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Types for elliptic non-discrete series representations of $\mathbf{SL_N(F)}$, N prime and F a p-adic field.

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Abstract

In this paper we give a classification of the elliptic non-discrete series representations of $G = SL_N(F)$, F p-adic and N prime, by the method of types, that is by restriction to compact open subgroups. We give an explicit construction of certain types and determine their Hecke algebras. We then prove that an irreducible smooth non-discrete series representation of G is elliptic if and only if it 'contains' one of our types.

1. Introduction

We recall that for any algebraically connected (group of F-rational points of a) reductive p-adic group, the Langlands quotient theorem (extended to p-adic groups by Silberger) gives a classification of irreducible smooth (hence admissible) representations in terms of the irreducible tempered representations [20]. These include the discrete series and consist in general of irreducible constituents of representations unitarily parabolically induced from discrete series. An element of the group is called elliptic if it is contained in a Cartan subgroup which is compact modulo the center. One says that an irreducible tempered representation is elliptic if its character has support (in the sense of distributions) on the subset of regular elliptic elements. The subset of elliptic representations also include the discrete series. It is known that for the groups SL, unlike the groups GL, this inclusion is strict. D. Goldberg has given in [12] criteria for the constituents of parabolically induced representations of $SL_k(F)$, where k is arbitrary, to be elliptic. This is based on the theory of the Knapp-Stein R-group and a result of J. Arthur in [1] (which are both valid for general p-adic groups). To be more precise we introduce some notations below.

Let χ denote a continuous character of the multiplicative group of the field F whose restriction to the group of integral units has order N. It determines canonically a character σ of the subgroup D of diagonal matrices in $SL_N(F)$; σ

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extends trivially across the subgroup U of upper unipotent matrices, to a character of the Borel subgroup $B = DU$ of upper triangular matrices (still denoted σ). It is a well known result of Howe and Silberger [16] that the unitarily induced representation $Ind(B, G, \sigma)$ decomposes simply. It follows from [12] that its N inequivalent constituents are elliptic, furthermore any non-discrete series elliptic representation of $SL_N(F)$ is a constituent of such an induced representation, since N is prime. We then construct a type (J, λ) which depends on χ. Our main Theorem is as follows:

Main Theorem. *An irreducible smooth representation π of $SL_N(F)$ is an irreducible constituent of the induced representation $Ind(B, G, \sigma)$ if and only if it contains (J, λ) upon restriction to J.*

Our methods are inspired from A. Borel [5] and W. Casselman [9], [10]. They require a quite explicit knowledge of the structure of the Hecke algebras associated to our types. To determine the support of these algebras we rely on orbit theory in the spirit of Howe and Moy [15]. This enables us to describe the Hecke algebras explicitly by generators and relations. In [10] Casselman uses the 'Iwahori' Hecke algebra to classify the irreducible smooth representations with an Iwahori fixed vector, the so called unramified principal series (see also Borel [5]). This algebra is known to have the nice structure of an affine Hecke algebra. In [15] Howe and Moy showed (in the tame and prime cases) that much of the representation theory of a given $GL_k(F)$ is controlled by the Iwahori Hecke algebras of (usually) smaller GL's. This observation was later confirmed by Bushnell and Kutzko in [6], with no restriction on k and the residual characteristic. The situation for other groups proved to be more complicated. Indeed the works of A. Moy [18] and L. Morris [17] provide us with examples of some 'degenerate' Hecke algebras.

For the group $SL_N(F)$ with N prime, we show that for each of our types, the Hecke algebra is isomorphic to the group algebra of some special subgroup in the affine Weyl group. We proceed by restriction from $GL_N(F)$, and as a consequence we construct a series of types for $GL_N(F)$ as well. We show that these are isomorphic to a tensor product of affine Hecke algebras. It is a conjecture of Bushnell and Kutzko that this should be true of any GL-type in 'general position'.

Numerous papers have been devoted to the representation theory of the groups $SL_k(F)$. Let us just mention the two most recent ones which we know of. In [8] Bushnell and Kutzko classify the supercuspidals by types. They proceed by restriction from $GL_k(F)$ exploiting their own earlier work [6] on $GL_N(F)$. There they constructed the so called simple types and gave another proof after L. Corwin [11] that the supercuspidals were induced from open and compact mod center subgroups. They extend this result to $SL_k(F)$ also. Marco Tadic gives in [21] a description of the tempered representations of $SL_k(F)$ by certain GL-parameters. His method is based on the Bernstein and Zelevinsky classification [3] of these representations for $GL_k(F)$.

We now discuss the organization of this paper. We introduce notations and review some general facts in the second section. The third section discusses Goldberg's theorem on elliptic representations. In the fourth section we give a construction of types and we state the main result (Theorem 1). We determine the structure of the Hecke algebra of a given type in section five. Section six deals with a discussion of 'closely related' types for $GL_N(F)$. The structure of the Hecke algebras of our types is determined in section seven. We prove our main theorem in section eight. In the last section, as a consequence of the methods used here, we describe the smooth representations of $GL_N(F)$ which contain the types constructed in section six.

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2. General background

Notation. We fix a p-adic field F. $\mathcal{O}_F$ is the ring of integers in F. $\mathcal{P}_F$ is the maximal ideal in $\mathcal{O}_F$ with generator ϖ. F^+ the additive group of the field is filtered by the ideals

$$\mathcal{P}_F^m = \langle \varpi^m \rangle,\ m \in \mathbb{Z}.$$

$\mathcal{U}_F$ is the group of units in $\mathcal{O}_F$. $\mathcal{U}_F$ is filtered by the groups

$$\mathcal{U}_F^{(i)} = 1 + \mathcal{P}_F^i,\ i \geq 1.$$

We let $k_F = \mathcal{O}_F/\mathcal{P}_F$ denote the residue field with cardinality $q = p^r$, where r is some positive integer and p is the residual characteristic. N denotes a fixed arbitrary prime number. $M_N(F)$ stands for the ring of N by N matrices with entries in F.

In this paper we let $\tilde{G}= GL_N(F)$ be the group of units in $M_N(F)$, the invertible N by N matrices with entries in F. We let $G = SL_N(F)$, the subgroup of $\tilde{G}$ consisting of matrices with determinant equal to 1, except otherwise noted.

We let $\tilde{B} = \tilde{D}\tilde{U}$ denote the standard Borel subgroup of upper triangular matrices with Levi component the subgroup $\tilde{D}$ of diagonal matrices and unipotent radical the subgroup $\tilde{U}$ of upper unipotent matrices. If $\tilde{H}$ is a subgroup of $\tilde{G}$ then we let $H = \tilde{H} \cap G$. In particular we get the standard Borel subgroup $B = DU$ of G, where D and U have the obvious meaning. Observe that $\tilde{U} = U$.

Particularly important for us are the *parahoric* subgroups of $\tilde{G}$ and G. Let $\tilde{G}_0 = GL_N(\mathcal{O}_F)$; this is a maximal compact open subgroup of $\tilde{G}$. We note in passing the *Iwasawa decomposition*

$$\tilde{G} = \tilde{B}\tilde{G}_0.$$

$\tilde{G}_0$ is filtered by the compact open subgroups

$$\tilde{C}_i = 1 + \varpi_F^i M_N(\mathcal{O}_F),\ i \geq 1.$$

These form a fundamental system of neighborhoods of the identity in $\tilde{G}$. We see that the reduction mod $\mathcal{P}_F$ homomorphism

$$\tilde{G}_0 \to GL_N(k_F)$$

yields an isomorphism:

$$\tilde{G}_0/\tilde{C}_1 \cong GL_N(k_F).$$

The inverse image in $\tilde{G}_0$ of a parabolic subgroup of $GL_N(k_F)$ is a parahoric subgroup of $\tilde{G}$. Among these is the standard *Iwahori* subgroup which is the inverse image of the standard Borel subgroup of $GL_N(k_F)$. Restricting to $SL_N(F)$ we obtain the parahoric subgroups of G.

The following realization of parahoric subgroups will be quite useful.
By a *lattice* in F^N, we mean a free $\mathcal{O}_F$-submodule of F^N of rank N. A *periodic lattice flag* $\mathcal{L}$ of period e is a sequence of lattices $\{L_i \mid i \in \mathbb{Z}\}$ such that

$$L_{i+1} \subseteq L_i \text{ and } L_{i+e} = \varpi L_i .$$

For a fixed lattice flag $\mathcal{L}$ let

$$A^m = A^m_{\mathcal{L}} = \{a \in M_N(F) \mid aL_i \subseteq L_{i+m}, \text{ all } i\}, m \in \mathbb{Z}.$$

Then $A = A^0$ is a subring of $M_N(F)$ (in fact an *hereditary $\mathcal{O}_F$-order* in $M_N(F)$) and A^1 is an ideal in A (in fact the *Jacobson radical* of A). For a useful account (in this context) on the theory of hereditary orders we refer to [6]. The group $\tilde{K} = \tilde{K}_{\mathcal{L}}$ of units of A is a parahoric subgroup of $GL_N(F)$. The correspondence $\mathcal{L} \to \tilde{K}_{\mathcal{L}}$ gives a parametrization of the parahoric subgroups of $\tilde{G}$. In fact more is true, index shifting determines an equivalence relation on the set of periodic lattice flags; and the correspondence above carries in a one to one fashion, an equivalence class to a G-conjugacy class of parahoric subgroups. Again the restriction to G yields the parahoric subgroups of G. Now we set

$$\tilde{K}_i = 1 + A^i,\ i \geq 1.$$

The $\tilde{K}_i$ are compact open subgroups of $GL_N(F)$ and normal in $\tilde{K}$.

Abelian characters and duality in $M_N(F)$. F^+ is a locally compact abelian group. $\mathcal{O}_F$ and the $\mathcal{P}_F^m$ are compact open subgroups of F^+. We fix once and for all a continuous character ψ of F^+ with conductor $\mathcal{P}_F$, that is ψ is trivial on $\mathcal{P}_F$ but not on $\mathcal{O}_F$ (such objects exist [22]). F^+ is self-dual, that is isomorphic to its group of characters $(F^+)^\wedge$ (the Pontryagin dual) under the map $a \to \psi_a$ where

$$\psi_a(y) = \psi(ay).$$

In particular the Pontryagin dual of the finite abelian group $\mathcal{P}_F^n/\mathcal{P}_F^{n+1}$ is given by

$$(\mathcal{P}_F^n/\mathcal{P}_F^{n+1})^\wedge \cong \mathcal{P}_F^{-n}/\mathcal{P}_F^{1-n}. \tag{1}$$

Similar facts hold for $M_N(F)$ which is also a locally compact abelian group under addition. With ψ as above we define $\psi_X,\ X \in M_N(F)$ by

$$\psi_X(Y) = \psi(\mathrm{Tr}(XY)),\ Y \in M_N(F)$$

where Tr denotes the usual trace function on $M_N(F)$. Then ψ_X is a character of $M_N(F)$ and the map $X \to \psi_X$ yields an isomorphism between $M_N(F)$ and its Pontryagin dual $M_N(F)^\wedge$.

We note below some useful consequences of these duality results. We let $\tilde{K} = \tilde{K}_{\mathcal{L}}$ be a parahoric subgroup of $\tilde{G}$ as in the previous section, with its natural filtration subgroups $\tilde{K}_i$, then we have an isomorphism of finite abelian groups

$$\tilde{K}_j/\tilde{K}_{i+1} \simeq A^j/A^{i+1} \quad \text{where } 2j \geq i+1 \geq j.$$

Thus the self duality of $M_N(F)$ yields an isomorphism

$$(\tilde{K}_j/\tilde{K}_{i+1})^\wedge \simeq A^{-i}/A^{1-j} \quad \text{where } 2j \geq i+1 \geq j. \tag{2}$$

If we set $K_i = \tilde{K}_i \cap SL_N(F)$ then we (now) have

$$(K_j/K_{i+1})^\wedge \simeq A^{-i}/\left(F \cap A^{-i} + A^{1-j}\right) \tag{3}$$

where we view F (abusing notation) as the scalar matrices in $M_N(F)$. This is Proposition 1.1 of [7].

Smooth representations. In this section we will state some standard results on smooth representations which hold for an arbitrary (group of F-rational points of a) reductive algebraic group G defined over F.

By a *representation* of G we mean a pair (π, V) where V is a complex vector space and π a homomorphism $G \to End_{\mathbb{C}}(V)$. We will often simply write π or V depending on context. If H is a subgroup of G we let $H^g = gHg^{-1}$ and if π is a *representation* of H we write π^g for the *representation* of H^g defined by

$$\pi^g(x) = \pi(g^{-1}xg), \; x \in H^g.$$

We say that a representation of G is *smooth* if for any vector $v \in V$ its $G-$stabilizer

$$\{g \in G \mid \pi(g)v = v\}$$

is open.

We will call a one dimensional smooth representation of G a *character* of G. Examples of characters include the continuous homomorphisms from G to $\mathbb{C}^\times$, where $\mathbb{C}^\times$ is given its usual topology. For $GL_N(F)$ a standard construction is to form the composite $\gamma\circ$ det, where γ is a continuous character of $F^\times$ [22] and det denotes the usual determinant homomorphism on $GL_N(F)$. A character θ of $F^\times$ is called *unramified* if it is trivial on $\mathcal{U}_F$. An *unramified* character of G is a character of the form $\theta \circ \eta$ where $\eta : G \to F^\times$ is a rational character of G (in the sense of algebraic groups) and θ is an unramified character of $F^\times$.

We say that (π, V) is *admissible* if it is smooth and if for each compact open subgroup $K \subseteq G$, the subspace

$$V^K = \{v \in V \mid \pi(k)v = v, \text{ all } k \in K\}$$

is finite dimensional. We say that (π, V) is *irreducible* if the only $G-$invariant subspaces are $\{0\}$ and V. It is a basic fact that

Theorem A. *An irreducible smooth representation of G is admissible* [9].

If $P = LU$ denotes a parabolic subgroup of G with unipotent radical U then a (smooth) representation (σ, W) of the Levi factor L extends (trivially across U) to a (smooth) representation of P also denoted (σ, W). G acts on the space $W(\sigma)$ of all functions $f : G \to W$ such that

$$f(hug) = \sigma(h)f(g),\ \forall h \in L,\ u \in U,\ g \in G.$$

by the formula

$$(\pi(g)f)(y) = f(yg),\ y, g \in G.$$

One usually refers to this as the action of π by right translations. The subspace of smooth vectors in $W(\sigma)$, which one may refer to as *smooth functions* is G-invariant so it affords a smooth representation, this is the *parabolically induced* representation denoted $\mathrm{Ind}_P^G\sigma$.

Let δ_P denote the modular character of P, it is well known that *if σ is unitarizable then $\mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \sigma)$ is itself unitarizable.* We will refer to it as the representation unitarily induced from σ. We will denote it $\mathrm{ind}_P^G\sigma$.

We will also need the notion of *compactly-induced* representation denoted c-$\mathrm{Ind}_H^G\sigma$ for which G acts by right translations on the subspace of $W(\sigma)$ consisting of compactly supported smooth functions. Clearly if G/H is compact then the two notions of induction coincide (this is the case for parabolic subgroups).

In this context, *Frobenius reciprocity* holds, that is

$$\mathrm{Hom}_H(\pi|_H, \sigma) \cong \mathrm{Hom}_G(\pi, \mathrm{c} - \mathrm{Ind}_H^G\sigma). \tag{4}$$

We now state the following important property of induced representations

Theorem B. [9] *A smooth representation of G induced from an irreducible smooth representation of a proper parabolic subgroup has finite composition length.*

By *constituents* of such a representation we mean the irreducible composition factors (or subquotients). The so called *supercuspidal* representations are those smooth representations which do not occur as constituents of these 'parabolically' induced representations.

We now review the construction of *Jacquet modules*.

Let $P = LU$ be a parabolic subgroup of G. The Jacquet module of a representation (π, V) of G with respect to P, denoted (π_U, V_U) is a representation of L on the quotient space

$$V_U = V/V(U)$$

where

$$V(U) = span\{\pi(u)v - v \mid v \in V,\ u \in U\}.$$

Write $\bar{v} = v + V(U)$ then L acts by

$$\pi_U(h)\bar{v} = \overline{\pi(h)v},\ h \in L, v \in V.$$

The subspace $V(U)$ is also characterized as the subspace

$$\{v \in V \mid \text{ for some compact open subgroup } U' \subseteq U, \int_{U'} \pi(u')v du' = 0\},$$

where $u' \in U'$ and du' is a Haar measure on U'. It is a useful technical fact (see [9]) that for any compact open subgroups $U_1 \subseteq U_2$,

$$\text{if} \int_{U_1} \pi(u)v du = 0 \text{ then} \int_{U_2} \pi(u)v du = 0\,. \tag{5}$$

The correspondence $\pi \to \pi_U$ defines an exact functor from smooth G-modules to smooth L-modules. Assuming now that π and σ (as above) are admissible; then the following second version of *Frobenius reciprocity* holds:

$$\mathrm{Hom}_L(\pi_U, \delta_P^{1/2} \otimes \sigma) \cong \mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma). \tag{6}$$

Finally we state the following important result of Jacquet and Harish-Chandra [9]:

Theorem C. *If π is an admissible representation of G then π_U is admissible.*

3. On elliptic representations

Let $P = LU$ be a (not necessarily proper) parabolic subgroup of G. Let $\tilde{L} \cong \prod_{j=1}^{k} GL_{N_j}$ (for some N_j's) be the Levi subgroup of $\tilde{G}$ such that $L = \tilde{L} \cap G$. Now we have the following results of David Goldberg.

Lemma 1. [12] *If for some i and j, $N_i \neq N_j$, then a representation of the form mbox*$\mathrm{ind}_P^G \sigma$ *can never have an elliptic constituent.*

We now specialize to the case that N is prime and $P = B$. We then have the following criteria:

Proposition 1. [12] *Suppose that all the N_j are equal* (*in fact they will then equal one, since N is prime*)*. Let σ be a unitary character of D, then the following are equivalent:*
a) $\mathrm{ind}_B^G \sigma$ *has an elliptic constituent,*
b) *Every constituent of* $\mathrm{ind}_B^G \sigma$ *is elliptic,*
c) *σ is of the form $\otimes_{j=1}^{N-1} \bar{\chi}^j$ where $\bar{\chi}$ is a character of $F^\times$ that has order N.*

The condition c) is formulated differently in [12]. The statement there is that the R-group is isomorphic to the cyclic group $\mathbb{Z}_N$. In particular $\mathrm{ind}_B^G \sigma$ decomposes into N inequivalent subrepresentations with multiplicity one. A simple verification shows that this condition is equivalent to the statement in c) above. This result motivates the construction given in the next section.

4. Construction of types

We consider a character $\overline{\chi}$ of $F^\times$ (the multiplicative group of F) whose restriction χ on $\mathcal{U}_F$ is nontrivial and satisfies $\chi^N = 1$. Now we have:

Lemma 2. a) *If* $p \neq N$ *then the powers* χ^j, $j \in \{1, \cdots, N-1\}$, *all have conductor* $\mathcal{P}_F$ (*that is, they are trivial on* $\mathcal{U}_F^{(1)} = 1 + \mathcal{P}_F$ *but not on* $\mathcal{U}_F$).
b) *If* $p = N$ *then the* χ^j *all have the same conductor* $\mathcal{P}_F^{n+1}$ *for some integer* $n \geq 1$.

Proof. a) If χ does not have conductor $\mathcal{P}_F$ then it has conductor $\mathcal{P}_F^{n+1}$ with $n \geq 1$, thus χ may be identified with a character of the group

$$\mathcal{U}_F^{(n)}/\mathcal{U}_F^{(n+1)} \cong k_F^+$$

which has order $q = p^r$. Hence $N = p$.
b) If $N = p$ then χ must have conductor $\mathcal{P}_F^{n+1}$, $n \geq 1$; otherwise χ would identify with a character of the group

$$\mathcal{U}_F/\mathcal{U}_F^{(1)} \cong k_F^\times$$

which has order $q - 1 = p^r - 1$. Since p does not divide $p^r - 1$, this is impossible.

We now appeal to the self-duality of F^+. Composing the isomorphism (1) with the following isomorphism

$$\begin{array}{ccc} \mathcal{U}_F^{(n)}/\mathcal{U}_F^{(n+1)} & \simeq & \mathcal{P}_F^n/\mathcal{P}_F^{n+1} \\ 1 + y & \to & y \end{array}$$

we see that as a character of $\mathcal{U}_F^{(n)}/\mathcal{U}_F^{(n+1)}$, χ is given by

$$\chi(1 + y) = \psi_a(y) \ \text{ where } a \in \mathcal{P}_F^{-n}/\mathcal{P}_F^{1-n}.$$

It follows that

$$\chi^j = \psi_{ja}.$$

Now each $j \in \{1, \cdots, N-1\}$ is a unit in $\mathcal{O}_F$ (since $p = N$); hence the ja all have the same valuation and the result follows. ∎

Our goal is to construct the type corresponding to the pair (D, σ) where σ is a character of D determined by the χ^j, $j \in \{1, \cdots, N-1\}$; σ is defined by:

$$\sigma \begin{pmatrix} d_1 & & & & \\ & \cdot & & 0 & \\ & & \cdot & & \\ & & & \cdot & \\ & 0 & & d_{N-1} & \\ & & & & \prod_{i=1}^{N-1} d_i^{-1} \end{pmatrix} = \prod_{j=1}^{N-1} \chi^j(d_j).$$

The construction which follows goes by cases.
Case $N \neq p$:

Here the χ^j, $j \in \{1, \cdots, N-1\}$ have conductor $\mathcal{P}_F$. We get a well defined character λ of the Iwahori subgroup as follows:

$$J = \begin{pmatrix} \mathcal{U}_F & & & & \\ & . & & \mathcal{O}_F & \\ & & . & & \\ & \mathcal{P}_F & & . & \\ & & & & \mathcal{U}_F \end{pmatrix} \cap SL_N(F)$$

($\mathcal{O}_F$, $\mathcal{U}_F$, $\mathcal{P}_F$ indicate where the entries belong)

$$\lambda \begin{pmatrix} u_1 & & & & \\ & . & & \star & \\ & & . & & \\ & \star & & . & \\ & & & & u_N \end{pmatrix} = \prod_{j=1}^{N-1} \chi^j(u_j),$$

(by $\star$ we mean some given entries which we do not need to show). We now state our main result:

Theorem 1. *An irreducible smooth representation (π, V) of G contains λ upon restriction to J if and only if there are unramified characters $\eta_j, j \in \{1, \cdots, N-1\}$ such that π embeds into $\mathrm{Ind}_B^G(\bigotimes_{j=1}^{N-1} \bar{\chi}^j \eta_j)$ where $\bigotimes_{j=1}^{N-1} \bar{\chi}^j \eta_j$ is a character of D defined by:*

$$\bigotimes_{j=1}^{N-1} \bar{\chi}^j \eta_j \begin{pmatrix} d_1 & & & & & \\ & . & & 0 & & \\ & & . & & & \\ & & & . & & \\ & 0 & & d_{N-1} & & \\ & & & & \prod_{i=1}^{N-1} d_i^{-1} \end{pmatrix} = \prod_{j=1}^{N-1} \bar{\chi}^j \eta_j(d_j).$$

Case $N = p$:

Here the χ^j have conductor $\mathcal{P}_F^{n+1}$, $n \geq 1$. We define a compact open subgroup J as follows: if $n = 2m$ then we let

$$J = \begin{pmatrix} \mathcal{U}_F & & & & \\ & . & & \mathcal{P}_F^m & \\ & & . & & \\ & \mathcal{P}_F^{m+1} & & . & \\ & & & & \mathcal{U}_F \end{pmatrix} \cap SL_N(F),$$

but if $n = 2m - 1$ then we let

$$J = \begin{pmatrix} \mathcal{U}_F & & & & \\ & . & & \mathcal{P}_F^m & \\ & & . & & \\ & \mathcal{P}_F^m & & . & \\ & & & & \mathcal{U}_F \end{pmatrix} \cap SL_N(F)\ .$$

Again by setting

$$\lambda\begin{pmatrix} u_1 & & & \\ & \cdot & \star & \\ & & \cdot & \\ & \star & \cdot & \\ & & & u_N \end{pmatrix} = \prod_{j=1}^{N-1} \chi^j(u_j),$$

we get a well defined character λ of J. Again we need to prove Theorem 1 for this choice of J.

5. Support of the Hecke algebra of a type

Recall that given a pair (J,λ) where J is a compact open subgroup of G and λ is a character of J, the convolution Hecke algebra $\mathcal{H}(G\|J,\lambda)$ consists of all compactly supported functions $f : G \to \mathbb{C}$ such that

$$f(hgk) = \lambda^{-1}(h)f(g)\lambda^{-1}(k),\ g \in G,\ h,\ k \in J.$$

Note that this definition still makes sense when J is not open in G, but in that case many of the nice properties of Hecke algebras will no longer hold.

It is convenient to (and we will always) normalize Haar measure on G so that $\mathrm{vol}(J) = 1$, when J is fixed.

Recall that the support of $\mathcal{H}(G\|J,\lambda)$ is the set

$$\{g \in G \mid JgJ \text{ supports a nonzero function in } \mathcal{H}(G\|J,\lambda)\}.$$

Since J is open it follows that there are only finitely many double cosets which support a given function in $\mathcal{H}(G\|J,\lambda)$. We will often denote f_g the unique function in $\mathcal{H}(G\|J,\lambda)$ (if any) such that $f_g(g) = 1$. An easy verification shows that the support of $\mathcal{H}(G\|J,\lambda)$ is the same as the G-intertwining of λ which is the set denoted $I_G(\lambda)$ defined by

$$I_G(\lambda) = \{g \in G \mid \lambda^g = \lambda, \ \text{ when restricted to } J \cap J^g\}.$$

(Recall our notations $J^g = gJg^{-1}$ and $\lambda^g(x) = \lambda(g^{-1}xg)$).

It is also clear, that

$$g \in I_G(\lambda) \ \text{ if and only if } \ JgJ \subseteq I_G(\lambda).$$

We now recall some standard facts on the structure of G which we shall need. If J is the Iwahori subgroup then we have the Bruhat decomposition:

$$G = JWJ = \coprod_{w\in W} JwJ \ \text{ (disjoint union)}, \tag{7}$$

where W is the *affine Weyl* group of G. Recall that if M is the normalizer of D in G and $M_0 = M \cap SL_N(\mathcal{O}_F)$ then $W = M/M_0$ and the Bruhat decomposition amounts to:

$$G = JMJ,$$

and if $w_1, w_2 \in M$ then

$$Jw_1J = Jw_2J \text{ if and only if } w_1 = w_2 \pmod{M_0}.$$

We take for generators of W the following (standard) representatives in M:

$$s_1 = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & I_{N-2} \end{pmatrix},\ s_2 = \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & I_{N-3} \end{pmatrix}, \cdots$$

$$s_{N-1} = \begin{pmatrix} I_{N-2} & & \\ & 0 & 1 \\ & -1 & 0 \end{pmatrix} \text{ and } s_N = \begin{pmatrix} & & \varpi^{-1} \\ & I_{N-2} & \\ -\varpi & & \end{pmatrix}.$$

There is defined on W a 'length' function l with respect to J. The length of $w \in W$ is the integer $l(w)$ such that:

$$[J : J \cap w^{-1}Jw] = q^{l(w)}.$$

In case J is the Iwahori subgroup (i.e $N \neq p$) $l(w)$ is equal to the minimum number of the s_i needed to express w. We have $l(s_i) = 1$ where $i \in \{1, ..., N\}$. We also note that

$$[J : J \cap w^{-1}Jw] = \mathrm{vol}(JwJ)/\mathrm{vol}(J).$$

We list some standard properties:

$$\begin{aligned} Jw_1w_2J &= Jw_1Jw_2J \text{ if } l(w_1w_2) = l(w_1) + l(w_2) \\ Js_iJs_iJ &= J \cup Js_iJ, \text{ where } i \in \{1, \cdots, N\}. \end{aligned}$$

We will denote $[d_1, \cdots, d_N]$ the diagonal matrix

$$\begin{pmatrix} d_1 & & & & \\ & . & & 0 & \\ & & . & & \\ & & & . & \\ & 0 & & & \\ & & & & d_N \end{pmatrix}$$

We introduce the Weyl element $w_0 = s_{N-1}s_{N-2}\cdots s_2s_1$. We have

$$\begin{aligned} w_0 &= \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & 0 & \\ & & . & . & & \\ & 0 & & . & . & \\ & & & & 0 & 1 \\ (-1)^{N+1} & & & & & 0 \end{pmatrix} \\ &= (-1)^{N+1}e_{N1} + \sum_{i=1}^{N-1} e_{i,i+1} \end{aligned}$$

where e_{ij} denotes the usual (δ_{ij}) matrix (δ_{ij} is the kronecker symbol). W acts by conjugation on the diagonal entries of elements in $M_N(F)$, as the symmetric group S_N. For this action the diagonal matrices in W act trivially and w_0 generates a subgroup of S_N consisiting of of cyclic permutations. We have $w_0^N = (-1)^{N+1} I_N$, where I_N denotes the N by N identity matrix. We also see by direct calculation that

$$w_0[d_1, \cdots, d_N]w_0^{-1} = [d_2, d_3, \cdots, d_N, d_1];$$

the action of w_0^k, $k \in \mathbb{N}$ is clear from this. We form the semi-direct product:

$$W_0 = \langle w_0 \rangle \ltimes D$$

We state:

Proposition 2. *$I_G(\lambda) = JW_0J$, where in case $p \neq N$, J is the Iwahori subgroup and in case $p = N$, J is the compact open subgroup defined in* Section 4.

Proof. The cases $p \neq N$ and $p = N$ require different techniques.
• Case $p \neq N$:
We first show that $JW_0J \subseteq I_G(\lambda)$ or equivalently $W_0 \subseteq I_G(\lambda)$. It is enough to show that $\langle w_0 \rangle \subseteq I_G(\lambda)$ (since the conjugation action of the diagonal matrices is trivial on diagonal entries). We show that $w_0 \in I_G(\lambda)$. We first observe the following simple fact. For $u = (u_{ij})$ in J, det(u)= 1 so we have:

$$\prod_{j=1}^{N} u_{ii} = 1 - \sum_{s \in S_N - \{1\}} \mathrm{sgn}(s) u_{1s(1)} \cdots u_{Ns(N)},$$

where sgn(s) denotes the sign of the permutation s in the symmetric group S_N. We clearly have

$$\sum_{s \in S_N - \{1\}} \mathrm{sgn}(s) u_{1s(1)} \cdots u_{Ns(N)} \in \mathcal{P}_F.$$

This implies that $\chi(u_{s(N)}) = \chi^{-1}(u_{s(1)} \cdots u_{s(N-1)})$ for any $s \in S_N$. Now we have

$$\begin{aligned}
\lambda^{w_0} \begin{pmatrix} u_1 & & & \\ & \cdot & \star & \\ & & \cdot & \\ & \star & \cdot & \\ & & & u_N \end{pmatrix} &= \lambda(w_0^{-1}[u_1, \cdots, u_N]w_0), \\
&= \chi(u_N)\chi^2(u_1) \cdots \chi^{N-1}(u_{N-2}) \\
&= \chi(u_1) \cdots \chi^{N-2}(u_{N-2})\chi^{-1}(u_N).
\end{aligned}$$

But $\chi^{-1} = \chi^{N-1}$, so that

$$\lambda^{w_0} = \lambda \text{ when restricted to } J \cap J^{w_0}.$$

We show in the same way that:

$$\prod_{j=1}^{N-1} \chi^j(u_{w_0^{k+1}(j)}) = \prod_{j=1}^{N-1} \chi^j(u_{w_0^k(j)}), \qquad k \in \mathbb{Z},$$

where we abuse notation by letting w_0^j, $j \in \mathbb{Z}$ denote the permutation in S_N which implements the conjugation action of w_0^j on diagonal entries. So we have shown that $JW_0J \subseteq I_G(\lambda)$.

To prove the converse we use again the fact that for $w \in W$, and any $d \in D$, $wd \in I_G(\lambda)$ if and only if $w \in I_G(\lambda)$. Now appealing to the Bruhat decomposition (7) we see that it is enough to show that if w is not in W_0 then w is not in $I_G(\lambda)$. Now let $w \in W - W_0$ then (up to multiplication by an appropriate power of w_0) we may assume that w conjugates $[u_1, u_2, \cdots, u_N]$ into $[u'_1, u'_2, \cdots, u'_N]$, where $u'_1 = u_1$ and $u'_k = u_2$ for some $k \in \{1, 3, \cdots, N\}$ (the point here is that $k \neq 2$).

Now we set $u_2 = u_1^{-1}$ and $u_j = 1$, for $j \in \{3, \cdots, N\}$. An easy calculation shows that λ^w and λ do not agree on $[u_1, u_1^{-1}, 1, \cdots, 1]$. This completes the proof in case $p \neq N$.

• Case $p = N$:

We see as above that $JW_0J \subseteq I_G(\lambda)$. To prove the converse we can no longer invoke the Bruhat decomposition. We shall rely instead, on orbit theory in $M_N(F)$ as developed by R. Howe (and exploited by Howe-Moy, Bushnell-Kutzko and others).

We consider the following standard lattice flags $\mathcal{L}_0 = \{L_j^0\}$, and $\mathcal{L}_1 = \{L_j^1\}$ of period respectively 1 and N defined as follows, let e_j denote the standard basis of F^N then set:

$$\begin{aligned}
L_j^0 &= \bigoplus_{i=1}^{N} \mathcal{P}_F^j e_i \,, j \in \mathbb{Z}. \\
L_0^1 &= \mathcal{O}_F e_1 \oplus \mathcal{O}_F e_2 \oplus \ldots \oplus \mathcal{O}_F e_{N-1} \oplus \mathcal{P}_F e_N, \\
L_1^1 &= \mathcal{O}_F e_1 \oplus \ldots \oplus \mathcal{O}_F e_{N-2} \oplus \mathcal{P}_F e_{N-1} \oplus \mathcal{P}_F e_N, \\
&\vdots \\
L_{N-1}^1 &= \mathcal{O}_F e_1 \oplus \mathcal{P}_F e_2 \oplus \ldots \oplus \mathcal{P}_F e_{N-1} \oplus \mathcal{P}_F e_N
\end{aligned}$$

We will denote A_0^j, A_1^j, $K_0^{(j)}$ and $K_1^{(j)}$, the corresponding filtrations of (compact open) ideals and subgroups, as in section 2.

In order to state a technical result on the geometry of conjugacy classes in A, we consider cases:

• Subcase n is odd: We let $n = 2m - 1$ and we set:

$$K_0 = K_0^{(m)} \cap G,$$

and

$$H_0 = K_0^{(n+1)} \cap G.$$

Clearly λ determines a character $\overline{\lambda}$ of K_0/H_0. It follows that $\overline{\lambda} = \psi_{a_0}$, where

$$a_0 \in A_0^{-n} / \left(A_0^{1-m} + F \cap A_0^{-n} \right).$$

• Subcase n is even: Here we set $n = 2m$ and we let

$$K_1 = K_1^{(Nm+1)} \cap G$$

and

$$H_1 = K_1^{(Nn+1)} \cap G.$$

Again we see that λ determines a character $\overline{\lambda} = \psi_{a_1}$ of K_1/H_1 where now

$$a_1 \in A_1^{-Nn} / \left(A_1^{-Nm} + F \cap A_1^{-Nn} \right).$$

In any case, we are in the 'separated' situation of Howe-Moy (see [15]). A direct adaptation of the proof of [15, Lemma 3.2], now yields:

Lemma 3. i) *For any $b \in A_0^{1-m}$, K_0 conjugates $a_0 + b$ to a diagonal matrix.* ii) *For any $b \in A_1^{-Nm}$, $K_1^{(Nm)} \cap G$ conjugates $a_1 + b$ to a diagonal matrix.*

Remark 1. i) *We note that K_0 and $K_1^{(Nm)} \cap G$ are subgroups of J.* ii) *Observe that a_0 and a_1 have the form: $[\alpha, 2\alpha, \ldots, (N-1)\alpha, 0]$, where $v_F(\alpha) = -n$.*

We now proceed with the proof of Proposition 2, as follows: We assume n is odd (the proof goes the same - with easy modifications - in case n is even).

It is enough to prove that $I_G(\psi_{a_0}) \subseteq JW_0J$. Calculations which are now standard (thanks to R. Howe [13]) show that

$$I_G(\psi_{a_0}) = \{x \in G \mid x(a_0 + A_0^{1-m} + F \cap A_0^{-n})x^{-1} \cap (a_0 + A_0^{1-m}) \neq \emptyset\}. \tag{8}$$

If $x \in I_G(\psi_{a_0})$ then there exists $y,\ y' \in A_0^{1-m}$, and $c \in F \cap A_0^{-n}$ such that

$$x(a+y)x^{-1} = a + c + y'. \tag{9}$$

By Lemma 3, there exists $z_1, z_2 \in K_0$, such that

$$z_1(a+y)z_1^{-1} = d \text{ and } z_2(a+y')z_2^{-1} = d',$$

for some diagonal matrices d and d'. Noting also that K_0 normalizes A_0^m and A_0^{1-m}, and K_0 conjugates a_0 to itself $(\mathrm{mod}\, A_0^{1-m})$, it follows that d and d' are congruent to $a_0 \pmod{A_0^{1-m}}$. In particular the entries of d (resp. d') are all distinct. Set $x_0 = z_2 x z_1^{-1}$; then

$$x_0 d x_0^{-1} = d' + c.$$

This shows that there exists $w \in W$ such that

$$d' + c = w d w^{-1}.$$

In particular

$$c \in \{\alpha, 2\alpha, \ldots, (N-1)\alpha, 0\} \pmod{A_0^{1-m}} \ (\alpha \text{ is taken as in Remark 1}).$$

(We trust the reader will distinguish between our use of c either as a scalar matrix or an element of the field). We also derive from this that

$$x_0^{-1} w d w^{-1} x_0 = d,$$

hence $w^{-1}x_0$ must be diagonal.

We claim that in fact $w \in W_0$. Indeed, if w is not in W_0, then (as noted before) we may assume that w^{-1} conjugates $d = [d_{11}, \ldots, d_{NN}]$ to $[d'_{11}, \ldots, d'_{NN}]$, where $d'_{11} = d_{11}$ but $d'_{1k} = d_{12}$ for some $k \neq 2$.

Now writing $d' = [d_{21}, \ldots, d_{2N}]$, we see that on one hand $d_{21} + c = d_{11}$ and hence $c \in \mathcal{P}_F^{1-n}$.

On the other hand we see that

$$c = d_{1k} - d_{12} \equiv (k-2)\alpha \pmod{\mathcal{P}_F^{1-n}}.$$

This is a contradiction. The proof is now complete in case $p = N$ also. ∎

6. Types for $GL_N(F)$

Implicit in our approach is the fact that our types may be constructed by restriction from similar types in $\tilde{G} = GL_N(F)$. We now discuss this explicitly and will later exploit the structure of two related $GL_N(F)$-algebras. In case $N \neq p$ we let $\tilde{J}$ denote the Iwahori subgroup of $\tilde{G}$. But in case $N = p$ we define it to be the following compact open subgroup:

$$\tilde{J} = \begin{pmatrix} \mathcal{U}_F & & & & \\ & \cdot & & \mathcal{P}_F^m & \\ & & \cdot & & \\ & \mathcal{P}_F^{m+1} & & \cdot & \\ & & & & \mathcal{U}_F \end{pmatrix} \text{ if } n = 2m;$$

$$\tilde{J} = \begin{pmatrix} \mathcal{U}_F & & & & \\ & \cdot & & \mathcal{P}_F^m & \\ & & \cdot & & \\ & \mathcal{P}_F^{m} & & \cdot & \\ & & & & \mathcal{U}_F \end{pmatrix} \text{ if } n = 2m-1.$$

(Recall that $n+1$ is the conductoral exponent of our fixed character $\bar{\chi}$ of $F^\times$). Clearly $J = \tilde{J} \cap G$. We define a character $\tilde{\lambda}$ of $\tilde{J}$ in a manner completely analogous to the definition of λ (see section 4). It is then clear that λ is just the restriction of $\tilde{\lambda}$ to J. We review some basic facts about the affine Weyl group and the Bruhat decomposition for $\tilde{G}$. For this account we refer to the excellent survey [14] and the bibliography there.

The Weyl group $\tilde{W}$ of $\tilde{G}$ is isomorphic to the symmetric group S_N and may be realized as the subgroup of permutation matrices. $\tilde{W}$ normalizes $\tilde{D}$. We denote by $\tilde{D}(\varpi)$ the subgroup consisting of diagonal matrices of the form

$$d_i = [\varpi^{i_1}, \ldots, \varpi^{i_N}],$$

for $i = (i_1, \ldots, i_N) \in \mathbb{Z}^N$. Clearly if $\sum_{k=1}^N i_k = 0$ then $d_i \in G$. We will always assume this condition whenever we view d_i as an element of G. The affine Weyl group $\tilde{W}^a$ may be realized as the semi-direct product

$$\tilde{W}^a = \tilde{W} \ltimes \tilde{D}(\varpi).$$

In case $N \neq p$ the group $\tilde{J}$ yields the Bruhat decomposition

$$\tilde{G} = \tilde{J}\tilde{W}^a\tilde{J}.$$

The length function l on $\tilde{W}^a$ with respect to $\tilde{J}$ is defined as before by the formula

$$\mathrm{vol}(\tilde{J}w\tilde{J}) = q^{l(w)}\mathrm{vol}(\tilde{J}), \qquad \text{for } w \in \tilde{W}^a.$$

Again in case $N \neq p$, l coincides with the 'Coxeter' length function. In any case a simple calculation gives

$$l_i = l(d_i) = \sum_{1 \leq k < j \leq N} |i_k - i_j|.$$

We next describe the structure of two algebras which we will need, namely $\tilde{\mathcal{H}} = \mathcal{H}(\tilde{G}\|\tilde{J},\tilde{\lambda})$ and $\overline{\mathcal{H}} = \mathcal{H}(\tilde{G}\|J,\lambda)$. We note however that J is not open in $\tilde{G}$ so that $\overline{\mathcal{H}}$ is not an Hecke algebra in the usual sense. Next we prove the following

Proposition 3. *The support of* $\mathcal{H}(\tilde{G}\|\tilde{J},\tilde{\lambda})$ *is equal to* $\tilde{J}\tilde{D}\tilde{J} = \tilde{J}\tilde{D}(\varpi)\tilde{J}$.

Proof. The technique of the proof is similar to that of Proposition 2. We will freely use some of the notations there. We indicate here only the changes that apply in this case. Again we consider cases.
Case $N \neq p$: One shows directly that $\tilde{D}$ (hence $\tilde{J}\tilde{D}\tilde{J}$) is in the support of $\tilde{\mathcal{H}}$. We prove the converse. By the decomposition of W^a as a semi-direct product and the Bruhat decomposition we see as before that it suffices to prove that no $w \in \tilde{\mathcal{H}}$ (except the identity) intertwines $\tilde{\lambda}$. For any $w \in \tilde{\mathcal{H}} - \{1\}$ there exists $j \in \{1,\ldots,N\}$ such that $w(j) \neq j$. Write $k = w(j)$; then there exists $u \in \mathcal{U}_F$ such that

$$\chi^j(u) \neq \chi^k(u).$$

But this means that λ^w and λ do not agree on the diagonal matrix $ue_{jj} + \sum_{i\neq j} e_{ii}$. This proves the result in this case.
Case $N = p$: We proceed as in the proof of Proposition 2. The important modification is that by (2) and (8), the scalar matrix c there is now zero. We complete the argument below.

We consider the same lattice flags $\mathcal{L}_0$ and $\mathcal{L}_1$ as before. These yield compact open subgroups $K_0^{(j)}$ and $K_1^{(j)}$, $j \geq 1$. Assuming that the conductoral exponent $n+1$ of χ is even (argument is the same otherwise), we set $n+1 = 2m$; then λ determines a character ψ_{a_0} of

$$K_0^{(m)}/K_0^{(n+1)}.$$

Now Lemma 3 and Remark 1 apply. As a consequence there exists $z_2, z_1 \in \tilde{J}$ and $w \in \tilde{W}$ such that

$$w^{-1}x_0 \text{ is diagonal}$$

where $x_0 = z_2xz_1$. Now the fact that c is zero forces w to be the identity. The proof is now clear. ∎

For $i \in \mathbb{Z}^N$ we let $\tilde{f}_i$ denote the unique function in $\tilde{\mathcal{H}}$ that takes value 1 on d_i. To state our version of the Hecke algebra isomorphism of Howe-Moy [15] we need another Hecke algebra, namely

$$\mathcal{H}' = \mathcal{H}(\tilde{D} \| \tilde{D} \cap \tilde{J}, \tilde{\lambda}|_{\tilde{D} \cap \tilde{J}}).$$

We let f_i' denote the unique function in $\mathcal{H}'$ that takes value 1 on d_i. Set $J' = \tilde{D} \cap \tilde{J}$. We now have:

Theorem 2. *Normalize Haar measures on $\tilde{G}$ and $\tilde{D}$ so that $\tilde{J}$ and J' have volume 1. The map $\Phi : \mathcal{H}' \to \tilde{\mathcal{H}}$, given by $\Phi(f_i') = q^{-l_i/2}\tilde{f}_i$, is an algebra isomorphism.*

Proof. See [15, Theorem 2.1]. Note also that under our normalizations of Haar measures we have $\mathrm{vol}(J' d_i J') = 1$ and $\mathrm{vol}(\tilde{J} d_i \tilde{J}) = q^{l_i}$. ∎

Corollary 1. *$\tilde{\mathcal{H}}$ is a commutative algebra and the following relation holds: For $i,\ j \in \mathbb{Z}^N$, $\tilde{f}_i * \tilde{f}_j = q^{1/2(l_i + l_j - l_{i+j})} \tilde{f}_{i+j}$.*

Proof. $\mathcal{H}'$ is clearly abelian (since $\tilde{D}$ is). As a consequence we have $f_i' * f_j' = f_{i+j}'$. Now apply Φ to get the desired relation. ∎

Corollary 2. *$\tilde{\mathcal{H}}$ is isomorphic to the N-fold tensor product of the 'one-dimensional' affine Hecke algebras $\mathcal{H}(F^\times \| \mathcal{U}_F, \chi^j)$, $j \in \{1, \ldots, N\}$.*

Proof. This result holds for $\mathcal{H}'$. Indeed we verify that the isomorphism is explicitly given by the map $f_i' \to \phi_{i_1} \otimes \ldots \otimes \phi_{i_N}$, where ϕ_{i_j} is the unique function in $\mathcal{H}(F^\times \| \mathcal{U}_F, \chi^j)$ such that $\phi_{i_j}(1) = 1$. ∎

In order to determine some useful relations in $\overline{\mathcal{H}}$ we now introduce the following Weyl element

$$\tilde{t} \;=\; \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ 0 & & & 0 & 1 \\ \varpi & & & & 0 \end{pmatrix}.$$

We let $\overline{W} = \langle \tilde{t} \rangle \ltimes \tilde{D}$. Then a calculation as in Proposition 2 shows that $\tilde{t}$ as well as the diagonals in $\tilde{G}$ intertwine λ so that we obtain following:

Proposition 4. *$J\overline{W}J$ is contained in the support of $\overline{\mathcal{H}}$.*

Let I_k denote the k by k identity matrix. Every $x \in \tilde{G}$ decomposes uniquely as zg where

$$z = \begin{pmatrix} \det(x) & 0 \\ 0 & I_{N-1} \end{pmatrix}$$

and $g \in G$. If we let Z be the subgroup consisting of diagonal matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & I_{N-1} \end{pmatrix}$$

where $\alpha \in F^\times$; then we have the decompositions

$$\tilde{G} = Z \ltimes G$$

and

$$\tilde{J} = (Z \cap \tilde{J}) \ltimes J\ .$$

We note that $Z \cong F^\times$. Furthermore the first decomposition allows for a decomposition of a Haar measure dx on $\tilde{G}$ as

$$dx = dzdg$$

where dz (resp. dg) is Haar measure on Z (resp. G).
The following simple result will be useful:

Lemma 4. a) *For any $d \in D$ we have $\tilde{J}d\tilde{J} \cap G = JdJ$. In particular the restriction of $\tilde{f}_d$ to G is equal to f_d.*
b) *Normalize Haar measures on $\tilde{G}$, Z, and G such that $\tilde{J}$, $Z \cap \tilde{J}$ and J all have volume 1. Then for $d,\ d' \in D$, the restriction of $\tilde{f}_d * \tilde{f}_{d'}$ to G is equal to $f_d * f_{d'}$.*

Proof. Let $g \in \tilde{J}d\tilde{J} \cap G$; then $g = xdy$ for some $x,\ y \in \tilde{J}$ such that $\det(xy)$=1. Now decomposing $x = z_1g_1$ and $y = z_2g_2$ as above we see that $g = g_1dg_2$. This proves a). To prove b) we compute

$$\begin{aligned}(\tilde{f}_d * \tilde{f}_{d'})(g) &= \int_{\tilde{G}} \tilde{f}_d(gx^{-1})\tilde{f}_{d'}(x)dx \\ &= \int_{Z\ltimes G} \tilde{f}_d(gy^{-1}z^{-1})\tilde{f}_{d'}(zy)dzdy.\end{aligned}$$

The integrand is nonzero only if $gy^{-1}z^{-1} \in \tilde{J}d\tilde{J}$ and $zy \in \tilde{J}d'\tilde{J}$. In particular $\det(z) \in \mathcal{U}_F$ hence $z \in Z \cap \tilde{J}$. This implies that

$$\begin{aligned}\tilde{f}_d(gy^{-1}z^{-1})\tilde{f}_{d'}(zy) &= \tilde{f}_d(gy^{-1})\tilde{\lambda}(z^{-1})\tilde{\lambda}(z)\tilde{f}_{d'}(y) \\ &= \tilde{f}_d(gy^{-1})\tilde{f}_{d'}(y) \\ &= f_d(gy^{-1})f_{d'}(y).\end{aligned}$$

This shows that the integral is equal to $(f_d * f_{d'})(g)$, the result is now clear. ∎

We observe that $\tilde{t}$ normalizes J. In particular $l(\tilde{t}) = 0$. Let $\bar{e}_\lambda$ denote the identity in $\overline{\mathcal{H}}$. Let $\bar{f}_i$ (resp. $\bar{f}_{\tilde{t}}$) denote the unique function in $\overline{\mathcal{H}}$ that takes value 1 on d_i (resp. $\tilde{t}$).

Remark 2. *We note that $\mathcal{H}(G\|J,\lambda)$ embeds into $\overline{\mathcal{H}}$, by extending functions to $\tilde{G}$ as zero off G. We will use the same notation for a function and its extension. Furthermore the restriction of $\tilde{f}_d$ to G may be regarded as an element of $\overline{\mathcal{H}}$ by* Lemma 4.

Next we will prove

Proposition 5. a) *For* $i,\ j \in \mathbb{Z}^N$, $\bar{f}_i * \bar{f}_j = q^{1/2(l_i+l_j-l_{i+j})}\bar{f}_{i+j}$.
b) $\bar{f}_{\tilde{t}}$ *is invertible and for any* $d \in \tilde{D}$, $\bar{f}_{\tilde{t}}^{-1} * \bar{f}_d * \bar{f}_{\tilde{t}} = \bar{f}_{\tilde{t}^{-1}d\tilde{t}}$.
c) $\bar{f}_{\tilde{t}}^k = \bar{f}_{\tilde{t}^k}$.

Proof. The relation in a) follows from Corollary 1, Lemma 4 and Remark 1. We prove the statements in b). We clearly have

$$J\tilde{t}J\tilde{t}^{-1}J = J.$$

This implies that

$$\bar{f}_{\tilde{t}} * \bar{f}_{\tilde{t}^{-1}} = \beta_1 \bar{e}_\lambda.$$

Evaluating $\bar{f}_{\tilde{t}} * \bar{f}_{\tilde{t}^{-1}}$ at 1 we get $\beta_1 = 1$. This means that $\bar{f}_{\tilde{t}^{-1}}$ is the inverse of $f_{\tilde{t}}$. For any $w \in \overline{W}$ we have

$$J\tilde{t}^{-1}JwJ = J\tilde{t}^{-1}wJ\ .$$

This shows that

$$\bar{f}_{\tilde{t}^{-1}} * \bar{f}_w = \beta_2 \bar{f}_{\tilde{t}^{-1}w}.$$

Fix Haar measures on G and J normalized by the condition that vol(J)=1. We compute

$$\begin{aligned}
\beta_2 &= (\bar{f}_{\tilde{t}^{-1}} * \bar{f}_w)(\tilde{t}^{-1}w) \\
&= \int_G \bar{f}_{\tilde{t}^{-1}}(\tilde{t}^{-1}wg^{-1})\bar{f}_w(g)dg \\
&= \mathrm{vol}(JwJ)\int_{J\times J} \bar{f}_{\tilde{t}^{-1}}(\tilde{t}^{-1}wk^{-1}w^{-1}h^{-1})\bar{f}_w(hwk)dhdk \\
&= \mathrm{vol}(JwJ)\int_J \bar{f}_{\tilde{t}^{-1}}(\tilde{t}^{-1}wk^{-1}w^{-1})\lambda(k)dk.
\end{aligned}$$

The integrand is nonzero only if $k \in w^{-1}Jw$. Under this condition we have

$$\begin{aligned}
\bar{f}_{\tilde{t}^{-1}}(\tilde{t}^{-1}wk^{-1}w^{-1}) &= \lambda(wk^{-1}w^{-1}) \\
&= \lambda(k^{-1})\ \text{(since } w \text{ intertwines } \lambda).
\end{aligned}$$

It follows that the integral is equal to $\mathrm{vol}(JwJ)\mathrm{vol}(J \cap w^{-1}Jw) = 1$. This shows the relation

$$\bar{f}_{\tilde{t}^{-1}} * \bar{f}_w = \bar{f}_{\tilde{t}^{-1}w}.$$

We prove similarly that

$$\bar{f}_w * \bar{f}_{\tilde{t}} = \bar{f}_{w\tilde{t}}.$$

Combining these two facts we get the desired relation. The proof of c) follows from the simple fact that the k-fold coset $J\tilde{t}J\tilde{t}\ldots\tilde{t}J$ is equal to $J\tilde{t}^kJ$, and an obvious inductive argument. ∎

This proposition clearly establishes the following

Theorem 3. *The subalgebra of* $\overline{\mathcal{H}}$ *generated by the* $\bar{f}_i$, $(i \in \mathbb{Z}^N)$ *and* $\bar{f}_{\tilde{t}}$ *is isomorphic to the group algebra* $\mathbb{C}[\overline{W}]$.

7. Structure of the Hecke Algebra

We now take up the task of proving the following:

Theorem 4. *As a* $\mathbb{C}$*-algebra* $\mathcal{H}(G\|J,\lambda)$ *is isomorphic to the group algebra* $\mathbb{C}[W_0]$.

First we make some remarks. By Proposition 2, we know that the support of $\mathcal{H}(G\|J,\lambda)$ is JW_0J. So the functions f_w, $w\in W_0$, generate $\mathcal{H}(G\|J,\lambda)$ as a $\mathbb{C}$-vector space. We now make use of the results of the previous section. We will freely use the notations there.

To determine relations in $\mathcal{H}(G\|J,\lambda)$ we may - by Remark 1 - compute inside $\overline{\mathcal{H}}$. We may also restrict relations from $\tilde{\mathcal{H}}$ by restricting functions to G. The Theorem will follow clearly from

Proposition 6. a) *For any* $i,j\in\mathbb{Z}^N$ *we have* $f_i*f_j=q^{1/2(l_i+l_j-l_{i+j})}f_{i+j}$.
b) f_{w_0} *is invertible and* $f_{w_0}^{-1}*f_d*f_{w_0}=f_{w_0^{-1}dw_0}$, $d\in D$.
c) $f_{w_0}^k=q^{\frac{k(k-1)}{2}}f_{w_0^k}$, *where* $k\in\mathbb{N}$. *In particular* $f_{w_0}^N=q^{\frac{N(N-1)}{2}}\chi((-1)^{N+1})e_\lambda$.
Proof. The relation in a) follows from Corollary 1 and Lemma 4. To prove the relations in b) and c) we consider the following diagonal matrix

$$\gamma_k=\begin{pmatrix}(-1)^{N+1}\varpi^{-1}I_k & 0\\ 0 & I_{N-k}\end{pmatrix},\ k\in\mathbb{Z}.$$

A calculation shows that

$$w_0^k=\tilde{t}^k\gamma_k.$$

In particular $l(w_0^k)=l(\gamma_k)=k(N-k)$. We now compute in $\overline{\mathcal{H}}$.
Computing directly as in the proof of Proposition 5 we get the following decompositions:

$$f_{w_0}=\bar{f}_{\tilde{t}}*\bar{f}_{\gamma_1}\qquad\text{and}\qquad f_{w_0^{-1}}=\bar{f}_{\gamma_1^{-1}}*\bar{f}_{\tilde{t}}^{-1}.$$

Hence

$$\begin{aligned}f_{w_0^{-1}}*f_{w_0}&=\bar{f}_{\gamma_1^{-1}}*\bar{f}_{\gamma_1}\\&=q^{l(\gamma_1)}\bar{e}_\lambda\ \text{(by Proposition 5 a))}\\&=q^{N-1}\bar{e}_\lambda.\end{aligned}$$

This shows in particular that f_{w_0} is invertible.
We make use of Proposition 5 and the simple fact that the conjugation action of $\tilde{t}$ does not change the length of a Weyl element; we compute:

$$\begin{aligned}f_{w_0^{-1}}*f_i*f_{w_0}&=\bar{f}_{\gamma_1^{-1}}*\bar{f}_{\tilde{t}}^{-1}*f_i*\bar{f}_{\tilde{t}}*\bar{f}_{\gamma_1}\\&=\bar{f}_{\gamma_1^{-1}}*\bar{f}_{\tilde{t}^{-1}d\tilde{t}}*\bar{f}_{\gamma_1}\\&=q^{1/2(l_i+l(\gamma_1)-l(\tilde{t}^{-1}d_i\tilde{t}\gamma_1))}\bar{f}_{\tilde{t}^{-1}d_i\tilde{t}\gamma_1}\\&=q^{1/2(l_i+l(\gamma_1)-l(\tilde{t}^{-1}d_i\tilde{t}\gamma_1))}q^{1/2(l(\gamma_1)+l(\tilde{t}^{-1}d_i\tilde{t}\gamma_1)-l_i)}\bar{f}_{\gamma_1^{-1}\tilde{t}^{-1}d_i\tilde{t}\gamma_1}\\&=q^{N-1}f_{w_0^{-1}d_iw_0}.\end{aligned}$$

We have thus shown the required relation in b), recalling also from above that

$$f_{w_0^{-1}} = q^{N-1} f_{w_0}^{-1}.$$

To prove the relations in c), we proceed by induction.

Using again the decomposition $f_{w_0} = \bar{f}_{\tilde{t}} * \bar{f}_{\gamma_1}$ we see that $f_{w_0}^2 = q f_{w_0^2}$. We now proceed with the induction as follows

$$\begin{aligned}
f_{w_0}^{k+1} &= f_{w_0}^k * f_{w_0} \\
&= q^{\frac{k(k-1)}{2}} f_{w_0{}^k} * f_{w_0} \\
&= q^{\frac{k(k-1)}{2}} \bar{f}_{\tilde{t}^k} * \bar{f}_{\gamma_k} * \bar{f}_{\tilde{t}} * \bar{f}_{\gamma_1} \\
&= q^{\frac{k(k-1)}{2}} \bar{f}_{\tilde{t}^{k+1}} * \bar{f}_{\tilde{t}^{-1}\gamma_k \tilde{t}} * \bar{f}_{\gamma_1} \\
&= q^{\frac{k(k-1)}{2}} q^{\frac{l(\gamma_k)+l(\gamma_1)-l(w_0^{k+1})}{2}} f_{w_0^{k+1}} \\
&= q^{\frac{(k+1)k}{2}} f_{w_0{}^{k+1}}
\end{aligned}$$

This proves the first relation in c). The second relation is an obvious consequence of this relation. ∎

8. Classification

Our goal here is to prove Theorem 1 (see section 4). We let $\mathcal{F}\colon V \to V_U$ denote the canonical Jacquet map with respect to the Borel subgroup $B = DU$. For simplicity of notations we will write $\bar{v} = \mathcal{F}(v)$ where $v \in V$. We note that J admits an Iwahori decomposition with respect to B:

$$J = (J \cap U)(J \cap D)(J \cap U^-),$$

a product which can be taken in any order. We let λ_D denote the restriction of λ on $J \cap D$ and λ_{DU^-} the restriction of λ to $(J \cap D)(J \cap U^-)$. We then prove the following result (known as Casselman's lemma):

Theorem 5. *The canonical map $\mathcal{F}\colon V \to V_U$ determines an isomorphism (of $\mathbb{C}$-vector spaces)*

$$V^\lambda \cong (V_U)^{\lambda_D}.$$

To prove this theorem we need the following results:

Proposition 7. (Casselman [9]) *Let G be a reductive connected p-adic group. If $P = LU$ is a proper parabolic subgroup and if U_1, U_2 are two compact open subgroups of U then there exists $a \in T$, where T is the maximal split torus in the center of L, such that $a^{-1}U_2 a \subseteq U_1$.*

Remark 3. *For $G = SL_N(F)$ we let $P = B = DU$ then $T = L = D \cong (F^\times)^{N-1}$.*

Lemma 5. *For any smooth representation (π, V) of G and any $d \in D$ the operator $\pi(e_\lambda) \circ \pi(d) \circ \pi(e_\lambda)$ is an automorphism of V^λ.*

Lemma 6. *V^λ and $V^{\lambda_{DU^-}}$ have the same image in V_U.*

We will give the proofs of these Lemmas later; now we use them to prove Theorem 5.
Proof of Injectivity:
The canonical map $\mathcal{F}\colon V \to V_U$ yields by restriction the map $V^\lambda \to V_U$ whose kernel is $V^\lambda \cap V(U)$. By way of contradiction, let $v \in V^\lambda \cap V(U)$, $v \neq 0$; there exists a compact open subgroup U_1 such that

$$\int_{U_1} \pi(u) v du = 0.$$

Proposition 7 implies that there exists $d \in D$ such that $d^{-1}U_1 d \subseteq J \cap U$. Now Lemma 5 says that: $(\pi(e_\lambda) \circ \pi(d) \circ \pi(e_\lambda))(v) = \pi(e_\lambda)\pi(d)v$ is a non-zero vector in V^λ. Observe that for any nonzero vector w in V^λ we have:

$$\int_{J \cap U} \pi(u) w du \neq 0,$$

since λ is trivial on $J \cap U$; but we have that

$$\begin{aligned} \int_{J \cap U} \pi(u) \circ \pi(e_\lambda) \circ \pi(d) v du &= \pi(e_\lambda) \int_{J \cap U} \pi(u)\pi(d) v du \\ &= \pi(e_\lambda) \circ \pi(d) \int_{J \cap U} \pi(d^{-1}ud) v du \\ &= \pi(e_\lambda) \circ \pi(d) \int_{d(J \cap U)d^{-1}} \pi(u) v du \\ &= 0 \ \ (\text{since } U_1 \subseteq d(J \cap U)d^{-1}, \text{ see } (1.6)). \end{aligned}$$

This is a contradiction.
Proof of Surjectivity:
We note that (by Theorem C) (π_U, V_U) is admissible since (π, V) is; hence $(V_U)^{\lambda_D}$ is finite dimensional. We also note the following facts:

- (f1) There exists a finite dimensional subspace $E \subseteq V^{\lambda_D}$ mapping onto $(V_U)^{\lambda_D}$ under the canonical map $\mathcal{F}\colon V \to V_U$.
- (f2) There exists a compact open subgroup $U_0^- \subseteq U^-$ that fixes E.
- (f3) For any $d \in D$, $\pi_U(d)$ is an automorphism of $(V_U)^{\lambda_D}$.

Proof of (f1):
Define an operator P on V by:

$$P(v) = \int_{J \cap D} (\pi \otimes \lambda_D^{-1})(x) v dx$$

where dx is normalized by the condition $\mathrm{vol}(J \cap D) = 1$. Choose a basis $\bar{f}_1, \ldots, \bar{f}_r$ of $(V_U)^{\lambda_D}$; then set

$$e_i = P(f_i)\ ,$$

where f_i is any representative in $\bar{f}_i$. We show that $\bar{e}_i = \bar{f}_i$. If K is a compact open subgroup of G that fixes f_i then:

$$e_i = \mathrm{vol}(J \cap D \cap K) \times \sum_{x \in J\cap D/J\cap D\cap K} (\pi \otimes \lambda_D^{-1})(x) f_i,$$

therefore

$$\overline{e}_i = \mathrm{vol}(J \cap D \cap K) \times \sum_{x \in J\cap D/J\cap D\cap K} \lambda_D^{-1}(x)\pi_U(x)\bar{f}_i = \bar{f}_i.$$

Also a direct computation shows that $P(v) \in V^{\lambda_D}$ for any $v \in V$, in particular $e_i \in V^{\lambda_D}$. Now we define $E = \mathrm{span}\{e_1, \ldots, e_r\}$. □

Proof of (f2):
Let $E = \mathrm{span}\{e_1, \ldots, e_r\}$ as above, let $U_i^- \subseteq U^-$ be a compact open subgroup that fixes e_i, now set $U_0^- = \bigcap_{i=1}^r U_i$. □

Proof of (f3):
It is enough to show that for all $d \in D$, $\pi_U(d)(V_U)^{\lambda_D} \subseteq (V_U)^{\lambda_D}$. Take $\overline{v} \in (V_U)^{\lambda_D}$, $d' \in D$ then:

$$\begin{aligned} \pi_U(d')\pi_U(d)\overline{v} &= \pi_U(d)\pi_U(d')\overline{v} \\ &= \pi_U(d)\lambda_D(d')\overline{v} \\ &= \lambda_D(d')\pi_U(d)\overline{v}. \end{aligned}$$

We now proceed with the proof of surjectivity. We choose $d \in D$ such that

$$d^{-1}(J \cap U)d \subseteq U_0^-$$

then we claim that $\pi(d)E \subseteq V^{\lambda_{DU^-}}$. To see this let $v \in E$, $d' \in J\cap D$, $u \in J\cap U^-$; then

$$\begin{aligned} \pi(d'u)\pi(d)v &= \pi(d')\pi(d)\pi(d^{-1}ud)v \\ &= \pi(d')\pi(d)v \\ &= \pi(d)\pi(d')v \\ &= \pi(d)\lambda(d')v \\ &= \lambda(d')\pi(d)v. \end{aligned}$$

We have

$$\pi_U(d)(V_U)^{\lambda_D} = \mathcal{F}(\pi(d)E) \subseteq \mathcal{F}(V^{\lambda_{DU^-}}) = \mathcal{F}(V^{\lambda})$$

(the last equality follows from Lemma 6). We also have

$$(V_U)^{\lambda_D} = \pi_U(d)(V_U)^{\lambda_D} \subseteq \mathcal{F}(V^{\lambda})$$

(the first equality follows from (f3)). This proves surjectivity. □

Proof of Lemma 5:
We define, for $\varphi \in \mathcal{H}(G,\lambda)$ and $x \in G$, ${}^x\varphi$ by

$$ {}^x\varphi(y) = \varphi(x^{-1}y). $$

A direct computation shows that

$$ {}^d e_\lambda = \delta_d * e_\lambda, $$

where δ_d is the Dirac function at d. We also have:

$$ \pi(d) \circ \pi(e_\lambda) = \pi({}^d e_\lambda), $$

hence

$$ \pi(e_\lambda) \circ \pi(d) \circ \pi(e_\lambda) = \pi(e_\lambda * {}^d e_\lambda). $$

It is then enough to show that $e_\lambda * {}^d e_\lambda$ is an invertible element of the algebra $e_\lambda * \mathcal{H}(G) * e_\lambda$. Clearly

$$ e_\lambda * {}^d e_\lambda = e_\lambda * \delta_d * e_\lambda $$

thus we see that $e_\lambda * {}^d e_\lambda$ has support JdJ. We have $JdJ = Jd_iJ$ for some $i \in \mathbb{Z}^N$. It follows that $e_\lambda * {}^d e_\lambda$ is a nonzero multiple of f_i, hence (by Proposition 4) it is invertible as required. □

Proof of Lemma 6:
It is enough to show that $\mathcal{F}(V^{\lambda_{DU^-}}) \subseteq \mathcal{F}(V^\lambda)$ (since $V^\lambda \subseteq V^{\lambda_{DU^-}}$). We need to show that given $v \in V^{\lambda_{DU^-}}$ there exists $v' \in V^\lambda$ such that $v' - v \in V(U)$. We let dx be the Haar measure on J normalized by the condition $\mathrm{vol}(J) = 1$. We define

$$ v' = \int_J (\pi \otimes \lambda^{-1})(x) v dx $$

then $v' \in V^\lambda$. A direct calculation and the fact that $v \in V^{\lambda_{DU^-}}$ show that $v' - v \in V(U)$. □

Proof of Theorem 1:
For convenience we write $\sigma = \bigotimes_{j=1}^{N-1} \chi^j \eta_j$, and $\pi = \mathrm{Ind}_B^G \sigma$. The restriction of π to J decomposes (by Mackey's Theorem) as

$$ \bigoplus_{x \in (J|G|B)} \text{c-Ind}_{J \cap B^x}^J \sigma^x, $$

where $(J|G|B)$ denotes the set of all $J-B$ double cosets in G. The restrictions of λ and σ to $J \cap B$ coincide, therefore by Frobenius reciprocity (4) $\text{c-Ind}_{J\cap B}^J \sigma$ contains λ, hence π contains λ.

We prove the converse. By assumption we have $V^\lambda \neq 0$, Theorem 5 then implies that $(V_U)^{\lambda_D} \neq 0$. But $(V_U)^{\lambda_D}$ is finite dimensional (by Theorem C, see section 2). Thus

$$ V_U = (V_U)^{\lambda_D} \oplus \tilde{V} $$

as $J \cap D$-spaces but also as D-spaces, where $\tilde{V}$ is some subspace in V_U, since D centralises $J \cap D$. D is abelian hence there exists a one dimensional D-quotient in $(V_U)^{\lambda_D}$. It follows by Frobenius reciprocity (6) that π embeds into $\mathrm{Ind}_B^G \gamma$ where γ is a character of D. We consider the restriction of $\mathrm{Ind}_B^G \gamma$ to J. It follows (by Mackey's theorem and Frobenius reciprocity again) that for some $x \in (J|G|B)$, the restrictions of λ and γ^x on $J \cap D^x$ coincide. We deduce from this that γ is of the form $\bigotimes_{j=1}^{N-1} \chi^j \eta_j$, where the η_j are unramified characters of $F^\times$. This concludes the proof of Theorem 1. $\square$

9. Concluding remarks

The analogue of Theorem 1 for $\tilde{G}$ is stated below. We denote $\otimes_{j=1}^{N} \chi^j \eta_j$ the character of $\tilde{D}$ defined by

$$\otimes_{j=1}^{N} \chi^j \eta_j([d_1, \ldots, d_N]) = \prod_{j=1}^{N} \chi^j \eta_j(d_j).$$

We then introduce the (Bernstein-Zelevinsky) notation

$$\times_{j=1}^{N} \chi^j \eta_j$$

for the induced representation

$$\mathrm{Ind}_{\tilde{B}}^{\tilde{G}} (\otimes_{j=1}^{N} \chi^j \eta_j) \ .$$

The Hecke algebras corresponding to the $GL_N(F)$-types introduced in section 6, are described by Corollary 2. The same methods used here to prove Theorem 1 yield now the following

Theorem 6. *An irreducible smooth representation (π, V) of $\tilde{G}$ contains $\tilde{\lambda}$ upon restriction to $\tilde{J}$ if and only if there are unramified characters η_j, $j \in \{1, \cdots, N\}$ such that π is equivalent to $\times_{j=1}^{N} \chi^j \eta_j$.*

Observe that by results of Bernstein and Zelevinsky [3], $\times_{j=1}^{N} \chi^j \eta_j$ is irreducible.

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