i\theta)| \leq \frac{\sqrt{2}(3+Q)Q}{d+1}\,\bar{c}\,\mu^d.$$

This estimate is deduced from the recursion formula (4); in fact for every complex λ in D_0, we have for the first term

$$\frac{m_{\mathfrak{v}}}{\mu} \sum_{j=0}^{\mu-1} |\Gamma_j(\lambda + i\theta)| \left| \frac{Q/2 + \theta + j - i\lambda}{\mu + 2\theta - 2i\lambda} \right| \leq \frac{\sqrt{2}\,m_{\mathfrak{v}}\,\bar{c}}{2\mu} \sum_{j=0}^{\mu-1} \max\left(1, \frac{Q/2 + \theta + j}{\mu/2 + \theta}\right) j^d.$$

Approximating the sum with an integral, we obtain

$$\begin{aligned} &\leq \frac{\sqrt{2}\,m_{\mathfrak{v}}\,\bar{c}}{2\mu} \int_0^\mu \left(\frac{Q + 4\theta + \mu}{2\theta + \mu}\,x^d + \frac{2}{2\theta + \mu}\,x^{d+1} \right) dx \\ &\leq \frac{\sqrt{2}\,m_{\mathfrak{v}}(3\mu + 4\theta + Q)}{2(d+1)(\mu + 2\theta)}\,\bar{c}\,\mu^d \leq \frac{\sqrt{2}\,m_{\mathfrak{v}}(3+Q)}{2(d+1)}\,\bar{c}\,\mu^d. \end{aligned}$$

For the second term we may compute in a similar way that

$$\left| \frac{2m_{\mathfrak{z}}}{\mu} \sum_{j=1}^{[\mu/2]} \Gamma_{\mu-2j}(\lambda+i\theta)\, \frac{Q/2+\theta+\mu-2j-i\lambda}{\mu+2\theta-2i\lambda} \right| \le \frac{\sqrt{2}\, m_{\mathfrak{z}}(3+Q)}{d+1}\, \bar{c}\, \mu^d.$$

Now we may choose d big enough so that $\sqrt{2}Q(3+Q) \le d+1$ and the desired estimate for $\Gamma(\lambda+i\theta)$ follows.

Notice that all the constants involved do not depend on θ, so the theorem is proved. ∎

4. Proof Theorem 1.1

Our first object is to prove that k is a left convolutor of $L^p(S)$. In order to do this, we proceed as in the paper by Giulini, Mauceri and Meda [14]. We split up the proof into two parts: the analysis of the kernel near the identity and the analysis of the kernel away from the identity (Proposition 4.3 and Proposition 4.5, respectively), where we use the results of the previous section; we deal with left convolutors first because of the restriction argument of Lemma 4.1.

Afterwards we exploit the duality argument ($\mathcal{M}_p(\mathbb{R}) = \mathcal{M}_{p'}(\mathbb{R})$) to prove that k is also a left convolutor of $L^{p'}(S)$; as radial left convolutors are not self-adjoint, this fact adds true information and the full result follows from the Stein interpolation Theorem.

Let k be the inverse spherical transform of m, i.e.,

$$k(x) = \int_{\mathbb{R}} m(\lambda)\, \phi_\lambda(x)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda$$

in distributional sense.

For the sake of brevity, denote by $M(\lambda)$ the function on $\mathbb{R}$

$$M(\lambda) = m(\lambda)\mathbf{c}^{-1}(-\lambda).$$

The function M extends to an holomorphic and bounded function in any strip of the form $\{\lambda \in \mathbb{C} : \varepsilon - Q/2 \le \operatorname{Im}(\lambda) < Q/2\}$, $\varepsilon > 0$; in particular M has nontangential limit on the line $\lambda + iQ/2$, $\lambda \in \mathbb{R}$, which we will call $(M)_{Q/2}$.

Our proof relies strongly on the following consequence of the Herz restriction Theorem [16]:

Lemma 4.1. *Let p be in $(1,\infty)$ and suppose that h^0 is an even distribution on $\mathbb{R}$ such that $h^0 A(|\cdot|)$ is in $Cv_p(\mathbb{R})$. Then the radial distribution h on S, defined by*

$$\langle\, h, \varphi \,\rangle_S = \langle\, h^0 A(|\cdot|), \varphi(|\cdot|) \,\rangle_{\mathbb{R}}$$

for every radial function φ in $C_c^\infty(S)$, is a left convolutor of $L^p(S)$ and

$$\|h\|_{LCv_p(S)} \le \|h^0 A(|\cdot|)\|_{Cv_p(\mathbb{R})}.$$

Proof. For every ω in the unit sphere Ω of the Lie algebra $\mathfrak{s}$ define the restriction operator $\mathcal{R}_\omega : C_c^\infty(S) \to C_c^\infty(\mathbb{R})$ restricting the values of a function on the group S to the one parameter group defined by the vector ω, i.e.,

$$(\mathcal{R}_\omega f)(r) = f(r\omega),$$

for every function f in $C_c^\infty(S)$ and every r in $\mathbb{R}$.

By the Herz restriction Theorem [16], each $\mathcal{R}_\omega$ extends to a bounded operator from $A_p(S)$ to $A_p(\mathbb{R})$, with norm uniformly bounded by 1.

Define the operator $\mathcal{R} : C_c^\infty(S) \to C_c^\infty(\mathbb{R})$ by

$$\mathcal{R}f = \int_\Omega \mathcal{R}_\omega f \, d\sigma(\omega).$$

Since

$$\|\mathcal{R}f\|_{A_p(\mathbb{R})} \le \int_\Omega \|\mathcal{R}_\omega f\|_{A_p(\mathbb{R})} \, d\sigma(\omega) \le C \, \|f\|_{A_p(S)},$$

$\mathcal{R}$ extends to a bounded operator from $A_p(S)$ to $A_p(\mathbb{R})$.

But for every radial test function φ on the group S, we have

$$\langle h, \varphi \rangle_S = \langle h^0 A(|\cdot|), \varphi(|\cdot|) \rangle_{\mathbb{R}} = \langle \, {}^t\mathcal{R}\left(h^0 A(|\cdot|)\right), \varphi \rangle_S .$$

Moreover the distribution ${}^t\mathcal{R}\,(h^0 A(|\cdot|))$ is radial, so that $h = {}^t\mathcal{R}\,(h^0 A(|\cdot|))$.

Therefore, as ${}^t\mathcal{R}$ is a bounded operator from the dual $A_p(\mathbb{R})^* = Cv_p(\mathbb{R})$ to $A_p(S)^* = LCv_p(S)$ (remember that S is solvable, hence amenable), we conclude that h is in $LCv_p(S)$ and

$$\|h\|_{LCv_p(S)} = \left\| {}^t\mathcal{R}\left(h^0 A(|\cdot|)\right) \right\|_{LCv_p(S)} \le \|h^0 A(|\cdot|)\|_{Cv_p(\mathbb{R})}$$

as required. ∎

We will have to estimate derivatives of the $\mathbf{c}$-function, so we prove the following Lemma.

Lemma 4.2. *The function $\lambda \mapsto |\mathbf{c}(\lambda)|^{-2}$ on $\mathbb{R}$ satisfies the estimate*

$$\left| \frac{d^\alpha}{d\lambda^\alpha} |\mathbf{c}(\lambda)|^{-2} \right| \le C_\alpha (1 + |\lambda|)^{n-1-\alpha}$$

for every integer $\alpha \ge 0$.

Proof. This is an immediate consequence of formula (6). Indeed we can easily check that

if $m_{\mathfrak{z}} = 2\kappa$ and $m_{\mathfrak{v}} = 2\ell$ are both even,

$$|\mathbf{c}(\lambda)|^{-2} = C \prod_{j=0}^{\kappa-1} \left((j + \ell/2)^2 + \lambda^2\right) \prod_{j=0}^{\ell-1} (j^2 + 4\lambda^2);$$

if $m_{\mathfrak{z}} = 2\kappa + 1$ is odd and $m_{\mathfrak{v}} = 4\ell$,

$$|\mathbf{c}(\lambda)|^{-2} = C\,\lambda \tanh(\pi\lambda) \prod_{j=0}^{\ell+\kappa-1} \Big((j+1/2)^2 + \lambda^2\Big) \prod_{j=0}^{\ell-1} \Big((j+1/2)^2 + \lambda^2\Big);$$

if $m_{\mathfrak{z}} = 2\kappa + 1$ is odd and $m_{\mathfrak{v}} = 4\ell + 2$,

$$|\mathbf{c}(\lambda)|^{-2} = C\,\lambda \coth(\pi\lambda) \prod_{j=0}^{\ell+\kappa} (j^2 + \lambda^2) \prod_{j=1}^{\ell} (j^2 + \lambda^2);$$

so the result follows by straightforward computations. ∎

We start by analyzing the local behaviour of the kernel k.

Let ψ be a smooth cut-off function on the group S supported in the ball of radius R_0 centered at the identity such that $0 \le \psi \le 1$, $\psi = 1$ in the ball of radius $R_0^{1/2}$, where R_0 is defined in Theorem 3.1.

Proposition 4.3. *Let m be an even function on $\mathbb{R}$ and suppose that M is a Euclidean Fourier multiplier of $L^p(\mathbb{R})$, for some p, $1 < p < \infty$. Then ψk is in $LCv_p(S)$ and*

$$\|\psi k\|_{LCv_p(S)} \le C\,\|M\|_{\mathcal{M}_p(\mathbb{R})}.$$

Proof. We claim that it is enough to prove the proposition when m is rapidly decreasing at infinity.

In fact, for general m, we define the multipliers on $\mathbb{R}$

$$m_t(\lambda) = m(\lambda) e^{-t(\lambda^2 + Q^2/4)} \qquad \forall t \in \mathbb{R}^+.$$

Then the inverse spherical transform k_t of the multiplier m_t is given by

$$k_t = h_t * k,$$

where h_t is the heat kernel corresponding to e^{tL} on S; moreover m_t is rapidly decreasing at infinity, therefore by this lemma applied to m_t, we obtain

$$\|\psi k_t\|_{LCv_p(S)} \le C\,\|M_t\|_{\mathcal{M}_p(\mathbb{R})}.$$

But $\|M_t\|_{\mathcal{M}_p(\mathbb{R})} = \|M\|_{\mathcal{M}_p(\mathbb{R})}$ for all t in $\mathbb{R}^+$, hence

$$\sup_t \|\psi k_t\|_{LCv_p(S)} \le C\,\|M\|_{\mathcal{M}_p(\mathbb{R})}$$

and we conclude

$$\|\psi k\|_{LCv_p(S)} = \lim_{t\to 0^+} \|\psi k_t\|_{LCv_p(S)} \le C\,\|M\|_{\mathcal{M}_p(\mathbb{R})},$$

as required.

From now on we will assume that m is rapidly decreasing at infinity. By Theorem 3.1 for $N=1$, we have that $\psi(r)k(r)$ equals

$$\begin{aligned}
&\psi(r)\, c_0 \left(\frac{r^{n-1}}{A(r)}\right)^{1/2} \int_{\mathbb{R}} m(\lambda)\left(\mathcal{J}_{\frac{n-2}{2}}(\lambda r) + r^2 a_1(r)\, \mathcal{J}_{\frac{n}{2}}(\lambda r)\right) |\mathbf{c}(\lambda)|^{-2} d\lambda \\
&\qquad\qquad + \psi(r) \int_{\mathbb{R}} m(\lambda)\, E_2(\lambda, r)\, |\mathbf{c}(\lambda)|^{-2} d\lambda \\
= \;& \Psi_0(r) + \Psi_1(r) + \Psi_2(r).
\end{aligned}$$

We split the integrals over $\mathbb{R}$ into the sum of the integrals where $|\lambda| < r^{-1}$ and $|\lambda| \geq r^{-1}$, so we rewrite the last sum as $\Psi_0^0 + \Psi_0^\infty + \Psi_1^0 + \Psi_1^\infty + \Psi_2^0 + \Psi_2^\infty$.

Ψ_0^0 is in $L^1(S)$, hence it is a left convolutor of $L^p(S)$, for every p in $[1,\infty]$; indeed, remembering that for small r, $A(r)$ behaves as r^{n-1}, $|\mathbf{c}^{-1}(\lambda)| \leq (1+|\lambda|)^{(n-1)/2}$ and $|\mathcal{J}_\mu(z)| \leq 1$ for $z \in [0,1]$, we have

$$\begin{aligned}
\int_S |\Psi_0^0(x)| dx &= c_0 \int_0^{R_0} \psi(r) \left(\frac{r^{n-1}}{A(r)}\right)^{1/2} \int_{|\lambda|<r^{-1}} M(\lambda)\, \mathcal{J}_{\frac{n-2}{2}}(\lambda r)\, \mathbf{c}^{-1}(\lambda) d\lambda A(r) dr \\
&\leq C\, \|M\|_{L^\infty(\mathbb{R})}.
\end{aligned}$$

In a similar way one can check that Ψ_1^0, Ψ_2^0, Ψ_2^∞, are in $L^1(S)$ with norm bounded by $\|M\|_{L^\infty(\mathbb{R})}$.

For the other terms, we use the following asymptotic expansions of Bessel functions,

$$\mathcal{J}_{\frac{n-2}{2}+j}(z) = \sqrt{\frac{2}{\pi}}\, z^{\frac{1-n}{2}-j} \left(\cos(z+\varepsilon) - \beta_n \frac{\sin(z+\varepsilon)}{2z} + O(z^{-2})\right) \qquad \forall z \in [1,\infty),$$

where $\beta_n = \left(\frac{n-2}{2}\right)^2 - \frac{1}{4}$ and $\varepsilon = \left(\frac{1-n}{4} - \frac{j}{2}\right)\pi$.

So Ψ_j^∞ $(j=0,1)$ may be written as the sum of the radial functions $k_{j,0}$, $k_{j,1}$ and an error term; more precisely

$$k_{j,\ell}(r) = c_\ell\, \psi(r) \left(\frac{r^{n-1}}{A(r)}\right)^{1/2} r^{2j} a_j(r) \int_{|\lambda|>r^{-1}} M(\lambda) |\lambda r|^{-\alpha}\, T_\ell(\lambda r + \varepsilon) \mathbf{c}^{-1}(\lambda)\, d\lambda,$$

where $\alpha = \frac{n-1}{2} + j + \ell$, $T_0(t) = \cos t$ and $T_1(t) = \sin t$.

$k_{1,1}$ and the error term are easily seen to be in $L^1(S)$, with norm controlled by $\|M\|_{L^\infty(\mathbb{R})}$.

So we have to deal with the case $j + \ell \leq 1$; we denote by η a smooth cut-off function on the real line such that $0 \leq \eta \leq 1$, $\eta(r) = 1$ for $|r| \leq 1/R_0$ and $\eta(r) = 0$ for $|r| > 2/R_0$.

We split $k_{j,\ell}$ into the sum of the functions $E_{j,\ell}$ and $K_{j,\ell}$, where

$$E_{j,\ell} = c_\ell\, \psi(r) \left(\frac{r^{n-1}}{A(r)}\right)^{1/2} r^{2j} a_j(r) \int_{|\lambda|>r^{-1}} M(\lambda)\, \eta(\lambda)\, |\lambda r|^{-\alpha}\, T_\ell(\lambda r + \varepsilon)\, \mathbf{c}^{-1}(\lambda)\, d\lambda$$

$$K_{j,\ell} = c_\ell\, \psi(r) \left(\frac{r^{n-1}}{A(r)}\right)^{1/2} r^{2j} a_j(r) \int_{\mathbb{R}} M(\lambda)\, (1-\eta(\lambda))\, |\lambda r|^{-\alpha}\, T_\ell(\lambda r + \varepsilon)\, \mathbf{c}^{-1}(\lambda)\, d\lambda.$$

One can compute that $E_{j,\ell}$ is in $L^1(S)$, so we are left to deal with $K_{j,\ell}$.

By Lemma 4.1, all we need to prove is that the even function on $\mathbb{R}$ defined by

$$K^0_{j,\ell}(r) = A(r)\, K_{j,\ell}(r) \qquad \forall r \in \mathbb{R}^+$$

is in $Cv_p(\mathbb{R})$.

Since $Cv_p(\mathbb{R}) = Cv_{p'}(\mathbb{R})$ and $\mathcal{M}_p(\mathbb{R}) = \mathcal{M}_{p'}(\mathbb{R})$, we can restrict ourselves to the situation $1 < p \leq 2$.

Equivalently, we shall prove that the even function $K^*_{j,\ell} : \mathbb{R} \to \mathbb{C}$, defined by the rule

$$K^*_{j,\ell}(r) = B_{j,\ell}(r) \int_{\mathbb{R}} M(\lambda)\,(1-\eta(\lambda))\,|\lambda r|^{-\alpha}\, e^{-i\lambda r}\, \mathbf{c}^{-1}(\lambda)\, d\lambda \qquad \forall r \in \mathbb{R}^+ \tag{7}$$

is a convolutor of $L^p(\mathbb{R})$. The function $B_{j,\ell}$ in formula (7) is of the form

$$B_{j,\ell}(r) = r^{-2\ell+\alpha}\, \psi'(r),$$

where ψ' is a bounded function on $\mathbb{R}$, with compact support and smooth away from the origin.

Denote by P_α the function on the real line

$$P_\alpha(\lambda) = (1-\eta(\lambda))\,|\lambda|^{-\alpha}\,\mathbf{c}^{-1}(\lambda).$$

By Lemma 4.2, P_α is an Hörmander multiplier for every $\alpha \geq (n-1)/2$.

Assume first that $j+\ell = 1$, i.e., $\alpha = (n+1)/2$. Then $B_{j,\ell}$ is in $L^q(\mathbb{R})$ for every q in $(1,\infty)$ and P_α is the Fourier transform of a function in $L^q(\mathbb{R})$ for every q in $[1,\infty)$.

In our hypotheses the function M is in $\mathcal{M}_p(\mathbb{R}) = \mathcal{M}_{p'}(\mathbb{R})$, thus applying Hölder inequality, we obtain

$$\|K^*_{j,\ell}\|_{L^\infty(\mathbb{R})} \leq C\, \|B_{j,\ell}\|_{L^p(\mathbb{R})}\, \|\mathcal{F}(MP_\alpha)\|_{L^{p'}(\mathbb{R})} \leq C\, \|M\|_{\mathcal{M}_p(\mathbb{R})}.$$

This proves that the compactly supported function $K^*_{j,\ell}$ is in $L^1(\mathbb{R})$, with the appropriate norm bound.

Finally, if $j = \ell = 0$, i.e., $\alpha = (n-1)/2$, the function MP_α is in $\mathcal{M}_p(\mathbb{R})$ and $B_{0,0}$ is in the Besov–Lipschitz space $\Lambda^{1/2}_{2,1}(\mathbb{R})$, therefore (see [3]) $B_{0,0}$ is in $A_2(\mathbb{R}) \subseteq A_p(\mathbb{R})$. Since $Cv_p(\mathbb{R})$ is an $A_p(\mathbb{R})$-module, we deduce that $K^*_{j,\ell}$ is in $Cv_p(\mathbb{R})$ and

$$\|K^*_{j,\ell}\|_{Cv_p(\mathbb{R})} \leq \|B_{0,0}\|_{A_p(\mathbb{R})}\, \|\mathcal{F}(MP_\alpha)\|_{\mathcal{M}_p(\mathbb{R})} \leq C\, \|M\|_{\mathcal{M}_p(\mathbb{R})}.$$

This concludes the proof of the proposition. ∎

We now investigate the behaviour of the kernel k at a large distance from the identity. We will need a refined version of Theorem 3.2, regarding multiplier-type estimates of the coefficients Γ_μ in the expansion of spherical functions (5).

Lemma 4.4. *Let $1 < p < \infty$. The functions $\Gamma_\mu(\cdot + iQ/2)$ are Euclidean Fourier multipliers of $L^p(\mathbb{R})$ and there exists a constant d such that*

$$\|\Gamma_\mu(\cdot + iQ/2)\|_{\mathcal{M}_p(\mathbb{R})} \leq C\,\mu^d \qquad \forall \mu \geq 1.$$

Proof. From Theorem 3.2 it follows that

$$\|\Gamma_\mu(\cdot + iQ/2)\|_{L^\infty(\mathbb{R})} \leq C\,\mu^d \qquad \forall \mu \geq 1.$$

If $\gamma_{\lambda+iQ/2}$ denotes the circle in the complex plane centered at $\lambda + iQ/2$ and of radius $Q/2$, an application of the Cauchy integral formula yields

$$\left|\frac{d}{d\lambda}\Gamma_\mu(\lambda + iQ/2)\right| = \left|\frac{1}{2\pi i}\int_{\gamma_{\lambda+iQ/2}} \frac{\Gamma_\mu(w)}{(w-(\lambda+iQ/2))^2}\,dw\right| \leq C\,\mu^d,$$

but we can give a better estimate for the derivative of $\Gamma_\mu(\cdot + iQ/2)$ for large values of $|\lambda|$.

Let $\gamma'_{\lambda+iQ/2}$ be the circle in the complex plane centered at $\lambda + iQ/2$ and of radius $|\lambda|/\sqrt{2}$. Notice that $\gamma'_{\lambda+iQ/2}$ lies inside the region D. Thus by using the Cauchy integral formula, we obtain that

$$\left|\frac{d}{d\lambda}\Gamma_\mu(\lambda + iQ/2)\right| = \left|\frac{1}{2\pi i}\int_{\gamma'_{\lambda+iQ/2}} \frac{\Gamma_\mu(w)}{(w-(\lambda+iQ/2))^2}\,dw\right| \leq C\,\frac{\mu^d}{|\lambda|}. \qquad \blacksquare$$

Proposition 4.5. *Let m be an even function on $\mathbb{R}$ and suppose that M and $(M)_{Q/2}$ are Euclidean Fourier multipliers of $L^p(\mathbb{R})$, for some p, $1 < p < \infty$. Then $(1-\psi)k$ is in $LCv_p(S)$ and*

$$\|(1-\psi)k\|_{LCv_p(S)} \leq C\left(\|M\|_{\mathcal{M}_p(\mathbb{R})} + \|(M)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}\right).$$

Proof. We may assume that M and $(M)_{Q/2}$ are rapidly decreasing at infinity, otherwise we can argue as in Proposition 4.3.

By Lemma 4.1, it is enough to show that the even function K on $\mathbb{R}$ defined by

$$K(r) = A(r)\,(1-\psi(r))\,k(r) \qquad \forall r \in \mathbb{R}^+$$

is in $Cv_p(\mathbb{R})$.

Using formula (5) and expanding $A(r) = e^{Qr}\sum_{j=0}^{2Q} c_j e^{-jr}$, we can write for every r in $\mathbb{R}^+$

$$K(r) = (1-\psi(r))\,e^{Qr/2}\sum_{\mu=0}^{\infty} e^{-\mu r}\sum_{j=0}^{2Q} c_j \int_{\mathbb{R}} M(\lambda)\,\Gamma_{\mu-j}(\lambda)\,e^{i\lambda r}\,d\lambda,$$

with $\Gamma_\mu \equiv 0$ if $\mu < 0$.

Define the functions on the real line

$$\begin{aligned} a_\mu^\pm(r) &= (1-\psi(|r|))\,\chi_{[0,+\infty)}(\pm r)\,e^{\mp\mu r} \\ b_{\mu-j}^\pm(r) &= \int_{\mathbb{R}} M(\lambda)\,\Gamma_{\mu-j}(\lambda)\,e^{\pm(i\lambda+Q/2)r}\,d\lambda \\ K_{\mu,j} &= a_\mu^- b_{\mu-j}^- + a_\mu^+ b_{\mu-j}^+ . \end{aligned} \tag{8}$$

Then $K=\sum_{\mu=0}^\infty \sum_{j=0}^{2Q} c_j K_{\mu,j}$.

As the function $\lambda\mapsto \mathbf{c}^{-1}(-\cdot)\Gamma_\mu(\cdot)$ is analytic for $\mathrm{Im}(\lambda)>-m_{\mathfrak{v}}/4-1/2$ and M decays rapidly at infinity, we may shift the path of integration in (8) to the upper edge of $\mathcal{T}$ and obtain

$$K_{\mu,j}(r)=a_\mu^-(r)\,\mathcal{F}((M\Gamma_{\mu-j})_{Q/2})(r)+a_\mu^+(r)\,\mathcal{F}\left(M(-\cdot+iQ/2)\Gamma_{\mu-j}(-\cdot+iQ/2)\right)(r),$$

where $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}$.

If $\mu>0$, the functions $a_\mu^\pm$ are in $A_p(\mathbb{R})$, with norm bounded above by $C\,\mu^{1/2}e^{-\mu}$, because we can control their A_p-norm with their Sobolev norm $\|a_\mu^\pm\|_{L^2(\mathbb{R})}+\|(a_\mu^\pm)'\|_{L^2(\mathbb{R})}$.

$Cv_p(\mathbb{R})$ is an $A_p(\mathbb{R})$-module, therefore by Lemma 4.4 we have that

$$\begin{aligned} \|K_{\mu,j}\|_{Cv_p(\mathbb{R})} &\le \|a_\mu^-\|_{A_p(\mathbb{R})}\,\|(M\Gamma_{\mu-j})_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}+\|a_\mu^+\|_{A_p(\mathbb{R})}\,\|(M\Gamma_{\mu-j})_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})} \\ &\le C\mu^{d+1/2}e^{-\mu}\,\|(M)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}. \end{aligned}$$

We are left with the case $\mu=0$.

Let $\eta_\pm$ be the functions on the real line defined by the rule

$$\eta_\pm(r)=\Big((1-\psi(r))\,\chi_{[0,\infty)}(\pm r)-1\Big)\,e^{\mp Qr};$$

then $K_{0,0}$ can be written as

$$\begin{aligned} K_{0,0} &= b_0^-+(\eta_--1)\,b_0^-+b_0^++(\eta_+-1)\,b_0^+ \\ &= \mathcal{F}((M)_{Q/2})+\eta_-\mathcal{F}(M)+\eta_+\mathcal{F}(M(-\cdot))+\mathcal{F}(M(-\cdot+iQ/2)). \end{aligned}$$

We notice that we have expressed $K_{0,0}$ as the sum of the kernels corresponding to the multiplier $(M)_{Q/2}$ and of other two terms which are the product of a function in $A_p(\mathbb{R})$ times the kernel corresponding to the multiplier M.

So $\|K_{0,0}\|_{Cv_p(\mathbb{R})}\le C(\|M\|_{\mathcal{M}_p(\mathbb{R})}+\|(M)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})})$ and

$$\begin{aligned} \|K\|_{Cv_p(\mathbb{R})} &\le C\sum_{j=0}^{2Q}\sum_{\mu=1}^{\infty}\mu^{C_1+1}e^{-\mu}\Big((\|M\|_{\mathcal{M}_p(\mathbb{R})}+\|(M)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}\Big) \\ &= C\Big(\|M\|_{\mathcal{M}_p(\mathbb{R})}+\|(M)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}\Big). \end{aligned}$$ ■

We are now ready to prove our result, Theorem 1.1.

Proof of Theorem 1.1.

We argue as in [14], Theorem 3.1. By Proposition 4.3 and Proposition 4.5, it is enough to show that in our hypotheses M and $(M)_{Q/2}$ are Euclidean Fourier multipliers of $L^p(\mathbb{R})$.

The function $(\omega m)_{Q/2}$ is in $\mathcal{M}_p(\mathbb{R})$ and, since it is even, the same is true of $(\omega m)_{-Q/2}$; hence their kernels $\nu_1 = \mathcal{F}^{-1}((\omega m)_{Q/2})$ and $\nu_{-1} = \mathcal{F}^{-1}((\omega m)_{-Q/2})$ are convolutors of $L^p(\mathbb{R})$.

For every complex z, with $|\mathrm{Re}(z)| \le 1$, define the distributions ν_z on $\mathbb{R}$ by

$$\nu_z = \mathcal{F}^{-1}((\omega m)_{zQ/2})$$

Since for every real y, $\nu_{1+iy} = e^{iyQ/2}\nu_1$ and $\nu_{-1-iy} = e^{-iyQ/2}\nu_{-1}$, ν_{1+iy} and ν_{-1-iy} are convolutors of $L^p(\mathbb{R})$, with norm bounded by $\|(\omega m)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}$.

We now invoke the Stein complex interpolation theorem, to conclude that ν_η is a convolutor of $L^p(\mathbb{R})$ for every η in $(-1,1)$; therefore $(\omega m)_{\eta Q/2} = \mathcal{F}(\nu_\eta)$ is a Euclidean Fourier multiplier of $L^p(\mathbb{R})$ for every $\eta \in (-1,1)$, with norm bounded by $\|(\omega m)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}$.

Define the function $w : \mathcal{T} \to \mathbb{C}$ by

$$w(\lambda) = \frac{\mathbf{c}^{-1}(-\lambda)}{\omega(\lambda)}.$$

Using Lemma 4.2, we may prove that w and $(w)_{Q/2}$ are Euclidean Fourier multipliers of $L^p(\mathbb{R})$, because they satisfy Hörmander conditions of arbitrary order on $\mathbb{R}$. But then $M = \omega m\, w$ and $(M)_{Q/2} = (\omega m)_{Q/2}(w)_{Q/2}$ are Euclidean Fourier multipliers of $L^p(\mathbb{R})$ with the appropriate norm bound.

So far we have proved that k is a left convolutor of $L^p(S)$; to complete the proof of the theorem, we need two simple considerations.

Let q be in $(1,\infty)$ and let h be a left convolutor of $L^q(S)$; it is easy to check that if T_h denotes the operator defined on $L^q(S)$ by $f \mapsto h * f$, then the transpose operator ${}^tT_h$ coincides with the operator $T_{\delta^{-1}\check{h}}$ defined on $L^{q'}(S)$.

The second argument is the relation between left and right convolutors on nonunimodular groups; more precisely, h is a left convolutor of $L^q(S)$ if and only if $\delta^{-1/q}\check{h}$ is a right convolutor of $L^q(S)$.

As $\mathcal{M}_p(\mathbb{R}) = \mathcal{M}_{p'}(\mathbb{R})$ (with equality of norms), we infer that k is also a left convolutor of $L^{p'}(S)$, hence, as k is radial, $\delta^{-1}\check{k} = \delta^{-1}k$ is in $LCv_p(S)$.

For every complex z, $\mathrm{Re}(z) \in [0,1]$, define the distributional kernels

$$\kappa_z = \delta^{-z}k.$$

Then if y is real

$$\kappa_{1+iy} = \delta^{-iy}\delta^{-1}k \qquad \text{and} \qquad \kappa_{iy} = \delta^{-iy}k.$$

Since δ^{-iy} is purely imaginary and is a homomorphism of the group S, κ_{1+iy} and κ_{iy} are left convolutors of $L^p(S)$, with the same norm bound $\|(\omega m)_{Q/2}\|_{\mathcal{M}_p(\mathbb{R})}$.

By the Stein complex interpolation Theorem, we conclude that κ_z is a left convolutor of $L^p(S)$, for every complex z such that $\mathrm{Re}(z) \in (0,1)$.

On the side of right convolutors, we conclude that $\delta^{-1/p}(\delta^{-z}k)^\vee = \delta^{z-1/p}k$ is a right convolutor of $L^p(S)$, for every complex z such that $\mathrm{Re}(z) \in [0,1]$. ∎

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