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# The density of the image of the exponential function and spacious locally compact groups

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A connected locally compact group $G$ is called spacious if there exists a nonempty open set $U \subseteq G$ such that $U^n \cap U^{n+1} = \emptyset$ for every positive integer $n$. $G$ is called weakly exponential if the union of its one-parameter subgroups is dense in $G$. We prove the following conjecture of Hofmann and Mukherjea [Math. Ann. **234**, 263-273 (1978)]: $G$ is spacious iff it fails to be weakly exponential.

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Let $G$ be a connected locally compact group. Following Hofmann and Mukherjea [3] we call $G$ *spacious* if it admits a nonempty open subset $U$ such that $U^n \cap U^{n+1} = \emptyset$ for every positive integer $n \in \mathbb{N}$. We call $G$ *weakly exponential* if the union of its one-parameter subgroups is dense in $G$.

When $G = \mathrm{SL}(n, \mathbb{R})$ and $U \subseteq G$ is the set of matrices with all entries strictly negative, then obviously $U^n \cap U^{n+1} = \emptyset$ for all $n$. $\mathrm{SL}(n, \mathbb{R})$ is thus an elementary example of a spacious connected locally compact group. On the other hand, when $G = \mathbb{R}$, $U \subseteq \mathbb{R}$ is open, and $x \in U$, there will be a positive integer $n$ such that $y = nx/(n+1) \in U$. So $nx = (n+1)y \in U^n \cap U^{n+1}$, and therefore $\mathbb{R}$ is not spacious. Using this result one immediately concludes that a weakly exponential locally compact group is not spacious. Hofmann and Mukherjea conjectured that the converse is also true, i.e., $G$ is spacious iff it fails to be weakly exponential [3, Conjecture 2.13]. The purpose of the present paper is to give a proof of this conjecture.

We begin by quoting two lemmas from [3].

**Lemma 1.** *A connected locally compact group $G$ is weakly exponential iff every neighbourhood of the identity $e \in G$ contains a compact normal subgroup $K$ such that $G/K$ is a weakly exponential Lie group.*

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Of course, a connected Lie group $G$ is weakly exponential iff the image of the exponential map is dense in $G$.

**Lemma 2.** *Let $R$ denote the radical of a connected locally compact group $G$. Then $G$ is weakly exponential iff $G/R$ is.*

It is easy to see that if $N$ is a closed normal subgroup of $G$ such that $G/N$ is spacious then $G$ is spacious. Therefore it is clear from Lemmas 1 and 2 that it suffices to prove the conjecture in the case that $G$ is a connected semisimple Lie group (as pointed out in [3]).

Let $G$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Recall that a Cartan subalgebra of $\mathfrak{g}$ is a maximal abelian subalgebra $\mathfrak{h}$ such that $\mathrm{ad}X$ is semisimple for every $X \in \mathfrak{h}$. The Cartan subgroup $H(\mathfrak{h})$ associated with $\mathfrak{h}$ is defined as the centralizer of $\mathfrak{h}$ in $G$: $H(\mathfrak{h}) = Z_G(\mathfrak{h}) = \{g \in G;\ \mathrm{Ad}(g)\restriction\mathfrak{h} = \mathrm{id}\}$. $H(\mathfrak{h})$ is a closed Lie subgroup with Lie algebra $\mathfrak{h}$. Thus if $H(\mathfrak{h})$ is connected then $H(\mathfrak{h}) = \exp(\mathfrak{h})$. Since the union of all Cartan subgroups of $G$ is dense in $G$ [5, Theorem 1.4.1.7], one obtains:

**Lemma 3.** *If all Cartan subgroups of a connected semisimple Lie group $G$ are connected then $G$ is weakly exponential.*

As shown in [3, Theorem 2.10] the converse is also true, but we do not need this in our argument. To prove the conjecture it suffices to show that if some Cartan subgroup is disconnected then $G$ is spacious. We will achieve this goal in two steps. First, we will show that spaciousness occurs when the minimal parabolic subgroups of $G$ are disconnected. Then we will show that if the minimal parabolic subgroups are connected then the Cartan subgroups are also connected.

Let $G$ be a real connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{l}$ be a Cartan decomposition ($\mathfrak{k}$ the subalgebra) and $\Theta$ the associated Cartan involution . We shall denote by $\langle\cdot,\cdot\rangle$ the bilinear form $\langle X, Y\rangle = -B(X, \Theta Y)$, $X, Y \in \mathfrak{g}$, where $B(\cdot,\cdot)$ is the Killing form. $\langle\cdot,\cdot\rangle$ is a scalar product on $\mathfrak{g}$ and we shall write $\|X\|$ for the norm

$$\|X\| = (\langle X, X\rangle)^{1/2}. \tag{1}$$

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{l}$. For any linear functional $\lambda$ let,

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g};\ (\mathrm{ad}Y)X = \lambda(Y)X \text{ for all } Y \in \mathfrak{a}\}. \tag{2}$$

When $\mathfrak{g}_\lambda \neq \{0\}$, $\lambda$ is called a root of $\mathfrak{g}$ (relative to $\mathfrak{a}$). One proves that $\mathfrak{g} = \bigoplus_{\lambda\in\Delta} \mathfrak{g}_\lambda$, where the sum is over the set $\Delta$ of roots. Moreover, the $\mathfrak{g}_\lambda$'s are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$. Let $\geq$ be any total vector order in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and denote by $\Delta_+$ set of positive (nonzero) roots. Set $\mathfrak{n} = \bigoplus_{\lambda\in\Delta_+} \mathfrak{g}_\lambda$. Then $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$ and one obtains an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Let $K$, $A$, and $N$ denote the connected Lie subgroups of $G$ with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$, respectively. Then the map $K \times A \times N \ni (k, a, n) \to kan \in G$ is a diffeomorphism onto $G$ and one obtains an

Iwasawa decomposition $G = KAN$. We note that $K$ contains the centre $C(G)$ and is compact iff $C(G)$ is finite.

We shall denote by $M$ the centralizer of $A$ in $K$. $M$ is either discrete or a closed Lie subgroup with Lie algebra $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$. The set $MAN$ is a closed cocompact subgroup of $G$ called the minimal parabolic subgroup associated with the Iwasawa decomposition $G = KAN$. All Iwasawa decompositions of $G$, and therefore also all minimal parabolic subgroups are conjugate.

Let $\tilde{\mathfrak{n}} = \bigoplus_{\lambda \in -\Delta_+} \mathfrak{g}_\lambda$. Then $\tilde{\mathfrak{n}}$ is a nilpotent subalgebra. We shall write $\tilde{N}$ for the connected Lie subgroup of $G$ with Lie algebra $\tilde{\mathfrak{n}}$. Given a subgroup $H \subseteq G$ we shall denote by $H_0$ the connected component of the identity $e$. We note that $(MAN)_0 = M_0AN$. The following lemma is a version of Lemma III.6 in [1, p. 66].

**Lemma 4.** *There exists an open semigroup $S \subseteq G$, an open neighbourhood $V$ of $e$ in $G$, and an element $a \in A$ such that $VaM_0 \subseteq S \subseteq \tilde{N}M_0AN$.*

**Proof.** Choose $Z \in \mathfrak{a}$ so that $\lambda(Z) > 0$ for all $\lambda \in \Delta_+$. Set $a = \exp(Z)$ and $\gamma = \exp\bigl(-\min\{\lambda(Z); \lambda \in \Delta_+\}\bigr) < 1$. Then for every $X \in \tilde{\mathfrak{n}}$ we have $\|\mathrm{Ad}(a)X\| \le \gamma\|X\|$. where $\|\cdot\|$ is the norm (1). Let $\Omega$ be an open ball in $\mathfrak{n}$ with centre 0. Put $E = \pi\bigl(\exp(\overline{\Omega})\bigr)$ and $E' = \pi\bigl(\exp(\Omega)\bigr)$ where $\pi\colon G \to G/M_0AN$ is the canonical map. $E$ is clearly a compact subset and $E' \subseteq E$. Now, $\exp(\Omega)$ is open because $\tilde{N}$ is a simply connected nilpotent Lie group. As the map $\tilde{N} \times MAN \ni (x,y) \to xy$ is a diffeomorphism onto an open subset of $G$ [2, pp. 406-407], it follows that $E' = \pi\bigl(\exp(\Omega)\bigr) = \pi\bigl(\exp(\Omega)M_0AN\bigr)$ is open. Furthermore, $aE = \pi\bigl(\exp\bigl(\mathrm{Ad(a)}\overline{\Omega}\bigr)\bigr) \subseteq \pi\bigl(\exp(\Omega)\bigr) = E'$.

Define $S = \{g \in G; gE \subseteq E'\}$. Then $S$ is a semigroup containing $a$. $S$ is open because $E$ is compact while $E'$ is open.

Let $V$ be a neighbourhood of $e$ in $G$ such that $Va \subseteq S$. Let $v \in V$ and $g \in M_0$. Then $vagE = va\pi\bigl(g\exp(\overline{\Omega})\bigr) = va\pi\bigl(\exp\bigl(\mathrm{Ad}(g)\overline{\Omega}\bigr)\bigr)$. Since $\mathrm{Ad}(g) \in \mathrm{Ad}(K)$ is an isometry, $\mathrm{Ad}(g)\overline{\Omega} = \overline{\Omega}$ and therefore $vagE = vaE \subseteq E'$. So $VaM_0 \subseteq S$.

It remains to show that $S \subseteq \tilde{N}M_0AN$. But $M_0AN = \pi(e) \in E$. Hence, $\pi(S) = S\pi(e) \subseteq E' \subseteq \pi(\tilde{N})$. So $S \subseteq \tilde{N}M_0AN$. ∎

**Lemma 5.** *If $M$ is disconnected then $G$ is spacious.*

**Proof.** By Lemma 4 there exists an open semigroup $S$, an open neighbourhood $V$ of $e$ in $G$, and $a \in A$ such that $VaM_0 \subseteq S \subseteq \tilde{N}M_0AN$. We can assume that $V = \exp(\Omega)$ where $\Omega = \{X \in \mathfrak{g}; \|X\| < \varepsilon\}$ for sufficiently small $\varepsilon > 0$. Let $x \in M - M_0$. As $\mathrm{Ad}(x)$ preserves the norm $\|\cdot\|$, we have $xVx^{-1} = V$. Furthermore, $VaM_0 \subseteq S$ and $xVaM_0x^{-1} = VaM_0$ (because $x$ centralizes $a$ and normalizes $M_0$). Consequently, $S_1 = \bigcup_{i=1}^{\infty}\bigl(VaM_0\bigr)^i \subseteq S \subseteq \tilde{N}MAN$ is an open semigroup such that $xS_1x^{-1} = S_1$. Hence, setting $U = xS_1$ we have $U^n \cap U^{n+1} \subseteq x^nS_1 \cap x^{n+1}S_1 = x^n\bigl(S_1 \cap xS_1\bigr) \subseteq x^n\bigl(\tilde{N}M_0AN \cap \tilde{N}xM_0AN\bigr)$ (because $x$ normalizes $\tilde{N}$). Using the fact that the map $\tilde{N} \times M \times AN \ni (g,h,k) \to ghk$ is one-to-one we conclude that $U^n \cap U^{n+1} = \emptyset$ for all $n \in \mathbb{N}$. So $G$ is spacious. ∎

Given a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ we shall write $\mathfrak{h}_{\mathfrak{k}}$ for $\mathfrak{h} \cap \mathfrak{k}$ and $\mathfrak{h}_{\mathfrak{l}}$ for $\mathfrak{h} \cap \mathfrak{l}$. If $\mathfrak{h}$ is stable under $\Theta$ then $\mathfrak{h} = \mathfrak{h}_{\mathfrak{k}} \oplus \mathfrak{h}_{\mathfrak{l}}$. We shall denote by $\mathcal{H}_{\mathfrak{a}}$ the set of all Cartan subalgebras containing $\mathfrak{a}$. $\mathcal{H}_{\mathfrak{a}}$ coincides with the set of maximal abelian subspaces of $\mathfrak{g}$ containing $\mathfrak{a}$. If $\mathfrak{h}$ is a member of $\mathcal{H}_{\mathfrak{a}}$ then $\mathfrak{h}$ is $\Theta$-stable, $\mathfrak{h}_{\mathfrak{l}} = \mathfrak{a}$ and $\mathfrak{h}_{\mathfrak{k}} \subseteq \mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{k}$.

**Lemma 6.** $\mathfrak{m} = \bigcup_{\mathfrak{h} \in \mathcal{H}_{\mathfrak{a}}} \mathfrak{h}_{\mathfrak{k}}$.

**Proof.** Let $X \in \mathfrak{m}$. If $\mathfrak{h}$ is a maximal abelian subspace containing both $X$ and $\mathfrak{a}$, then $\mathfrak{h} \in \mathcal{H}_{\mathfrak{a}}$ and $X \in \mathfrak{h}_{\mathfrak{k}}$. ∎

**Lemma 7.** *There exist*:

(*i*) *a compact connected semisimple Lie group $G_{\#}$ with Lie algebra $\mathfrak{g}_{\#}$ and trivial centre*;

(*ii*) *a linear isomorphism $F : \mathfrak{g} \to \mathfrak{g}_{\#}$ such that $F[X,Y] = [FX, FY]$ for $X \in \mathfrak{k}$, $Y \in \mathfrak{g}$, and $F[X,Y] = -[FX,FY]$ for $X, Y \in \mathfrak{l}$*;

(*iii*) *an analytic homomorphism $\varphi\colon K \to G_{\#}$ such that $F \restriction \mathfrak{k}$ is the differential of $\varphi$ at $e$ and $F\mathrm{Ad}(g) = \mathrm{Ad}\big(\varphi(g)\big)F$ for all $g \in K$.*

**Proof.** Let $\tilde{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$. Then $\mathfrak{k} \oplus i\mathfrak{l}$ is a compact real form of $\tilde{\mathfrak{g}}$. Set $G_{\#} = \mathrm{Int}(\mathfrak{k} \oplus i\mathfrak{l})$. Identify $\mathfrak{k} \oplus i\mathfrak{l}$ with the Lie algebra of $G_{\#}$ and define $F$ by $F(X+Y) = X + iY$, $X \in \mathfrak{k}$, $Y \in \mathfrak{l}$. Given $g \in K$ define $\varphi(g)$ to be the complexification of $\mathrm{Ad}(g)$ restricted to $\mathfrak{k} \oplus i\mathfrak{l}$. ∎

**Remark 8.** $\mathrm{Ker}\varphi = C(G)$. $[F\mathfrak{k}, F\mathfrak{l}] \subseteq F\mathfrak{l}$, $[F\mathfrak{l}, F\mathfrak{l}] \subseteq F\mathfrak{k}$.

**Lemma 9.** *Let $W \subseteq \mathfrak{g}$ be a subset. Denote by $Z_K(W)$ (resp., $Z_{G_{\#}}(FW)$) the centralizer of $W$ (resp., $FW$) in $K$ (resp., in $G_{\#}$). Then $Z_K(W) = \varphi^{-1}\big(Z_{G_{\#}}(FW)\big)$.*

**Proof.** Obvious by Lemma 7(iii). ∎

We shall write $K_{\#}$ for the connected Lie subgroup of $G_{\#}$ with Lie algebra $F\mathfrak{k}$. Clearly, $K_{\#} = \varphi(K)$.

**Lemma 10.** *If $M$ is connected then $K_{\#} \cap \exp(F\mathfrak{a}) \subseteq \exp(F\mathfrak{m})$.*

**Proof.** Let $g \in K_{\#} \cap \exp(F\mathfrak{a})$. Clearly, $g \in Z_{G_{\#}}(F\mathfrak{a})$. But $g = \varphi(g')$ for some $g' \in K$. By Lemma 9, $g' \in Z_K(\mathfrak{a}) = M$. As $M$ is connected and has compact Lie algebra $\mathfrak{m}$, $M = \exp(\mathfrak{m})$. So $g = \varphi(g') \in \varphi\big(\exp(\mathfrak{m})\big) = \exp(F\mathfrak{m})$ by Lemma 7(iii). ∎

For the next lemma see [5, Proposition 1.4.1.2].

**Lemma 11.** *Let $\mathfrak{h}$ be a $\Theta$-stable Cartan subalgebra. Set $H_K(\mathfrak{h}) = H(\mathfrak{h}) \cap K$. Then $H(\mathfrak{h}) = H_K(\mathfrak{h}) \exp(\mathfrak{h}_{\mathfrak{l}})$ and $H(\mathfrak{h})$ is connected iff $H_K(\mathfrak{h})$ is connected.*

**Theorem 12.** *The following conditions are equivalent for a connected semisimple Lie group $G$:*

(*a*) *the minimal parabolic subgroups of $G$ are connected;*

(*b*) *the Cartan subgroups of $G$ are connected;*

(*c*) *G is weakly exponential;*
(*d*) *G is not spacious.*

**Proof.** (a) $\Rightarrow$ (b): Every Cartan subalgebra $\mathfrak{h}$ is conjugate in $\mathrm{Int}(\mathfrak{g})$ to a $\Theta$-stable Cartan subalgebra $\mathfrak{h}'$ such that $\mathfrak{h}'_{\mathfrak{l}} \subseteq \mathfrak{a}$ [5, Proposition 1.3.1.1]. Therefore it suffices to consider Cartan subgroups $H = H(\mathfrak{h})$ where $\mathfrak{h}$ is $\Theta$-stable and $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{a}$. Due to Lemma 11 it suffices to show that $H_K = \exp(\mathfrak{h}_{\mathfrak{k}})$.

Let $g \in H_K$. Then $\hat{g} = \varphi(g) \in Z_{G_\#}(F\mathfrak{h})$ (Lemma 9). It is easy to see that $F\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}_\#$. As $\mathfrak{g}_\#$ is compact, this means that $F\mathfrak{h}$ is a Cartan subalgebra [4, Lemma 4.12.1]. As $G_\#$ is compact and connected, $H(F\mathfrak{h}) = \exp(F\mathfrak{h})$ [4, Theorems 4.12.3 and 4.12.5]. Therefore $\hat{g} = \exp(FX)$ for some $X \in \mathfrak{h}$. Write $X = X_1 + X_2$, $X_1 \in \mathfrak{h}_{\mathfrak{k}}$, $X_2 \in \mathfrak{h}_{\mathfrak{l}}$. Then $\hat{g} = \exp(FX_1)\exp(FX_2)$, and $FX_2 \in F\mathfrak{h}_{\mathfrak{l}} \subseteq F\mathfrak{a}$. As $\hat{g} \in K_\#$, $\hat{g}_2 = \exp(FX_2) = \exp(-FX_1)\hat{g} \in K_\# \cap \exp(F\mathfrak{a})$. By Lemma 10 $\exp(FX_2) = \exp(FX_2')$ where $X_2' \in \mathfrak{m}$. As $\mathfrak{m} = \bigcup_{\mathfrak{h}' \in \mathcal{H}_{\mathfrak{a}}} \mathfrak{h}'_{\mathfrak{k}}$ (Lemma 6), there exists $\mathfrak{h}' \in \mathcal{H}_{\mathfrak{a}}$ with $X_2' \in \mathfrak{h}'_{\mathfrak{k}}$. Then $\mathfrak{h}$ and $\mathfrak{h}'$ are two $\Theta$-stable Cartan subalgebras such that $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{h}'_{\mathfrak{l}} = \mathfrak{a}$. By [5, Proposition 1.3.1.3] there exists $k \in K$ such that $\mathrm{Ad}(k)\restriction\mathfrak{h}_{\mathfrak{l}} = \mathrm{id}$ and $\mathrm{Ad}(k)\mathfrak{h}'_{\mathfrak{k}} \subseteq \mathfrak{h}_{\mathfrak{k}}$. Now, $\varphi(k)\hat{g}_2\varphi(k)^{-1} = \varphi(k)\exp(FX_2)\varphi(k)^{-1} = \exp\big(\mathrm{Ad}(\varphi(k))FX_2\big) = \exp\big(F\mathrm{Ad}(k)X_2\big) = \exp(FX_2) = \hat{g}_2$ (by Lemma 7(iii)). Hence, $\hat{g}_2 = \varphi(k)\hat{g}_2\varphi(k)^{-1} = \varphi(k)\exp(FX_2')\varphi(k)^{-1} = \exp\big(\mathrm{Ad}(\varphi(k))FX_2'\big) = \exp\big(F\mathrm{Ad}(k)X_2'\big) = \exp(FX_2'')$, where $X_2'' \in \mathfrak{h}_{\mathfrak{k}}$. So $\varphi(g) = \hat{g} = \exp(FX_1)\exp(FX_2'') = \exp(FY) = \varphi\big(\exp(Y)\big)$, where $Y \in \mathfrak{h}_{\mathfrak{k}}$.

Since $\mathrm{Ker}\varphi = C(G)$, $g = \exp(Y)c$ where $c \in C(G)$. But $C(G) \subseteq M = \exp(\mathfrak{m})$ because $M$ is connected and $\mathfrak{m}$ is compact. So $c = \exp(Z)$ for some $Z \in \mathfrak{m}$. Then by Lemma 6 there exists $\mathfrak{h}' \in \mathcal{H}_{\mathfrak{a}}$ such that $Z \in \mathfrak{h}'_{\mathfrak{k}}$. Again, $\mathfrak{h}$ and $\mathfrak{h}'$ are two $\Theta$-stable Cartan subalgebras such that $\mathfrak{h}_{\mathfrak{l}} \subseteq \mathfrak{h}'_{\mathfrak{l}} = \mathfrak{a}$. So there exists $k \in K$ such that $\mathrm{Ad}(k)\restriction\mathfrak{h}_{\mathfrak{l}} = \mathrm{id}$ and $\mathrm{Ad}(k)\mathfrak{h}'_{\mathfrak{k}} \subseteq \mathfrak{h}_{\mathfrak{k}}$ [5, Proposition 1.3.1.3]. We then have $c = kck^{-1} = \exp(\mathrm{Ad}(k)Z) = \exp(Z')$ with $Z' \in \mathfrak{h}_{\mathfrak{k}}$. Consequently, $g = \exp(Y + Z')$, and $Y + Z' \in \mathfrak{h}_{\mathfrak{k}}$. Thus $H_K = \exp(\mathfrak{h}_{\mathfrak{k}})$.

(b) $\Rightarrow$ (c): Lemma 3.

(c) $\Rightarrow$ (d): explained at the beginning of the paper.

(d) $\Rightarrow$ (a): Lemma 5. ∎

**Corollary 13.** *A connected locally compact group fails to be weakly exponential iff it is spacious.*

**Remark 14.** Using Lemma 7 one can prove an analog of Lemma 6 for the group $M$. Namely, $M = \bigcup_{\mathfrak{h} \in \mathcal{H}_{\mathfrak{a}}} H_K(\mathfrak{h})$. This, together with Lemma 11 allows to obtain the implication (b) $\Rightarrow$ (a) of Theorem 12 directly, without involving (c) and (d).

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