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A connected complex simple centerfree Lie group whose exponential function is not surjective

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Abstract. We discuss the example of a complex simple Lie group G, with trivial center, whose exponential map is not surjective and with $\dim_{\mathbb{C}} G = 10$.

In [2] I have characterized solvable Lie groups and complex semisimple Lie groups with surjective exponential function. In particular I have proven the following Theorem (See [2, Satz IV.3.27] and [3]):

Theorem 1. *A semisimple complex Lie groups has surjective exponential function if and only if it is isomorphic to a finite product of groups* $\mathrm{PSl}(n(i),\mathbb{C})$.

Independently, in [1] Moskowitz also has proven that finite products of $\mathrm{PSl}(n(i),\mathbb{C})$ have surjective exponential map. In the context of this work it is informative to observe that for the exponential map on a connected complex simple Lie group to be surjective it is not enough that it is centerfree. In the following I shall discuss the smallest counterexample known, namely $\mathrm{PSP}(2,\mathbb{C})$. Here $\mathrm{PSP}(2,\mathbb{C}) \cong \mathrm{SP}(2,\mathbb{C})/Z(\mathrm{SP}(2,\mathbb{C}))$ is the adjoint Lie group of $\mathrm{SP}(2,\mathbb{C})$ of all matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sl}(4,\mathbb{C})$ with A, B, C, $D \in \mathbb{C}^{2\times 2}$ such that $C^TA - A^TC = 0$, $D^TB - B^TD = 0$, $A^TD - C^TB = I$, where I is the identity of $\mathrm{Gl}(2,\mathbb{C})$.

Theorem 2. *The complex simple Lie group* $\mathrm{PSP}(2,\mathbb{C})$ *is centerfree and its exponential map is not surjective.*

Proof. The center of $\mathrm{PSP}(2,\mathbb{C})$ is trivial since it is an adjoint Lie group. We claim that there is an element $\gamma \in \mathrm{PSP}(2,\mathbb{C})$, for which there is no $x \in \mathrm{sp}(2,\mathbb{C})$ such that $\gamma = \exp_{\mathrm{PSP}(2,\mathbb{C})} x$. First, we consider the group $\mathrm{SP}(2,\mathbb{C})$.

Let $p\colon \mathrm{SP}(2,\mathbb{C}) \to \mathrm{PSP}(2,\mathbb{C})$ be the quotient map. We claim that there is no $g \in p^{-1}(\gamma)$ which is in the image of $\exp_{\mathrm{SP}(2,\mathbb{C})}$. Then we will have finished the proof. We shall prove the claim in the following lemma. ■

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Lemma 3. *There is a* $\gamma \in \mathrm{PSP}(2,\mathbb{C})$ *such that no element of* $p^{-1}(\gamma)$ *lies in the image of* $\exp_{\mathrm{SP}(2,\mathbb{C})}$.

Proof. We set $g = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. The proof that g is in $\mathrm{SP}(2,\mathbb{C})$ is straightforward. Now we set $\gamma := p(g)$.

The elements of the Lie algebra $\mathrm{sp}(2,\mathbb{C})$ of $\mathrm{SP}(2,\mathbb{C})$ are exactly those elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ of $\mathrm{sl}(4,\mathbb{C})$ with 2×2 complex matrtices A, B, C and D, for which the following conditions hold: $A = -D^T$, $B = B^T$, $C = C^T$.

We assume that there is an $x \in \mathrm{sp}(2,\mathbb{C})$ such that $g = \exp_{\mathrm{SP}(2,\mathbb{C})} x$ and consider $\mathrm{SP}(2,\mathbb{C})$ as a group of endomorphisms of $\mathbb{C}^4$. Since a subspace which is invariant under x must be invariant under g and since g possesses two 2-dimensional eigenspaces for two different eigenvalues, respectively, x must possess two 2-dimensional eigenspaces. By the Spectral Mapping Theorem one eigenvalue must be in $2\pi i\mathbb{Z}$ and the other must be of the form $(2z+1)\pi i$ with $z \in \mathbb{Z}$. But then the trace of x is equal to $(4z_1 + 2(2z_2+1))\pi i$ with $z_1, z_2 \in \mathbb{Z}$. On the other hand, $\operatorname{tr} x = 0$ because $x \in \mathrm{sp}(2,\mathbb{C}) \subseteq \mathrm{sl}(4,\mathbb{C})$. But this implies $2 \in 4\mathbb{Z}$ and this is impossible. Now we consider $\mathrm{PSP}(2,\mathbb{C})$. The center of $\mathrm{SP}(2,\mathbb{C})$ is equal to $\{1,-1\}$. So, $p^{-1}(\gamma) = \{g,-g\}$. We saw a moment ago that $g \notin \mathrm{im}(\exp_{\mathrm{SP}(2,\mathbb{C})})$. But the proof of the assertion $-g \notin \mathrm{im}(\exp_{\mathrm{SP}(2,\mathbb{C})})$ is analogous to that for g. ∎

The argument shows that g and $-g$ have no preimage in $\mathrm{sl}(4,\mathbb{C})$. But in this case, the center of $\mathrm{Sl}(4,\mathbb{C})$ is equal to $\{I,-I,iI,-iI\}$ and, indeed, ig has a preimage in $\mathrm{sl}(4,\mathbb{C})$, namely $\begin{pmatrix} \frac{\pi}{2}i & 0 & -1 & 0 \\ 0 & -\frac{\pi}{2}i & 0 & 1 \\ 0 & 0 & \frac{\pi}{2}i & 0 \\ 0 & 0 & 0 & -\frac{\pi}{2}i \end{pmatrix}$. So, of course, we have not constructed a counterexample to the result in Theorem 1 that $\mathrm{PSl}(n,\mathbb{C})$ possesses a surjective exponential function.

References

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