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Regularization of birational group operations in sense of Weil.

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1. Introduction.

The present paper deals with the classical results of A. Weil [11] on regularization of pre-groups and pre-transformation spaces (see Definitions 3.1 and 4.1). As pointed out in [4], those purely algebraic results appear to be very useful in the following complex analytic setting.

Let $D \subset \mathbb{C}^n$ be a bounded domain and $Aut(D)$ the group of all holomorphic automorphisms of D. By a theorem of H. Cartan ([1], see also [8]), $Aut(D)$ is a real Lie group. In [10], Webster gave the conditions on D, such that all automorphisms extend to the birational transformations of the ambient $\mathbb{C}^n$. Moreover, as shown in [13], the group $Aut(D)$ has finitely many components in this case. Such properties are also valid for the automorphisms of bounded homogeneous domains, if they are realized as Siegel domains ([5]). The graph of every birational transformation defines an n-dimensional compact cycle in $\mathbb{P}_{2n}$. Thus we obtain an embedding of $Aut(D)$ in the space C_n of n-dimensional cycles in $\mathbb{P}_{2n}$ (the Chow scheme). The space C_n is a countable collection of projective varieties parameterized by the degrees of the cycles. In fact, it is proven in [4], that the degree of possible cycles is bounded, which means that $Aut(D)$ lies in finitely many components of C_n (Theorem 3). The group operation of $Aut(D)$ extends rationally to the Zariski closure Z of it in C_n and endows Z with a structure of a *pre-group*, which is in general not a group. The action $Aut(D) \times D \to D$ extends also to a rational action $Z \times \mathbb{C}^n \to \mathbb{C}^n$. Again, this is a *pre-transformation space* which is not a transformation space in general.

The pre-groups and pre-transformation spaces can be obtained by passing from algebraic groups and their regular actions on algebraic varieties to birationally equivalent algebraic varieties. The mentioned results of Weil imply that in this way we obtain all possible pre-groups and pre-transformation spaces ([11], p. 375). In the case of connected pre-groups and homogeneous pre-transformation spaces, a similar proof is given in the book of Merzlyakov [6]. The regularizations of pre-

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transformation spaces were generalized by Rosenlicht to the case of non-connected algebraic groups ([9], page 404). Both papers [11, 9] and the book [6] utilize the "algebraic language of generic points" which is mainly developed in the classical book of Weil [12].

In view of the above applications of analytic nature, we propose here a more geometrical way to study pre-groups and pre-transformation spaces. This allows to reprove the above classical results without use of generic points and to generalize them to the case of several components (Theorems 3.4 and 4.9).

These results are used in [4] to establish the following type of linearization (see Theorems 1 and 2 there):

Let G be a real connected Lie group operating on a domain $D \subset \mathbb{C}^n$ by holomorphic automorphisms, which extend to birational transformations of the ambient $\mathbb{C}^n$. Then there exists a linear representation of G on some $\mathbb{C}^{N+1}$ together with a birational (onto its closed image) G-equivariant mapping from $\mathbb{C}^n$ into $\mathbb{P}_N$.

Furthermore, we investigate the points of pre-groups and pre-transformation spaces, where the above regularizations are biregular (Propositions 3.10 and 4.10). These are the regular points of certain rational mappings. This description allows to construct the above linearization of G, such that the embedding of D into $\mathbb{P}_N$ is biholomorphic.

Another application of Theorem 4.9 and Proposition 4.10 is the following type of a complexification:

Let G be a real algebraic group operating on a complex algebraic variety X by complex regular transformations. Then X is embedded as a Zariski open and dense subset in a (complex) algebraic variety Y such that the G action on X extends to a regular action of the complexification $G^{\mathbb{C}}$ on Y.

In fact, the action of G extends uniquely to a rational action of the complexification $G^{\mathbb{C}}$, which makes X to a pre-transformation $G^{\mathbb{C}}$-space. The space Y is obtained by Theorem 4.9 as a regularization of X and Proposition 4.10 implies that the birational mapping between X and Y is an embedding. In case G is a compact Lie group operating by holomorphic transformations on a Stein space X, such complexification was constructed by Heinzner [2] (see [3] for X holomorphically convex).

Finally we study the uniqueness of the regularizations. The regularization of a pre-group is unique up to isomorphisms. In case of pre-transformation spaces this is not true but can be achieved by the consideration of a restricted class of minimal in some sense regularizations (Theorem 4.12).

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2. Conventions.

All our objects will be defined over an algebraically closed ground field k. All open and closed sets are considered with respect to the Zariski topology. We follow the book of Mumford [7] in the main terminology. For the convenience of the reader we recall here the basic definitions. By saying that some property is satisfied for generic $(x_1, \ldots, x_n) \subset X_1 \times \cdots \times X_n$, where $X_1, \ldots, X_n$ are arbitrary algebraic varieties, we always mean that it is satisfied for all $(x_1, \ldots, x_n)$ in some open dense subset of $X_1 \times \cdots \times X_n$.

Let V, W be two algebraic varieties. A *rational mapping* $f: V \to W$ is an equivalence class of the pairs (U, ϕ) of an open dense subset $U \subset V$ and a morphism $\phi: U \to W$. Two such pairs are said to be equivalent if the morphisms coincide on the intersection of the corresponding sets. For every such class one can choose a canonical representative (U, ϕ) with the maximal possible subset $U \subset V$, such that, for every other representative (U', ϕ'), $U' \subset U$. This defines a one-to-one correspondence between the rational mappings and their canonical representatives. Such a representative will be also called a rational mapping and will be said to be *regular* on the set U and the latter will be referred as the regular set of f. A rational mapping is said to be *regular* at some point $v \in V$ if $v \in U$.

For arbitrary subsets $A \subset V$ and $B \subset W$ we define the *image* of A and the *preimage* of B by $f(A) := f(U \cap A)$ and $f^{-1}(B) := (f|U)^{-1}(B)$.

In order to have the notion of composition of two rational mappings, we need to introduce dominating mappings. We say that a rational mapping is *dominating* if the preimages of all open dense subsets are open dense. In special case of a morphism between two irreducible algebraic varieties this notion coincides with given in [7].

Let $f: V \to V'$ and $g: V' \to V''$ be two rational mappings. If f is dominating, the composition $g \circ f: V \to V''$ is defined by the representative $(f^{-1}(U') \cap U, g \circ f)$, where $U \subset V$ and $U' \subset V'$ are the regular sets of f and g respectively.

More generally, the representative $(f^{-1}(U') \cap U, g \circ f)$ defines a rational composition if and only if the preimage $f^{-1}(U')$ is dense in V. A rational mapping f is dominating if and only if the compositions $g \circ f$ exist as the rational mappings for all possible rational mappings $g: V' \to V''$.

By an *expression* we understand an arbitrary *formal* composition $f_n(f_{n-1}(\cdots f_2(f_1(a)) \cdots))$, where $f_i: V_{i-1} \to V_i, i = 1, \ldots, n$ are rational mappings and $a \in V_0$ is a point. Such expression is said to be *defined* if, f_1 is regular at a and, for all $i = 2, \ldots, n$, f_i is regular at $f_{i-1}(\cdots f_1(a) \cdots)$.

The *birational mapping* $f: V \to W$ is similarly defined as an equivalence class of the triples (U_1, U_2, ϕ) of open dense subsets $U_1 \subset V$, $U_2 \subset W$ and isomorphisms $\phi: U_1 \to U_2$. Again, there is a one-to-one correspondence between the birational mappings and their canonical representatives with the maximal subsets U_1 and U_2. Such a representative will be also called a birational mapping and will be said to be *biregular* on the set U_1. A birational mapping is said to be *biregular* at some point $v \in V$ if $v \in U_1$. The birational mappings are always dominating.

We should say also several words about restrictions. Let $S \subset V$ be a locally closed subvariety. The restriction to S of a rational mapping $f: V \to W$ may be in general nowhere defined. We say that the restriction $f|S$ is *rational* if the set of all points $v \in S$, where f is regular, i.e. $S \cap U$, is dense in S. In this case we call the rational mapping from S into W, defined by the restriction $f|S \cap U$, the *restriction* $f|S$. It may be in general regular on a larger subset of S than $S \cap U$.

If $f: V \to W$ is birational, the restriction $f|S$ is said to be *birational* if the set of all points $v \in S$, where f is biregular, i.e. $S \cap U_1$, is dense in S. In this case the restriction $f|S$ defines a birational mapping from S onto the image $f(S \cap U_1)$.

3. Algebraic pre-groups.

Similarly to the algebraic groups we call the pre-groups algebraic in order to emphasize their algebraic nature. As was pointed out above, the algebraic pre-groups are obtained from algebraic groups via birational mappings. The original definitions of them by Weil [11] make use of the language of generic points. Here we reformulate these definitions in the terminology of [7]. Our definitions coincide with given in [11] if all varieties are irreducible.

The standard group axioms usually involve the unit. Since we wish to allow birational transformations, the unit may "disappear". Therefore, we start with another set of group axioms, namely with the associativity condition and the existence and uniqueness of the left and right divisions.

Definition 3.1. An **algebraic pre-group** is an algebraic variety V with a rational mapping $V \times V \to V$, written as $(v, w) \mapsto vw$, such that:

1. for generic $(u, v, w) \in V \times V \times V$, both expressions $(uv)w$ and $u(vw)$ are defined and equal (generic associativity condition);
2. the mappings $(v, w) \mapsto (v, vw)$ and $(v, w) \mapsto (v, wv)$ from $V \times V$ into itself are birational (generic existence and uniqueness of left and right divisions).

Remarks.

1. Of course, an algebraic group is a special case of an algebraic pre-group.
2. As noted above, an algebraic pre-group may have no unit, e.g. $V = (\mathbb{C} \setminus \{0\}, +)$.
3. We write $(v, w) \mapsto (v, v^{-1}w)$ and $(v, w) \mapsto (v, wv^{-1})$ for the inverses of the mappings $(v, w) \mapsto (v, vw)$ and $(v, w) \mapsto (v, wv)$ respectively. These birational mappings are the analogons of the left and right divisions.
4. Condition 2) in Definition 3.1 implies that the pre-group operation $(v, w) \mapsto vw$ is dominating.

Warning. If wv^{-1} is defined, it does not mean that the "inverse" v^{-1} is defined. Since, in general, an algebraic pre-group has no unit, we cannot define the inverse elements directly.

The following Lemma due to Weil [11] introduces the inverse mapping $\phi: V \to V, v \mapsto v^{-1}$ as a birational mapping.

Lemma 3.2. *There exists a birational mapping $\phi: V \to V$ with following properties:*

1. $\phi \circ \phi = \mathrm{id}$,

2. $\phi(v)w = v^{-1}w$,

3. $w\phi(v) = wv^{-1}$,

4. $\phi(vw) = \phi(w)\phi(v)$,

where $(v, w) \in V \times V$ is generic.

Proof. We are looking for $\phi(v)$ in the form $w(vw)^{-1}$. This expression appears as a component of the mapping $f\colon (v, w) \mapsto (vw, w(vw)^{-1})$ from $V \times V$ into itself. The mapping f is birational as a composition of two birational mappings from $V \times V$ into itself: $(v, w) \mapsto (vw, w)$ and $(u, w) \mapsto (u, wu^{-1})$.

We first wish to prove that $\phi(v, w) := w(vw)^{-1}$ is independent of $w \in V$. For this we multiply both sides of the equality $\phi(v, w)(vw) = w$ by some $u \in V$ and use the associativity condition. Since, for generic (w, u), the right-hand side wu is defined, the left-hand side $(\phi(v, w)(vw))u$ is also defined and one has

$$(\phi(v, w)(vw))u = wu. \tag{1}$$

In order to apply the associativity condition to (1), we have to prove that the expression $(\phi(v, w)((vw)u)$ is also defined. For this consider the birational mapping

$$g\colon (v, w, u) \mapsto (f(v, w), u) = (vw, \phi(v, w), u)$$

from $V \times V \times V$ into itself. Then, for $h\colon (a, b, c) \mapsto b(ac)$, the composition $h \circ g(v, w, u) = \phi(v, w)((vw)u)$ is defined for generic (v, w, u) and one has the associativity:

$$\phi(v, w)((vw)u) = (\phi(v, w)(vw))u. \tag{2}$$

The relations (1) and (2) imply

$$\phi(v, w)((vw)u) = wu. \tag{3}$$

By the associativity, the expressions $(vw)u$ and $v(wu)$ are defined and equal for generic $(v, w, u) \in V \times V \times V$. Therefore, (3) implies

$$\phi(v, w)(v(wu)) = wu. \tag{4}$$

Since the mapping $(v, w, u) \mapsto (v, w, wu)$ is birational, one can write

$$\phi(v, w)(vt) = t \tag{5}$$

for generic (v, w, t). This implies

$$\phi(v, w) = t(vt)^{-1} = \phi(v, t),$$

which means that $\phi(v, w)$ is independent of $w \in V$. Thus we can write $\phi(v) := \phi(v, w)$.

The property 2) is now implied by $\phi(v)(vw) = w$ for generic (v, w). The latter relation means that the left multiplications by v and by $\phi(v)$ are inverse to each other. This implies the property 1) and therefore the birationality of ϕ.

The fact that the above left multiplications are inverse to each other can be also expressed as follows:

$$v(\phi(v)w) = w. \tag{6}$$

For proving property 3) we use similar calculations as for proving the independence of $\phi(v,w)$ of $w \in V$. Here we multiply (6) by $u \in V$ on the left:

$$u(v(\phi(v)w)) = uw. \tag{7}$$

By similar arguments, we see that, for generic (v,w,u), the associativity condition can be applied and we obtain

$$((uv)\phi(v))w = uw. \tag{8}$$

This can be reduced to

$$((uv)\phi(v)) = u. \tag{9}$$

This means that the right multiplication by $\phi(v)$ is inverse to that by v, which implies property 3).

Finally, property 4) is obtained by the following calculation:

$$(\phi(w)\phi(v))((vw)u) = \phi(w)(\phi(v)(v(wu))) = \phi(w)(wu) = u, \tag{10}$$

where all expressions are defined for generic (v,w,u). □

Remark. We write $\phi(v) = v^{-1}$. By properties 2) and 3), the expressions $v^{-1}w$ and wv^{-1} are then well-defined. If v^{-1} and these products are defined, they are also defined in the previous sense but the converse may not be true. When saying that such expressions are defined we understand them in the previous sense.

The morphisms in the category of algebraic pre-groups are defined as follows.

Definition 3.3. A **rational (resp. birational) homomorphism** between two algebraic pre-groups V and W is a rational (resp. birational) mapping $f\colon V \to W$ such that the expressions $f(uv)$ and $f(u)f(v)$ are defined and equal for generic $(u,v) \in V \times V$.

Remarks.

1. Since the mapping $(u,v) \mapsto uv$ is dominating, the expression $f(uv)$ is always defined for generic $(u,v) \in V \times V$. The expression $f(u)f(v)$, on the contrary, may be nowhere defined in general.

2. Of course, if $f\colon V \to W$ is birational, the second expression is also defined for generic $(u,v) \in V \times V$. In this case the inverse mapping $f^{-1}\colon W \to V$ is also a birational homomorphism.

3. If $f\colon V \to V', g\colon V' \to V''$ are two rational homomorphisms and f is dominating, than the composition $g \circ f\colon V \to V''$ is defined and is also a rational homomorphism.

For algebraic groups such morphisms coincide with the usual ones:

Proposition 3.4. *Let $f: V \to W$ be a rational homomorphism between two algebraic groups V and W. Then f is a (regular) homomorphism.*

Proof. Let $U \subset V$ be the open dense subset, where f is regular and $v \in V$ an arbitrary point. Since V is an algebraic group, one has $u := w^{-1}v \in U$ for generic $w \in U$. By the homomorphic property,

$$f(wu') = f(w)f(u')$$

for generic $(w, u') \in V \times V$. Since $w, u \in U$ and W is a group, the right-hand side is defined for $u' = u$. Then the left-hand side $f(wu) = f(v)$ is also defined. Since $v \in V$ is arbitrary, f is regular. $\square$

On the contrary to the algebraic group structures, the algebraic pre-group structures can be transformed via birational mappings, which follows directly from the definition:

Proposition 3.5. *For every birational mapping f from an algebraic pre-group V into an algebraic variety W there exists unique algebraic pre-group structure on W such that $f: V \to W$ is a birational homomorphism with respect to this structure.*

We call this algebraic pre-group structure *induced* by f. If an algebraic pre-group, obtained in such a way, is a group, it is called the regularization:

Definition 3.6. A **regularization** of an algebraic pre-group V is an algebraic group $\tilde{V}$ with a birational homomorphism $\phi: V \to \tilde{V}$.

One of the main results of [11] is the existence and uniqueness of such regularizations. We reprove and generalize this result for algebraic pre-groups with several irreducible components.

Theorem 3.7. *For every algebraic pre-group V there exists a regularization $\tilde{V}$ which is unique up to isomorphisms.*

The proof will be given in section 5.. Now we characterize the points of V, where the regularization homomorphism $\varphi: V \to \tilde{V}$ is biregular.

Definition 3.8. Let V be an algebraic pre-group. A point $v \in V$ is called a **point of regularity** if the following conditions are satisfied:

1. the mapping $w \mapsto wv$ from V into itself is birational;
2. the mapping $v' \mapsto uv'$ from V into itself is biregular at $v' = v$ for generic $u \in V$;

Lemma 3.9. *The set of points of regularity of an algebraic pre-group V is an open dense subset of V.*

Proof. By definition of an algebraic pre-group, the right multiplication $(v, w) \mapsto (v, wv)$ is birational. It is biregular on an open dense subset $U \subset V \times V$. Condition 1) is satisfied if the intersection $U \cap (\{v\} \times V)$ is dense in $\{v\} \times V$. This is valid for all v from the open dense set $V' := \bigcap_j \pi_1((V \times V_j) \cap U)$, where

V_j's are the irreducible components of V and $\pi_1: V \times V \to V$ is the projection on the first component.

The left multiplication $(u, v') \mapsto (u, uv')$ is also birational. Again, denote by $U \subset V \times V$ the biregular set of it. Condition 2) is satisfied, if the intersection $U \cap (V \times \{v\})$ is dense in $V \times \{v\}$. This is valid for all v from the open dense set $V'' := \bigcap_j \pi_2((V_j \times V) \cap U)$, where V_j's are as above and $\pi_2: V \times V \to V$ is the projection on the second component.

The set of points of regularity is now $V' \cap V''$ which is open dense as an intersection of two open dense subsets. □

Proposition 3.10. *Let V be an algebraic pre-group and $\phi: V \to \tilde{V}$ be its regularization. A point $v \in V$ is a point of regularity if and only if the mapping ϕ is biregular at v.*

Proof. Let ϕ be biregular at v. Then the mapping $w \mapsto wv$ can be decomposed as a composition of the following three birational mappings: $w \mapsto \phi(w) \mapsto \phi(w)\phi(v) \mapsto \phi^{-1}(\phi(w)\phi(v))$. Indeed, by definition of a birational homomorphism, $\phi(w)\phi(v') = \phi(wv')$, i.e. $\phi^{-1}(\phi(w)\phi(v')) = wv'$ for generic $(w, v') \in V \times V$. If $v' = v$, both sides depend birationally on w. Thus, for generic $w \in V$ they are defined and equal. This proves that the mapping $w \mapsto wv$ is birational.

For generic $(v', u) \in V \times V$, the second mapping $v' \mapsto uv'$ can be decomposed as follows: $v' \mapsto \phi(v') \mapsto \phi(u)\phi(v') \mapsto \phi^{-1}(\phi(u)\phi(v'))$. Here the first mapping is ϕ, which is biregular at $v' = v$ by the assumption. The second one is biregular as a right multiplication in the algebraic group W. Finally, the third mapping is biregular at $\phi(u)\phi(v)$ for generic $u \in V$. Thus the required mapping $v' \mapsto uv'$ is biregular at $v' = v$ for generic $u \in V$ as a composition of biregular mappings.

Suppose now that $v \in V$ is a point of regularity. By definition of a birational homomorphism, $\phi(wv') = \phi(w)\phi(v')$ for generic $(w, v') \in V \times V$. Since $\tilde{V}$ is an algebraic group, $\phi(v') = \phi(w)^{-1}\phi(wv')$ and the mapping $v' \mapsto \phi(v')$ can be decomposed as the composition of the following three birational mappings: $v' \mapsto wv' \mapsto \phi(wv') \mapsto \phi(w)^{-1}\phi(wv')$. Since v is a point of regularity, the first mapping is biregular at $v' = v$ if $w \in V$ is generic. Since the mapping $w \mapsto wv$ is birational, ϕ is biregular at wv for generic $w \in V$. Since $\tilde{V}$ is an algebraic group, the last mapping $\tilde{V} \to \tilde{V}, u \mapsto \phi(w)^{-1}u$ is biregular whenever $\phi(w)$ is defined. For generic $w \in V$, this is the case. Then ϕ is biregular at v as a composition of biregular mappings. □

The regularization satisfies also the following functorial property:

Proposition 3.11. *Let $f: V \to W$ be a rational homomorphism between two algebraic pre-groups V and W such that $f(v) \in W$ is a point of regularity for generic $v \in V$. Let $\phi_V: V \to \tilde{V}, \phi_W: W \to \tilde{W}$ be their regularizations. Then there exists unique homomorphism $\tilde{f}: \tilde{V} \to \tilde{W}$, such that the following diagram is commutative:*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle \phi_V} & & \downarrow{\scriptstyle \phi_W} \\ \tilde{V} & \xrightarrow{\tilde{f}} & \tilde{W} \end{array}$$

Remark. The condition that $f(v)$ is a point of regularity is essential. This is shown on the example of the inclusion $f: V \to W$, where $V = (\{0\}, +)$, $W = (\mathbb{C}, \times)$. Here $\tilde{V} = V$ and $\tilde{W} = \mathbb{C}^*$.

Proof. By Proposition 3.10, the mapping ϕ_W is biregular at $f(v) \in W$ for generic $v \in V$. Therefore, the composition $\tilde{f}(v') := \phi_W(f(\phi_V^{-1}(v')$ is defined for generic $v' \in \tilde{V}$. Moreover, it is a rational homomorphism as a composition of rational homomorphisms. By Proposition 3.4, $\tilde{f}: \tilde{V} \to \tilde{W}$ is a (regular) homomorphism. □

4. Algebraic pre-transformation spaces.

Together with the pre-groups, A. Weil ([11]) introduced also their birational "actions" on algebraic varieties, which he referred as pre-transformation spaces. Again, we reformulate this notion, where several irreducible components are also allowed.

The standard axioms for the group action involve the unit of the group. Since an algebraic pre-group may have no unit, we pass to another set of axioms, namely to the associativity condition and the existence and uniqueness of the "left divisions" by group elements.

Definition 4.1. Let V be an algebraic pre-group. An **algebraic pre-transformation V-space** is an algebraic variety X with a rational mapping $V \times X \to X$, written as $(v, x) \mapsto vx$, such that:

1. for generic $(v, w, x) \in V \times V \times X$, both expressions $(vw)x$ and $v(wx)$ are defined and equal (generic associativity condition);

2. the mapping $(v, x) \mapsto (v, vx)$ from $V \times X$ into itself is birational (generic existence and uniqueness of left divisions).

Remark. Condition 2) implies, in particular, that the operation $(v, x) \mapsto vx$ is dominating.

In Lemma 3.2 we introduced the inverse mapping $v \mapsto v^{-1}$. Thus, for generic $(v, x) \in V \times X$, the expression $v^{-1}x$ is defined.

Lemma 4.2. *The mapping $(v, x) \mapsto (v, v^{-1}x)$ from $V \times X$ into itself is the inverse of the operation $(v, x) \mapsto (v, vx)$.*

Proof. Both mappings are birational. Therefore, it is enough to prove that $v(v^{-1}x) = x$ for generic $(v, x) \in V \times X$. For generic $(w, v, x) \in V \times V \times X$, one has the following relations:

$$w(v(v^{-1}x)) = (wv)(v^{-1}x) = ((wv)v^{-1})x = wx. \tag{11}$$

Since, for generic $w \in V$, the mapping $x \mapsto wx$ from X into itself is birational, (11) implies the required relation $v(v^{-1}x) = x$. □

Lemma 4.3. *Let V be an algebraic pre-group and X an algebraic pre-transformation V-space. Let $v \in V$ be a point of regularity. Then the mappings $w \mapsto vw$ from V into itself and $x \mapsto vx$ from X into itself are birational.*

Proof. Since V is a pre-transformation V-space with respect to the left multiplication, it is sufficient to prove the birationality of the second mapping $f_v: x \mapsto vx$. By definition of a pre-transformation V-space, the mapping $f_u: X \to X, x \mapsto ux$, is birational for generic $u \in V$. Let $U \subset V$ be the open dense subset of such elements u. We express the required mapping f_v as a composition $f_{u^{-1}} \circ f_{uv}$. Since the mappings $u \mapsto u^{-1}$ and $u \mapsto uv$ are birational, one has $u^{-1} \in U, uv \in U$ for generic $u \in V$. By the choice of U, both $f_{u^{-1}}, f_{uv}: X \to X$ are birational for generic $u \in V$.

By the associativity condition,

$$(f_{u^{-1}} \circ f_{uv'})(x) = u^{-1}((uv')x) = (u^{-1}(uv'))x = v'x \tag{12}$$

for generic $(u, v', x) \in V \times V \times X$. If $(u, x) \in V \times X$ is generic, the left-hand side is defined for $v' = v$. Then the right-hand side $vx = f_v(x)$ is also defined and $f_v = f_{u^{-1}} \circ f_{uv}$. This proves that f_v is birational. □

Definition 4.4. Let V be an algebraic pre-group and X and Y two algebraic pre-transformation V-spaces. A rational (resp. birational) mapping $f: X \to Y$ is called V-**equivariant** if the expressions $f(vx)$ and $vf(x)$ are defined and equal for generic $(v, x) \in V \times X$.

Remarks.

1. Since the mapping $(v, x) \mapsto vx$ is dominating, the expression $f(vx)$ is defined for generic $(v, x) \in V \times X$. The expression $f(u)f(v)$, on the contrary, may be nowhere defined in general.

2. Of course, if $f: X \to Y$ is birational, the second expression is also defined for generic $(v, x) \in V \times X$. In this case the inverse mapping $f^{-1}: Y \to X$ is also V-equivariant.

3. If $f: X \to X', g: X' \to X''$ are two V-equivariant rational mappings and f is dominating, than the composition $g \circ f: X \to X''$ is defined and is also V-equivariant.

Similarly to algebraic pre-groups, the structure of algebraic pre-transformation spaces can be transformed via rational mappings:

Proposition 4.5. *1. For every birational mapping f from an algebraic pre-transformation V-space X into an algebraic variety Y there exists unique algebraic pre-transformation V-space structure on Y, such that $f: X \to Y$ is V-equivariant with respect to this structure.*

2. For every rational homomorphism ϕ between two algebraic pre-groups V and W and an algebraic pre-transformation W-space there exists unique algebraic pre-transformation V-space structure on X, such that $\phi(v)x = vx$ for generic $(v, x) \in V \times X$.

This follows directly from definitions. We call such algebraic pre-transformation V-space structures **induced** by f and ϕ respectively. For instance, the regularization $\phi: V \to \tilde{V}$ induces an algebraic pre-transformation $\tilde{V}$-space structure.

Similarly to algebraic pre-groups, we would like to introduce regularizations of pre-transformation V-spaces X. However, if we wish the uniqueness, it is not sufficient to require the induced operation $V \times Y \to Y$ to be regular. For example, let $V = \mathbb{C}^*$ and $X = \mathbb{C} \times \mathbb{C}^*$, where V acts by multiplication, which is already regular. The inclusion into $Y_1 = \mathbb{C}^2$ and restriction to $Y_2 = (\mathbb{C}^*)^2$ induce other regular actions.

Therefore, we call regularizations only some special regular actions. They should not "loose" the "good" points as in the above example with Y_2. These "good" points will be called the points of regularity. The definition of them is analogous to the second condition in the corresponding definition for algebraic pre-groups (see Definition 3.8).

Definition 4.6. Let V be an algebraic pre-group and X an algebraic pre-transformation V-space. A point $x \in X$ is called a **point of regularity** if the mapping $x' \mapsto ux'$ from X into itself is biregular at $x' = x$ for generic $u \in V$.

Remark. If $\phi: V \to W$ is a birational homomorphism into another algebraic pre-group W, the sets of regularity of X with respect to the actions of V and W coincide.

Similarly to Lemma 3.9, one obtains the following

Lemma 4.7. *The set of points of regularity of a pre-transformation V-space X is an open dense subset of X.*

Definition 4.8. Let V be an algebraic pre-group and X an algebraic pre-transformation V-space. **A regularization** of X is an algebraic pre-transformation V-space Y together with a V-equivariant birational mapping $\psi: X \to Y$, such that

1. if $v \in V$ is a point of regularity, vy is defined for all $y \in Y$;

2. if $x \in X$ is a point of regularity, ψ is biregular at x.

Remark. If V is an algebraic group, it follows from condition 1) that the action $V \times Y \to Y$ on the regularization is regular.

The following is a generalization of a result of A. Weil ([11]).

Theorem 4.9.
For every algebraic pre-group V and algebraic pre-transformation V-space X, there exists a regularization $\psi: X \to \tilde{X}$.

The proof will be given in section 5.. The set of regularity is exactly the set of points where a regularization ψ is biregular:

Proposition 4.10. *Let $\psi: X \to \tilde{X}$ be a regularization which is biregular at some point $x \in X$. Then x is a point of regularity.*

Proof. For generic $(v, x') \in V \times X$, the mapping $x' \mapsto vx'$ can be decomposed as follows: $x' \mapsto \psi(x') \mapsto v\psi(x') \mapsto \psi^{-1}(v\psi(x'))$. The first mapping ψ is biregular at $x' = x$ by the assumption. The second one is biregular for generic $v \in V$, by condition 1) of the definition of a regularization. Finally, the third mapping ψ^{-1} is biregular at $v\psi(x)$ for generic $v \in V$. Thus the required mapping $x' \mapsto vx'$ is biregular at $x' = x$ for generic $v \in V$, as a composition of biregular mappings. □

Unfortunately, such regularizations are still not unique as the above example with Y_1 shows. To obtain the uniqueness we introduce some special regularizations which are in some sense minimal.

Definition 4.11. A regularization $\psi: X \to \tilde{X}$ of an algebraic pre-transformation V-space is called **minimal** if there are no proper open subsets of $\tilde{X}$ which are also regularizations of X with respect to the induced algebraic pre-transformation V-space structure.

Theorem 4.12. *For every algebraic pre-group V and algebraic pre-transformation V-space X there exists a minimal regularization $\psi: X \to \tilde{X}$ which is unique up to isomorphisms.*

Proof. Let $\phi: V \to \tilde{V}$ be the regularization of V and $\psi: X \to \tilde{X}'$ be an arbitrary regularization of X as a $\tilde{V}$-space. Further, let $U \subset X$ be the set of regularity, where ψ is biregular. Then the image $\psi(U)$ is open dense in $\tilde{X}'$ and the union

$$\tilde{X} := \tilde{V}\psi(U) = \bigcup_{v \in \tilde{V}} (v\psi(U))$$

is also open dense. Since $\tilde{V}$ is an algebraic group, $\tilde{X}$ is also a $\tilde{V}$-regularization and therefore a V-regularization of X.

We claim that $\tilde{X}$ is a minimal V-regularization. Assume that there exists a smaller one $X' \subset \tilde{X}$. By definition, it contains the image $\psi(U)$ of the set of regularity U. Let $W \subset V$ be the set of all points of regularity. By condition 1) of Definition 4.8, $W\psi(U) \subset X'$ and

$$W^2\psi(U) := \{w_1\phi(w_2) \mid w_1, w_2 \in W\} \subset X'.$$

This is equivalent to

$$\phi(W)^2\psi(U) \subset X'.$$

Since $\phi(W)$ is open dense in the algebraic group $\tilde{V}$, one has

$$\phi(W)^2 = \tilde{V}.$$

This implies $\tilde{X} = \tilde{V}\psi(U) = \phi(W)^2\psi(U) \subset X'$. Thus $\tilde{X} = X'$ and the minimality is proven.

It remains to prove the uniqueness. Let $\psi_i: X \to \tilde{X}_i$, $i = 1, 2$ be two minimal regularizations. Then $\psi := \psi_2 \circ \psi_1^{-1}$ is a birational mapping between $\tilde{X}_1$ and $\tilde{X}_2$. Since both ψ_1 and ψ_2 are biregular at U, ψ is biregular at $\psi_1(U)$. We have seen that $\phi(W)^2\psi_i(U)$, $i=1,2$ are also regularizations which coincide with $\tilde{X}_i$'s because of minimality. Since both operations on $\tilde{X}_1$ and $\tilde{X}_2$ are regular, the mapping ψ extends to a biregular mapping from $\phi(W)^2\psi_1(U)=\tilde{X}_1$ onto $\phi(W)^2\psi_2(U)=\tilde{X}_2$ by setting

$$\psi(\phi(w_1)(\phi(w_2)x))=\phi(w_1)(\phi(w_2)\psi(x)).$$

Thus $\tilde{X}_1$ and $\tilde{X}_2$ are isomorphic. □

We have proven in fact the following

Corollary 4.1. *Let $\psi: X \to \tilde{X}$ be the minimal regularization. If $U \subset X$, $W \subset V$ are the sets of regularity, one has $\phi(W)^2\psi(U)=\tilde{X}$. If V is an algebraic group, $V\psi(U)=\tilde{X}$.*

5. Construction of regularizations.

Our goal here is to prove Theorems 3.7 and 4.9. We first reduce the general algebraic pre-groups and pre-transformation spaces to the case where all point of them are points of regularity.

Let V be an algebraic pre-group and X an algebraic pre-transformation V-space. Let $V' \subset V$ and $X' \subset X$ denote the sets of points of regularity. By Lemmas 3.9 and 4.7, these sets are open dense. Then V' and X' can be considered as an algebraic pre-group and an algebraic V'-space with respect to the induced structures.

Proposition 5.1. *All points of V' and X' are points of regularity.*

Remark. The statement is essentially that the concept of points of regularity is invariant under passing to open dense subsets.

Proof. Let x be a point from X', i.e. a point of regularity of X. This means that the mapping $x' \mapsto ux'$ from X into itself is biregular at $x'=x$ for generic $u \in V$. We have to prove that it is also biregular there as mapping from X' into itself. For this it is sufficient to show that $ux \in X'$, i.e. $y := ux$ is again a point of regularity for generic $u \in V$.

This means, by definition, that the mapping $y' \mapsto vy'$ from X into itself is biregular at $y'=y$ for generic $v \in V$. For generic $(u,v,y') \in V \times V \times X$, this mapping can be decomposed as

$$y' \mapsto u^{-1}y' \mapsto (vu)(u^{-1}y'), \tag{13}$$

which is implied by generic associativity. Since x is a point of regularity, the first mapping is biregular at $y'=y$ for generic $u \in V$. By the same reason, the second mapping $x' \mapsto (vu)x'$ is biregular at $x'=x$. Then the composition $y' \mapsto vy'$ is biregular at $y'=y$.

This proves the statement for X'. We can apply it to the case $X=V$ where V "acts" by the left multiplications. Then condition 2) in Definition 3.8 is satisfied for all $v \in V$. To prove condition 1) we decompose the mapping $f': w \mapsto wv$ from V' into itself as $i \circ f \circ i^{-1}$, where f is the same mapping from V into itself and

$i: V' \to V$ denotes the birational inclusion. Then f' is birational as a composition of three birational mappings. Thus condition 1) is also satisfied. $\square$

Proposition 5.1 reduces the general case to the case where all points of V and X are points of regularity. Thus we take this case for granted in the following Proposition and Lemmas.

Proposition 5.2. *There exists an algebraic pre-transformation V-space $\tilde{X}$ with a V-equivariant open dense embedding $\psi: X \to \tilde{X}$, such that*

1. *the induced operation $V \times \tilde{X} \to \tilde{X}$ is regular;*
2. *$V\psi(X) := \{v\psi(x) \mid v \in V, x \in X\} = \tilde{X}$.*

For the proof we need several lemmas. Since every $v \in V$ is a point of regularity, the mapping $f_v: X \to X$ is birational by Lemma 4.3.

Lemma 5.3. *Let $v \in V$ be arbitrary and $\Gamma \subset X \times X$ be the closed graph of the rational mapping $f_v: x \mapsto vx$. Let $(a, b) \in \Gamma$ be an arbitrary point. Then f_v is biregular at a and $va = b$.*

Remark. Lemma 5.3 implies that, if X is complete, the action of V on X is already regular.

Proof. By definition of points of regularity, the mapping $w \mapsto wv$ is birational. By Lemma 4.3, both mappings $V \to V, w \mapsto vw$, and $X \to X, x \mapsto vx$, are also birational. Then the mappings $(t, x) \mapsto (tv, x)$ and $(t, x) \mapsto (t, vx)$ are birational and we can compose them with the action. Together with associativity, this implies that the expressions $(tv)x$ and $t(vx)$ are defined and equal for generic $(t, x) \in V \times X$.

By condition 2) in the definition of points of regularity, the expressions ta and tb are defined for generic $t \in V$. Since the mapping $t \mapsto tv$ is birational, the expression $(tv)a$ is also defined for generic $t \in V$.

The relation $t(vx) = (tv)x$ can be written as $ty = (tv)x$, where $y = vx$. It is valid for generic $(t, x) \in V \times X$ or, equivalently, for generic $(t, x, y) \in V \times \Gamma$. Both sides are defined at $(x, y) = (a, b) \in \Gamma$ as rational mappings on Γ. Since they are equal for generic $(t, x, y) \in V \times \Gamma$, they are also equal for $(x, y) = (a, b)$: $tb = (tv)a$.

Since, by condition 2) of the definition of points of regularity, $t^{-1}(tb)$ is defined and equal to b for generic $t \in V$, one has

$$b = t^{-1}((tv)a). \tag{14}$$

Consider the mapping $(t, v', a') \mapsto t^{-1}((tv')a')$. It follows by associativity, that it is equal to $(t, v', a') \mapsto v'a'$ for generic $(t, v', a') \in V \times V \times X$. But we know by (14), that $t^{-1}((tv')a')$ is defined for $v' = v$ and $a' = a$ and is equal to b there. Then the equal rational mapping $(t, v', a') \mapsto v'a'$ is also defined for $v' = v$ and $a' = a$ and is equal to b there. Thus f_v is defined at a and $va = b$.

To prove that f_v is biregular at a we apply the same considerations to the relation

$$a = (tv)^{-1}(tb) \tag{15}$$

which is valid for generic $t \in V$, because of relation $(tv)a = tb$, condition 2) of the definition of points of regularity applied to a and the fact, that the mapping $t \mapsto tv$ is birational. The mapping $(t, v', b') \mapsto (tv')^{-1}(tb')$ is equal to $(t, v', b') \mapsto (v')^{-1}b'$. They are defined for $v = v'$, $b = b'$ and equal to a there. Thus f_v is biregular at a. □

Lemma 5.4. *Let A, B be algebraic varieties and $U \subset A \times B$ an open subset with non-empty fibres $U_a := \{b \in B \mid (a, b) \in U\} \neq \emptyset, a \in A$. Then there exists a finite collection of points $b_i \in B, i \in I$, such that, for every $a \in A$, one has $b_i \in U_a$ for some $i \in I$.*

Proof. Without loss of generality, A is irreducible. We prove the statement by induction on dimension of A. If $\dim A = 0$, the statement is obvious. Assume it to be proven for $\dim A < d$. Let $(a_0, b_0) \in U$ be an arbitrary point. The set A' of points $a \in A$, such that $b_0 \in U_a$, can be expressed as

$$A' = \{a \in A \mid (a, b_0) \in U\}.$$

This is an open subset of A. Since A is assumed to be irreducible, $\dim(A \setminus A') < \dim A = d$.

Now we use induction for all irreducible components of the complement $A \setminus A'$. This yields a number of points $b_i, i \in I$. These points together with b_0 satisfy the required condition. □

Definition 5.5. Let $t_i \in V, i \in I$ be any finite collection of points. Let $t_{ij}: X \to X, i, j \in I$ denote the composition of two birational mappings $z \mapsto t_j z$ and $z' \mapsto t_i^{-1} z'$. **A trace** of X, generated by these points, is the quotient $\tilde{X}(\{t_i\}) := (X \cup (X \times I))/\sim$, where $x \sim (y, j)$ (resp. $(x, i) \sim (y, j)$) if the mapping $z \mapsto t_j z$ (resp. $z \mapsto t_{ij} z$) from X into itself is biregular at $z = y$ and $x = t_j y$ (resp. $x = t_{ij} y$).

Warning. If the mapping t_{ij} is biregular at y, the mappings $z \mapsto t_j z$, $z' \mapsto t_i^{-1} z'$ may not be in general biregular at corresponding points.

Lemma 5.6. *The relation "$\sim$" is an equivalence relation.*

Proof. The symmetrical property has to be checked only for (x, i) and (y, j), where $x, y \in X, i, j \in I$. In this case it follows from the fact that the mappings t_{ij} and t_{ji} are inverse to each other as birational mappings.

The transitivity has to be checked for the four different cases: $x \sim (y, j) \sim z$, $x \sim (y, j) \sim (z, k)$, $(x, i) \sim y \sim (z, k)$, $(x, i) \sim (y, j) \sim (z, k)$. Consider birational mappings $t_{ij}, t_i: z \mapsto t_i z$. The transitivity follows from the relations $t_j t_j^{-1} = \mathrm{id}$, $t_j t_{jk} = t_k$, $t_i^{-1} t_k = t_{ik}$, $t_{ij} t_{jk} = t_{ik}$ respectively. □

Lemma 5.7. *Let V and X satisfy the conditions of Proposition 5.2. Then the trace $\tilde{X}(\{t_i\})$ is an algebraic variety. The restriction $\psi: X \to X(\{t_i\})$ of the canonical projection $X \cup (X \times I) \to \tilde{X}(\{t_i\})$ is an open dense embedding.*

Proof. Since the relation "$\sim$" is an equivalence relation, the set $Y := \tilde{X}(\{t_i\})$ makes sense and satisfies the definition of an algebraic prevariety (see [7]). It is a variety, if the hausdorff property of the closedness of the diagonal $\Delta(Y) \subset Y \times Y$ is satisfied. We identify the sets X and $X \times \{i\}$ with their images under the canonical projection $X \cup (X \times I) \to \tilde{X}(\{t_i\})$. We have the following decomposition

$$\begin{gathered} Y \times Y = (X \times X) \cup (\bigcup_i ((X \times \{i\}) \times X)) \cup \\ (\bigcup_j (X \times (X \times \{j\}))) \cup (\bigcup_{i,j} ((X \times \{i\}) \times (X \times \{j\}))). \end{gathered} \tag{16}$$

The intersection of the diagonal $\Delta(Y)$ with the sets $X \times X$, $(X \times \{i\}) \times X$, $X \times (X \times \{j\})$ and $(X \times \{i\}) \times (X \times \{j\})$ is equal to the graphs of the mappings id, $x \mapsto t_i x$, $x \mapsto t_j^{-1} x$, $x \mapsto t_{ji}$ respectively. By Lemma 5.3, these graphs are closed. Thus the diagonal $\Delta(Y)$ is closed. This proves that the trace Y is a variety.

The variety X, considered as a subset of Y, intersects every $X \times \{i\}$ along some open dense subsets $U_i \subset X \times \{i\}$ where the mappings $z \mapsto t_i z$ are biregular. This implies that the closure of X contains every $X \times \{i\}$ and therefore coincides with Y. Thus X is open dense embedded into Y. □

Proof of Proposition 5.2. We are looking for the required pre-transformation space $\tilde{X}$ among the traces $\tilde{X}(\{t_i\})$. We prove that the induced operation $V \times \tilde{X} \to \tilde{X}$ is regular for an appropriate choice of the points $\{t_j\}_{j \in I}$.

The regularity for such traces means that tx is always defined for $t \in V, x \in \tilde{X}(\{t_i\})$. We fix an element of $X \cup (X \times I)$ which represents the class of x in the quotient $\tilde{X}(\{t_i\})$ and denote it also by $x \in X$ or by $(x, i) \in X \times I$.

We wish to find some $j \in I$ such that the induced operation $V \times X \to X \times \{j\}$ (resp. $V \times (X \times \{i\}) \to X \times \{j\}$) is defined at (t, x) (resp. at (t, x, i)). The induced operation can be obtained by passing to equivalent elements in X, applying the given operation there and passing to the equivalent elements in the required component $V \times j$. It is given by

$$(t, x) \mapsto tx \sim (t_j^{-1}(tx), j) \tag{17}$$

in the first case and by

$$(t, x, i) \sim (t, t_i x) \mapsto t(t_i x) \sim (t_j^{-1}(t(t_i x)), j) \tag{18}$$

in the second.

The regularity follows now from the following statement:

Statement. There exists a collection of points $t_i \in V, i \in I$, such that, for all $t, t' \in V, x \in X$, each of the expressions $((t_j^{-1} t) t') x$ and $(t_j^{-1} t) x$ is defined at least for one choice of $j \in I$.

Proof of the Statement. Since, by Lemma 4.3, the mappings $v \mapsto vx$ from V into X are rational, they are defined on open dense subsets $U_x \subset V, x \in X$. Then the family $U \subset X \times V$ of these subsets is open dense as a set of definition of the rational mapping $(x, v) \mapsto vx$ from $X \times V$ into X.

By the definition of points of regularity, the mappings $v \mapsto vt'$ from V into itself are birational and therefore defined on some open dense subsets. Moreover, one has $vt' \in U_x$ for all v from open dense subsets $U'_{t',x} \subset V, t' \in V, x \in X$. The family $U' \subset V \times X \times V$ of these subsets is open dense as the preimage of the family U under the rational mapping $(t', x, v) \mapsto (x, vt')$ from $V \times X \times V$ into $X \times V$.

The mappings $v \mapsto v^{-1}t$ from V into itself are also birational and therefore defined on some open dense subsets. Moreover, one has $v^{-1}t \in U'_{x,t'}$ for all v from open dense subsets $U''_{t,t',x} \subset V, t, t' \in V, x \in X$. The family $U'' \subset V \times V \times X \times V$ of these subsets is open dense as the preimage of the family U' under the rational mapping $(t, t', x, v) \mapsto (t, t', x, v^{-1}t)$ from $V \times V \times X \times V$ into $V \times X \times V$.

We constructed an open dense family $U'' \subset V \times V \times X \times V$, such that $U''_{t,t',x} \subset V$ is open dense and $((v^{-1}t)t')x$ is defined for all $t, t' \in V, x \in X$. Now we set in Lemma 5.4 $A := V \times V \times X$, $B := V$, $U := U''$. Then the conditions of Lemma 5.4 are satisfied and we obtain a finite collection of points $t_j = b_j \in V, i \in I$ such that, for all $t, t' \in V, x \in X$, the expression $((t_j^{-1}t)t')x$ is defined at least for one $j \in I$.

By the same considerations we prove that, for all $t \in V, x \in X$, the expression $(t_j^{-1}t)x$ is also defined at least for one $j \in I$. This finishes the proof of the above statement.

By the construction of the trace $\tilde{X}(\{t_i\})$, we have the property $t_iX = X \times \{i\}$ for all $i \in I$. This implies the required property $VX = \tilde{X}$. □

Proof of Theorem 3.7. Let V be an algebraic pre-group and V' the open dense subset of all points of regularity. Then V' is an algebraic pre-transformation V'-space, which satisfies conditions of Proposition 5.2 by Proposition 5.1. Proposition 5.2 yields an open dense V'-equivariant embedding $\psi: V' \to \tilde{V}$.

We wish to prove that $\tilde{V}$ is the required regularization. For this we need to show that the operation $\tilde{V} \times \tilde{V} \to \tilde{V}, (v, w) \mapsto vw$ and the inverse mapping $\tilde{V} \to \tilde{V}, v \mapsto v^{-1}$ are regular.

The operation is regular for $v \in \psi(V')$ because of property 1) in Proposition 5.2. In general case, by property 2), every $v \in \tilde{V}$ is a product of $v_1, v_2 \in \psi(V')$. By the generic associativity,

$$(v_1'v_2')w' = v_1'(v_2'w')$$

for generic (v_1', v_2', w'). Since $v_1, v_2 \in \psi(V')$, the right-hand side is regular at $(v_1', v_2', w') = (v_1, v_2, w)$. Then the left-hand side is also regular there, which means that vw is defined.

To prove the regularity of the inverse mapping we establish the birationality of the mapping $f_v: w \mapsto wv$ from V' into $\tilde{V}$. Since V' consists of points of regularity, f_v is birational for all $v \in \psi(V')$. Again, in general case, $v = v_1v_2$, $v_1, v_2 \in \psi(V')$ and, by the generic associativity,

$$w(v_1'v_2') = (wv_1')v_2'.$$

The right-hand side is equal to $(f_{v_2'} \circ f_{v_1'})(w)$, which is a birational mapping in the variable w for $(v_1', v_2') = (v_1, v_2)$. Thus the left-hand side $f_v(w)$ is also birational.

By property 3) in Lemma 3.2, $(wv)v^{-1} = w$, i.e. $v^{-1} = (wv)^{-1}w$ for generic $(v, w) \in \psi(V')$. We wish to prove that v^{-1} is defined for all $v \in \tilde{V}$. Since the operation $V' \times \tilde{V} \to \tilde{V}$ is regular, it is enough to show that $(wv)^{-1}$ is defined for generic $w \in \psi(V')$. We have just proven that the mapping $f_v: w \mapsto wv$ from V' into $\tilde{V}$ is birational. Then the required mapping $w \mapsto (wv)^{-1}$ is the composition of f_v and the inverse $u \mapsto u^{-1}$, which are birational. Thus the inverse mapping is also regular on $\tilde{V}$. □

Proof of Theorem 4.9. Let V be an algebraic pre-group and X an algebraic pre-transformation V-space. Let $V' \subset V$ and $X' \subset X$ be the open dense subsets of regularity. By Proposition 5.1, V' and X' consists of points of regularity. Therefore we can apply Proposition 5.2 and obtain an open dense V'-equivariant embedding $\psi : X \to \tilde{X}$. The conclusions of Proposition 5.2 imply that this is the required regularization. $\square$

References

[1] H. Cartan., "Sur les groupes de transformations analytiques", Act. Sc. et Int., Hermann, Paris, 1935.

[2] P. Heinzner., *Geometric invariant theory on stein spaces*, *Math. Ann.*, **289**:631–662, 1991.

[3] P. Heinzner and A. Ianuzzi., *Integration of local actions on holomorphic fiber spaces*, Preprint Bochum, 1995.

[4] A. T. Huckleberry and D. Zaitsev, *Actions of the groups of birational automorphisms*, to appear in Proc. Geom. Complex Analysis, 1995.

[5] S. Kaneyuki, *On the automorphism groups of homogeneous bounded domains*, J. Fac. Sci. Univ. Tokyo, **14**:89–130, 1967.

[6] Y. Merzlyakov, "Rational groups", Moscow, Nauka, 1987.

[7] D. Mumford, "The red book of varieties and schemes", Lect. Notes in Math., **1358**, Springer, 1980.

[8] R. Narasimhan, "Several complex variables", Chicago Lectures in Mathematics. Univ. of Chicago Press, 1971.

[9] M. Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math., **78**:401–443, 1956.

[10] S. Webster, *On the mapping problem for algebraic real hypersurfaces*, Inventiones math., **43**:53–68, 1977.

[11] A. Weil, *On algebraic groups of transformations*, Amer. J. of Math., **77**:355–391, 1955.

[12] A. Weil, "Foundation of algebraic geometry", Amer. Math. Soc., 1962.

[13] D. Zaitsev, *On the automorphism groups of algebraic bounded domains*, Math. Ann., **302**:105–129, 1995.

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