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Determinant Functions and the Geometry of the Flag Manifold for SU(p,q)

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Abstract. Let $G_0 = \mathrm{SU(p,q)}$ with $q \leq p$, $K_0 = \mathrm{S(U(p)\times U(q))}$ a maximal compact subgroup, and let G, K be their complexifications. Finally, let B be a Borel subgroup of G. We define a number of algebraic functions on $G/B \times G/K$ and use them to describe the closures of codimension one K orbits of the flag manifold G/B. We show how the underlying geometry of the flag manifold interacts with these functions. In particular, we shall use these functions to construct a Stein extension of the Riemannian symmetric space G_0/K_0, whose connected component turns out to be the space of linear cycles in most cases.

1. Introduction

In this paper we define a number of functions called the determinant functions and explain their relationship with the underlying geometry of the flag manifold. We will concentrate on the special case of the Lie group $G_0 = \mathrm{SU(p,q)}$ with $q \leq p$. We let $K_0 = \mathrm{S(U(p) \times U(q))}$, a maximal compact subgroup, and $G = \mathrm{SL(p+q, \mathbb{C}), K}$ be the complexifications. The $q+1$ determinant functions $D_j(x, z)$ are defined on the product $X \times Z$, where X is the flag manifold associated to G and Z the complex symmetric space G/K. Closely related to these $q+1$ determinant functions are the zero sets $D_j(\ \ , z) = 0$ for each fixed $z \in Z$. We call these sets $\mathfrak{D}_j(z)$ the determinant varieties. These determinant varieties contain very interesting geometric information.

We would like to indicate briefly why the determinant functions and varieties are interesting. First, consider the set $\mathfrak{D}(z) = \cup_k \mathfrak{D}_k(z)$, called a configuration. It turns out to be the complement of the unique open orbit of the stabilizer G_z of the point z of the complex symmetric space G/K. Second, the determinant functions are special functions associated to the flag variety. To give an idea why they are important, we wold like to mention that we can write down Szegö kernels for generalized flag manifolds. In principle, the more "degenerate" a generalized flag manifold is, the less is the number of factors involved in the singularities of

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the kernels. As limiting cases, the Grassmanians $G(p, p+q), G(q, p+q)$ involve only two factors. They are exactly D_0, D_q. The remaining $q-1$ functions are genuienely new. Third, the determinant functions pin down all the codimensional one orbits of G_z. Indeed, we expect a suitable G_z invariant stratification of $\mathfrak{O}(z)$ would yield various higher codimension G_z orbits. Fourth, we know that G_0 admits an Iwasawa decomposition $K_0A_0N_0$. When we complexify the decomposition, it is no longer true that $G = KAN$, but the equations $D_k = 0$ encode the obstruction. Finally, as shown in [2] and alluded to just now, the functions D_k capture the singularities when we try to extend meromorphically the Szegö kernels.

From a different direction, we can understand the geometric information contained in the determinant functions as follows. From Matsuki's work ([4]) we know that it is important to study the interaction between K orbits and G_0 orbits of X. We have now a small variation of the same theme. We have two families of subgroups. On one hand, we have the conjugacy class (in G) of K, and correspondingly the configurations $\mathfrak{O}(z)$. On the other hand, we have the conjugacy class of G_0, and correspondingly the G translates of X_0, the unique closed G_0 orbit on X. These two classes of objects interact as follows. For a set $C \subseteq Z$, call its polar $\widehat{C}$ to be $\{x \in X | (\forall z \in C) x \in \mathfrak{O}(z)\}$. Likewise, for $A \subseteq X$, the polar $\widehat{A}$ is $\{z \in Z | (\forall x \in A) x \in \mathfrak{O}(z)\}$. It turns out that X_0 is its double polar, i.e., $\widehat{\widehat{X_0}} = X_0$.

From a third direction, we can appreciate the importance of the determinant functions as follows. For various reasons people are interested in G_0 invariant Stein extensions of G_0/K_0. Wolf and Zierau have found, in explicit terms, the Stein extension as the space of linear cycles. We will show that the connected component of $\widehat{X_0}$ is the space of linear cycles. This suggests another, though equivalent, manner of obtaining the Stein extension.

In [2] we explain the origin of the determinant functions as the singularties of Szegö kernels. An immediate consequence of that paper is that the solutions of Schmid equations on the Riemmanian symmetric space G_0/K_0 extend holormophically to the space of linear cycles.

Here is a brief summary of the paper. In Section 2 we define the $q+1$ determinant functions in order to describe the $q+1$ codimensional one K orbits of the flag manifold in Sections 3 and 4. In Section 5 we will explain how to find a Stein extension for the Riemannian symmetric space G_0/K_0 using the determinant functions. We relate this extension to the space of linear cycles, first studied by R.O. Wells and J. Wolf, later also by R. Zierau (see the citations in Section 5), in Section 6. In the same section, we also prove that $X_0 = \widehat{\widehat{X_0}}$. Section 7 is the appendix, in which we explain how to obtain some of our results by Lie-algebraic method, and this should have some bearing on [7, Problem 9, p.740].

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2. Determinant Functions

In [2], a number of determinant functions are defined. We would like to repeat the definition here for the reader's convenience.

The space X denotes the set of all sequences $(V_i; 1 \le i \le n)$ of subspaces of $\mathbb{C}^n$, where $\dim V_i = i$ and $V_i \subseteq V_j$ if $i \le j$. We can represent a point of X by $\omega = (\omega_i, 1 \le i \le p+q)$, where $\omega_i \in \Lambda^i \mathbb{C}^{p+q}$. Strictly speaking, we should look at the $n-$tuple $(\widetilde{\omega_i}) \in \Pi_{i=1}^n \mathbb{P}(\Lambda^i \mathbb{C}^n)$, where $\widetilde{\omega_i}$ is the projective point in $\mathbb{P}(\Lambda^i \mathbb{C}^n)$ represented by ω_i. The space Z consists of pairs (L_p, L'_q) of disjoint subspaces of dimensions p and q. We can likewise think of L_p, L'_q as two elements in $\Lambda^\bullet \mathbb{C}^n$. Sometimes we use an ordered basis $(v_i, 1 \le i \le n)$ to represent a flag, by putting $\omega_i = v_1 \wedge \ldots \wedge v_i$.

For the sake of simplicity, we find the following notation $\doteq$ useful. It means that the two sides are equal up to a non-zero scalar multiple.

We introduce two more pieces of notations, for convenience's sake. On $\Lambda^\bullet \mathbb{C}^{p+q}$ we have a "star" operator. Suppose we have an ordered basis $(v_i, 1 \le i \le p+q)$ for $\mathbb{C}^{p+q}$, then $*v_{k_1} \wedge \ldots \wedge v_{k_i} := \epsilon v_{k_{i+1}} \wedge \ldots \wedge v_{k_{p+q}}$. Here ϵ is $+1$ or -1 according to whether $(k_1, \ldots, k_{p+q})$ is an even or odd permutation of $(1, \ldots, p+q)$. Now suppose $u \in \Lambda^a \mathbb{C}^{p+q}, v \in \Lambda^b \mathbb{C}^{p+q}$, and suppose $a+b \ge p+q$, define $u \sqcap v := \iota(*u)v$. Here $\iota(\ )$ means the interior product. Geometrically, if u, v represent vector subspaces then $u \sqcap v \ne 0$ iff the subspaces intersect transversally and in that case $u \sqcap v$ represents their intersection.

Next, for any forms ω, θ, define $\omega \sqcup \theta := \omega \wedge \theta$. If ω, θ represent subspaces, then $\omega \sqcup \theta \ne 0$ iff they do not intersect, and in that case $\omega \sqcup \theta$ represents their linear span.

We will take the convention that $a \sqcup b \sqcup c := (a \sqcup b) \sqcup c$, likewise for $a \sqcap b \sqcap c$.

We will take $(e_1, \ldots, e_{p+q})$ as the standard ordered basis of $\mathbb{C}^{p+q}$.

Given any forms u of the top degree, we can identify it as a scalar $[u]$, provided we have made a choice of a "standard" top form, as we have already done.

We define $q+1$ determinant functions on $X \times Z$ as follows.

Definition 2.1. For $0 \le j \le q$, define $D_j = D_j(\omega_1, \ldots, \omega_{p+q}; L_p, L'_q)$ as follows.

$$D_j = \begin{cases} [\omega_p \sqcup L'_q] & \text{if } j = 0, \\ [(\omega_{p+q-j} \sqcap L_p) \sqcup \omega_j \sqcup L'_q] & \text{if } 1 \le j \le q-1, \\ [\omega_q \sqcup L_p] & \text{if } j = q. \end{cases} \tag{1}$$

3. Codimension One Orbits

We take as base point $l_p := e_1 \sqcup \ldots \sqcup e_p$ and $l'_q := e_{p+1} \sqcup \ldots \sqcup e_{p+q}$. We then define polynomial functions δ_j on X by

$$\delta_j(\omega) := D_j(\omega; l_p, l'_q). \tag{2}$$

We can now define $q+1$ subsets $\mathfrak{O}_j^0; 0 \le j \le q$ of X. They turn out to be the codimensional one K orbits. For $1 \le j \le q-1$,

$$\mathfrak{O}_j^0 := \{(\omega_i, 1 \le i \le p+q) | \delta_j(\omega) = 0, \delta_a(\omega) \ne 0 \text{ if } a \ne j\}. \tag{3}$$

For the case $j = 0$, we would like to imitate the definition in Equation 3 by requiring $\delta_a(\omega) = 0$ iff $a = 0$. However, a moment's thought reveals that this still consists of several orbits. We have to add more conditions. We define

$$\mathfrak{O}_0^0 := \{\omega | \delta_j(\omega) \neq 0 \text{ if } j > 0, \omega_{p-1} \sqcup l'_q \neq 0, \delta_0(\omega) = 0\}. \tag{4}$$

Likewise, define

$$\mathfrak{O}_q^0 := \{\omega | \delta_j(\omega) \neq 0 \text{ if } j < q, \delta_q(\omega) = 0, \omega_{q-1} \sqcup l_p \neq 0, \omega_{q+1} \sqcap l_p \neq 0\}. \tag{5}$$

There are obvious geometric descriptions for the orbits $\mathfrak{O}_j^0$. When $1 \leq j \leq q-1$, note that $\delta_0, \delta_q \neq 0$ implies $\omega_{p+q-k} \sqcap l_p \neq 0$ and $\omega_k \sqcup l'_q \neq 0$ for $1 \leq k \leq p-1$. So $\delta_k \neq 0$ iff the subspace $\omega_{p+q-k} \cap l_p$ (viewing ω_{p+q-k} as a subspace) is disjoint from the subspace $\widetilde{\omega_k}$ spanned by ω_k and l'_q. Hence, $\omega \in \mathfrak{O}_j$ iff ω_p (ω_q) is disjoint from l'_q (l_p) and that $\omega_{p+q-k} \cap l_p$ is disjoint from $\widetilde{\omega_k}$ if $1 \leq k \leq q-1$ and $k \neq j$. (Indeed, when $k = j$, we can even assume that the intersection of the two subspaces is of dimension one.)

For the case $j = 0, q$, we can proceed likewise. For all $1 \leq k \leq q-1, \omega_{p-1} \sqcup l'_q \neq 0$ implies $\omega_k \sqcup l'_q \neq 0$, and $\delta_q \neq 0$ means $\omega_{p+q-k} \sqcap l_p \neq 0$. Hence $\omega \in \mathfrak{O}_0^0$ iff ω_{p-1} is disjoint from l'_q and ω_q from l_p, and that for all $1 \leq k \leq q-1$, $\omega_{p+q-k} \cap l_p$ is disjoint from $\widetilde{\omega_k}$. Likewise, $\omega \in \mathfrak{O}_q^0$ iff ω_p is disjoint from l'_q, ω_{q-1} from l_p, and ω_{q+1} is transversal to l_p, and for $1 \leq k \leq q-1, \omega_{p+q-k} \cap l_p$ is disjoint from $\widetilde{\omega_k}$.

The following results relate the functions δ_j with the geometry of the flag manifold X. Notice that K has a natural action on X. It is well known that there are only finitely many orbits and there is a unique open orbit ([4] and [8]). It is trivial to check that the sets $\mathfrak{O}_j^0$ are K invariant.

Proposition 3.1. *The set $\mathfrak{O} := \{(\omega) | \delta_j(\omega) \neq 0$ for all $j\}$ is the unique open K orbit.*

Proposition 3.2. *There are exactly $q+1$ codimensional one K orbits. They are $\mathfrak{O}_j^0$.*

Lemma 3.3. *The Zariski closure $\mathfrak{O}_j$ of $\mathfrak{O}_j^0$ is the set $\{\omega | \delta_j(\omega) = 0\}$.*

We will prove these results in the next section.

4. Proofs of Results on the Orbits

For the sake of convenience, define $\widetilde{K}$ to be the subgroup of GL(p + q) which consists of matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, as in the case of $K \subseteq G$. $\widetilde{K}$ acts naturally on X and Z.

First, we prove Proposition 3.1.

Proof. (Proposition 3.1). The set $\mathfrak{O}$ is clearly K invariant and open, we only have to show that K acts transitively on it. Let us pick a base point. Recall we can represent a flag by an ordered basis. The following is a good base point ω_0 to pick: $(e_1 + e_{p+1}, \ldots, e_q + e_{p+q}, \overbrace{e_{q+1}, \ldots, e_p,} e_q - e_{p+q}, \ldots, e_1 - e_{p+1})$ (the part under the overbrace is absent if $p = q$). It is trivial to verify that such a flag lies inside $\mathfrak{O}$.

Suppose we have any $\omega \in \mathfrak{D}$, it suffices to find an element $k \in \widetilde{K}$ such that $k\omega_0 = \omega$. Now such an element k must be of the form

$$k = \begin{pmatrix} u_1 & \cdots & u_p & 0_p & \cdots & 0_p \\ 0_q & \cdots & 0_q & v_1 & \cdots & v_q \end{pmatrix}, \tag{6}$$

where $u_i \in \mathbb{C}^p, v_j \in \mathbb{C}^q$ are column vectors, and $0_p, 0_q$ are the zero vectors. We will show how to choose the vectors u_i, v_j.

First of all, pick any ordered basis $(z_i, 1 \leq i \leq p+q)$ that represents ω. Suppose $z_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ for $1 \leq i \leq q$. The requirement that $\delta_q(\omega) \neq 0$ implies that $v_1, \ldots, v_q$ form a basis for $\mathbb{C}^q$. Hence we can assume that $z_i = \begin{pmatrix} u_i \\ 0_q \end{pmatrix}$ for $q+1 \leq i \leq p$ (if $p = q$, there is nothing to do). The fact that $\delta_0(\omega) \neq 0$ implies that $u_1, \ldots, u_p$ form a basis of $\mathbb{C}^p$.

We would like to argue that we can assume that $z_i = \begin{pmatrix} u_{p+q-i+1} \\ -v_{p+q-i+1} \end{pmatrix}$ for $p+1 \leq i \leq p+q$. We can do this by an inductive argument. We will argue carefully for $i = p+1$, the general inductive step involves a similar reasoning and will be omitted.

We can assume that $z_{p+1} = \begin{pmatrix} u_{p+1} \\ 0_q \end{pmatrix}$. The fact that $\delta_i \neq 0$ for $i = q-1$ implies that $u_1 \wedge \ldots u_{q-1} \wedge (u_{q+1} \ldots u_p \wedge) u_{p+1} \neq 0$. So $u_1, \ldots, u_{q-1}, (u_{q+1}, \ldots u_p,) u_{p+1}$ form a basis of $\mathbb{C}^p$, as do $u_1, \ldots u_p$ (in both cases, if $p = q$, the portions inside the parentheses are absent). We can assume that $u_{p+1} \doteq u_q + \sum_{1 \leq i \leq p, i \neq q} a_i u_i$. Now replace u_q by $u_q + \sum_{1 \leq i \leq q-1} a_i u_i$ and v_q by $v_q + \sum_{1 \leq i \leq q-1} a_i v_i$. (This replacement does not change the flag represented!) Hence we can rewrite ω_{p+1} as $u_{p+1} \doteq u_q + \sum_{i=q+1}^{p} a_i u_i$ (if $p = q$, the summation from $q+1$ to p is treated as zero). Thus $z_{p+1} \doteq \frac{1}{2}(\begin{pmatrix} u_q \\ v_q \end{pmatrix} + \begin{pmatrix} u_q \\ -v_q \end{pmatrix}) + \sum_{i=q+1}^{p} a_i \begin{pmatrix} u_i \\ 0_q \end{pmatrix}$. Thus, without loss of generality, we can redefine $z_{p+1} = \begin{pmatrix} u_q \\ -v_q \end{pmatrix}$ and we still represent the same flag.

By a similar argument, we know that we can choose $z_i = \begin{pmatrix} u_{p+q-i+1} \\ -v_{p+q-i+1} \end{pmatrix}$ when $p+1 \leq i \leq p+q$. Thus the element $k \in \widetilde{K}$ moves ω_0 to ω. Hence, $\widetilde{K}$ acts transitively on $\mathfrak{D}$. ■

The following elementary observation can be proved by direct verification, and we will omit the proof.

Lemma 4.1. *The following ordered basis represents a base point $\omega_0(j)$ of $\mathfrak{D}_j^0$. For $1 \leq j \leq q-1$, it is $(e_1 + e_{p+1}, \ldots, e_q + e_{p+q}, \overbrace{e_{q+1}, \ldots, e_p}, e_q - e_{p+q}, \ldots, e_{j+2} - e_{p+j+2}, e_j - e_{p+j}, e_{j+1} - e_{p+j+1}, e_{j-1} - e_{p+j-1}, \ldots, e_1 - e_{p+1})$. For $j = 0$ it is $(e_1 + e_{p+1}, \ldots, \overbrace{e_q + e_{p+q}, e_{q+1}, \ldots, e_{p-1}}, e_{p+q}, e_p, e_{q-1} - e_{p+q-1}, \ldots, e_1 - e_{p+1})$. For $j = q$, it is $(e_1 + e_{p+1}, \ldots, e_{q-1} + e_{p+q-1}, e_q, \overbrace{e_{q+1} + e_{p+q}, e_{q+2}, \ldots, e_p}, e_q - e_{p+q}, \ldots, e_1 - e_{p+1})$. (When $p = q$, the portions under the overbraces are skipped.)*

Lemma 4.2. *The group $\widetilde{K}$ acts transitively on $\mathfrak{D}^0_j$.*

Proof. We will constantly refer to the proof of Proposition 3.1. We will write ω_0 instead of $\omega_0(j)$. We would like to find an element $k \in \widetilde{K}$ that moves ω_0 to any given element $\omega \in \mathfrak{D}^0$, where k is as in Equation 6.

First, we prove the case $j = q$. The case $j = 0$ is similarly proved and thus omitted. Pick any ordered basis (z_i) that represents ω. Suppose $z_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ for $1 \leq i \leq q-1$. The fact that $\omega_{q-1} \sqcup l_p \neq 0$ implies that $v_1, \dots v_{q-1}$ is linearly independent. As $\delta_q(\omega) = 0$, we can assume that $z_q = \begin{pmatrix} u_q \\ 0_q \end{pmatrix}$. We also let $z_{q+1} = \begin{pmatrix} u_{q+1} \\ v_q \end{pmatrix}$ (if $p = q$, let $u_{q+1} = 0_p$ and we may as well let $z_{q+1} = \begin{pmatrix} 0_p \\ -v_q \end{pmatrix}$). The fact that $\omega_{q+1} \sqcap l_p \neq 0$ implies that $v_1, \dots, v_q$ form a basis of $\mathbb{C}^q$. We can therefore assume that $z_i = \begin{pmatrix} u_i \\ 0_q \end{pmatrix}$ for $q+2 \leq i \leq p$ (there is nothing to do if $p = q$ or $p = q+1$). The fact that $\delta_0 \neq 0$ implies that $u_1, \dots u_p$ form a basis of $\mathbb{C}^p$.

At this point we can carry over the argument for the proof of Proposition 3.1 verbatim and conclude that we can choose $z_{p+1} = \begin{pmatrix} u_{q+1} \\ -v_q \end{pmatrix}$ (if $p = q$, this step is redundant) and then let $z_i = \begin{pmatrix} u_{p+q-i+1} \\ -v_{p+q-i+1} \end{pmatrix}$ for $p+2 \leq i \leq p+q$ (if $q \geq 2$, otherwise there is nothing to do). The proof is complete at this point.

When $1 \leq j \leq q-1$, we pick any ordered basis representing ω. Let $z_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$ for $1 \leq i \leq q$. The fact that $\delta_q \neq 0$ implies that $v_1, \dots v_q$ form a basis of $\mathbb{C}^q$. Therefore we can choose $z_i = \begin{pmatrix} u_i \\ 0_q \end{pmatrix}$ for $q+1 \leq i \leq p$. Since $\delta_0 \neq 0$, we know that $u_1, \dots, u_p$ form a basis of $\mathbb{C}^p$. By the same sort of argument as in Proposition 3.1, we can assume that $z_i = \begin{pmatrix} u_{p+q-i+1} \\ -v_{p+q-i+1} \end{pmatrix}$ for $p+1 \leq i \leq p+q-(j+2)+1$. Next, let $z_{p+q-j} = \begin{pmatrix} u_{p+q-j} \\ 0_q \end{pmatrix}$. The fact that $\delta_j = 0$ implies that u_{p+q-j} is a linear combination of $u_1, \dots, u_j, u_{j+2}, \dots, u_p$. However, $\delta_{j-1} \neq 0$ implies that the coefficient of u_j is non-zero. Therefore, without loss of generality, we can assume that $u_{p+q-j} \doteq u_j + \sum_{m=1}^{j-1} a_m u_m + \sum_{k=j+2}^{p} b_k u_k$. Redefine u_j to be $u_j + \sum_{m=1}^{j-1} a_m u_m$ and v_j to be $v_j + \sum_{m=1}^{j-1} a_m v_m$. Hence we have $2\begin{pmatrix} u_{p+q-j} \\ 0_q \end{pmatrix} = \begin{pmatrix} u_j \\ v_j \end{pmatrix} + \begin{pmatrix} u_j \\ -v_j \end{pmatrix} + \sum_{k=j+2}^{p} b_k (\begin{pmatrix} u_k \\ v_k \end{pmatrix} + \begin{pmatrix} u_k \\ -v_k \end{pmatrix})$. Therefore, without loss of generality, we can choose $z_{p+q-j} = \begin{pmatrix} u_j \\ -v_j \end{pmatrix}$.

We can argue in a similar fashion that we can assume $z_{p+q-j+1} = \begin{pmatrix} u_{j+1} \\ -v_{j+1} \end{pmatrix}$, and $z_i = \begin{pmatrix} u_{p+q-i+1} \\ -v_{p+q-i+1} \end{pmatrix}$ for $p+q-j+2 \leq i \leq p+q$. ■

We can now prove Proposition 3.2.

Proof. of Proposition 3.2. To show that $\mathfrak{O}_j^0$ are orbits, it is enough to see that $\widetilde{K}$ acts transitively on them, which we have shown. The $q+1$ orbits $\mathfrak{O}_j^0$ are clearly distinct and have codimension 1. So we know there are at least $q+1$ orbits.

We need to see that the complement of $\cup_j \mathfrak{O}_j^0$ is a union of orbits with codimension other than 1. The complement is contained inside the union of $\mathfrak{O}$ with $\cup_j \mathfrak{O}_j \backslash \mathfrak{O}_j^0$. The orbit $\mathfrak{O}$ is open (Proposition 3.1). It remains to check that $\mathfrak{O}_j \backslash \mathfrak{O}_j^0$ has higher codimension. Essentially, the only way for a point ω to be contained in $\mathfrak{O}_j \backslash \mathfrak{O}_j^0$ is to satisfy, apart from $\delta_j(\omega) = 0$, at least some additional K invariant system of polynomial equations. For example, for $j = q$, it may be, amongst others, $\delta_{q-1} = 0$ or the system $\omega_{q+1} \sqcap l_p = 0$. The proof is complete at this point. ■

Proof. Proof of Lemma 3.3. We can view the flag manifold as a projective subvariety of $\mathbb{P}_{n-1} \times \mathbb{G}(1, n-1) \times \cdots \times \mathbb{G}(n-2, n-1)$ in the standard way.

For any j, we can find homogeneous polynomials $P_0, P_1, \ldots, P_m$ (m depends on j), such that the set $\mathfrak{O}_j^0 = \{\omega | P_0(\omega) = 0, P_j(\omega) \neq 0 \text{ for } i > 0\}$. For example, when $j = q$, put $P_0 = \delta_q, P_i = \delta_i$ for $1 \leq i \leq q-1$, and $P_q = \delta_0$. Note that $\omega_{q-1} \sqcup l_p$ and $\omega_{q+1} \sqcap l_p$ have various components, and we let P_i, for $i > q$, be these components.

Since K is connected, the K-orbits $\mathfrak{O}_j^0$ are irreducible. Now the lemma follows as a basic result of algebraic geometry. ■

5. Stein Extension and Determinant Functions

Let G_0 be a general connected semisimple linear Lie group, K_0 its maximal compact subgroup, and let G and K be the complexifications. For various reasons, we are interested in G_0 invariant Stein extensions of $G_0/K_0 \subseteq G/K$. Work done in this direction includes [1], [11], [9], [10]. Recently, J. Wolf and R. Zierau, ([12]) have shown that if G_0/K_0 admits a Hermitian symmetric structure, then $G_0/K_0 \times \overline{G_0/K_0}$ can be viewed as sitting inside G/K and this embedding is a G_0 invariant Stein extension of $G_0/K_0 \subseteq G/K$ ([12]). The motivation behind their results is to find concrete realization of the space of linear cycles of a measurable open G_0 orbit of a generalized flag manifold associated to G. We will now show, in the special case of $G_0 := \mathrm{SU(p,q)}$ and $K_0 := \mathrm{S(U(p) \times U(q))}$, another way of obtaining the same Stein extension.

First of all, we identify G_0/K_0 as

$$Z_0 := \{L_p \in G(p, p+q) | \langle\ ,\ \rangle \text{ is positive definite on } L_p\}.$$

Here we define $\langle z, z\rangle := \sum_{i=1}^{p} |z_i|^2 - \sum_{j=p+1}^{p+q} |z_j|^2$. We shall identify the conjugate manifold $\overline{G_0/K_0}$ as

$$\overline{Z_0} := \{L_q' \in G(q, p+q) | \langle\ ,\ \rangle \text{ is negative definite on } L_q'\}.$$

Hence there is the obvious identification of $Z_0 \times \overline{Z_0} \subseteq Z = G/K$. If $p = q$ we have also the following identification:

$$\begin{aligned}\overline{Z_0} \times Z_0 = \ & \{(L_p, L'_q) \in G(p, p+q) \times G(q, p+q) | \\ & \langle \ , \ \rangle \text{ is negative definite on } L_p \\ & \text{and positive definite on } L'_q\}.\end{aligned}$$

We sometimes identify

$$Z_0 = \{(L_p, (L_p)^\perp) | \langle \ , \ \rangle \text{ is positive definite on } L_p\} \subseteq Z_0 \times \overline{Z_0}.$$

Here $\perp$ denotes the orthogonal complement with respect to $\langle \ , \ \rangle$. This identification sets Z_0 as the real form of Z.

It is well known that X has a unique closed G_0 orbit X_0 which is K_0 homogeneous ([8]). It can be realized geometrically as follows. On $\mathbb{P}_{n-1}$ are three G_0 orbits, two of them are open (those points on which $\langle \ , \ \rangle$ is positive, resp. negative), the remaining one Σ is closed. A flag $\omega = (\omega_j, 1 \leq j \leq n)$ belongs to X_0 iff the following is true: $\omega_q \subseteq \Sigma$ (here, we view ω_q as a projective plane of dimension $q-1$ and Σ a subset in $\mathbb{P}_{n-1}$) and ω_{n-1}, when viewed as a hyperplane of $\mathbb{P}_{n-1}$, is tangent to Σ at any point of ω_q. Indeed, the family of flags just described is G_0- invariant. On the other hand, it is easy to check that its diemnsion coincides with the diemnsion of the closed orbit, X_0.

Now consider the set

Definition 5.1. Define $\widehat{X_0} \subseteq Z$ as follows.

$$\widehat{X_0} := \{(L_p, L'_q) | D_j(\omega; L_p, L'_q) \neq 0 \text{ for all } j \text{ and for all } \omega \in X_0\}.$$

If $D := \Pi_j D_j$, then

$$\widehat{X_0} = \{(L_p, L'_q) | D(\omega; L_p, L'_q) \neq 0 \text{ for all } \omega \in X_0\}. \tag{7}$$

Since X_0 is G_not-invariant, $\widehat{X_0}$ is a G_0 invariant subset of Z. We want to show that $\widehat{X_0}$ is an invariant Stein space containing Z_0. In [2], we give a proof whose method we believe is capable of generalization. Here, instead, we will compute $\widehat{X_0}$ explicitly, and it will be manifest that the space is Stein.

Lemma 5.2. *The set $Z_0 \times \overline{Z_0}$ is contained in $\widehat{X_0}$. If $p = q$, then we also have $\overline{Z_0} \times Z_0 \subseteq \widehat{X_0}$. In particular, $\widehat{X_0} \subseteq Z$ is non-empty.*

Proof. While it is not strictly necessary, it is simpler to recall that X_0 is indeed a K_0 orbit. In fact, if we let $\mathrm{U(p,q)}$ act on X and Z in the obvious way, $\mathrm{U(p)} \times \mathrm{U(q)}$ preserves X_0. We use the ordered basis $(e_1 + e_{p+1}, \ldots, e_q + e_{p+q}, \overbrace{e_{q+1}, \ldots, e_p}, e_q - e_{p+q}, \ldots, e_1 - e_{p+1})$ to represent a base point on X_0 (for $p = q$, omit the overbrace). Since X_0 is a K_0-orbit, a point on X_0 is of the form $k_0(e_1 + e_{p+1}, \ldots, e_q + e_{p+q}, \overbrace{e_{q+1}, \ldots, e_p}, e_q - e_{p+q}, \ldots, e_1 - e_{p+1})$ with $k_0 \in K_0$ as in (6). Thus, any point on X_0 can be represented by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} u_q \\ v_q \end{pmatrix};$$
$$\begin{pmatrix} u_{q+1} \\ 0_q \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} u_p \\ 0_q \end{pmatrix}; \tag{8}$$
$$\begin{pmatrix} u_q \\ -v_q \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} u_1 \\ -v_1 \end{pmatrix},$$

where $u_1, \ldots, u_p$ form an orthonormal basis of $\mathbb{C}^p$ (with respect to the standard Hermitian inner product) and $v_1, \ldots, v_q$ form an orthonormal basis for $\mathbb{C}^q$ (for $p = q$, omit the middle row).

Suppose (ω_j) represents a point in X_0 and $(L_p, L'_q) \in Z_0 \times \overline{Z_0}$. Since ω_q represents a totally null subspace (i.e., the restriction of $\langle\ ,\ \rangle$ to ω_q is identically zero) and L_p a positive definite subspace, $\omega_q \cap L_p = 0$, hence $D_q = [\omega_q \sqcup L_p] \neq 0$. Likewise, ω_p is a positive semi-definite subspace whereas L'_q is negative definite. By similar reasoning $D_0 \neq 0$.

For $1 \leq j \leq q-1$, assume, for the moment, that $\omega_{p+q-j} \sqcap L_p \neq 0$. Under this assumption, $\omega_{p+q-j} \sqcap L_p$ represents a positive definite subspace. However, ω_j is a totally null subspace, and L'_q a negative definite subspace. Hence, these three subspaces intersect trivially pairwisely, hence $D_j = [(\omega_{p+q-j} \sqcap L_p) \sqcup \omega_j \sqcup L'_q] \neq 0$.

It remains to show that $\omega_{p+q-j} \sqcap L_p \neq 0$. In other words, we want to show that the dimension of $\omega_{p+q-j} \cap L_p$ is $p-j$. By general consideration, it is at least $p-j$. On the other hand, the maximum possible dimension of a positive definite subspace of ω_{p+q-j} is $p-j$, as ω_{p+q-j} represents a subspace containing the totally null subspace ω_q of dimension q. As L_p is positive definite, the dimension of $\omega_{p+q-j} \cap L_p$ cannot be more than $p-j$.

The additional possibility when $p = q$ is similarly proved and we omit it. ■

Proposition 5.3. *If $p \neq q$, $\widehat{X_0} = Z_0 \times \overline{Z_0}$. If $p = q$, $\widehat{X_0} = (Z_0 \times \overline{Z_0}) \cup (\overline{Z_0} \times Z_0)$.*

Proof. In view of Lemma 5.2, it suffices to show that $\widehat{X_0} \subset Z_0 \times \overline{Z_0}$ if $p \neq q$ and $\widehat{X_0} \subset (Z_0 \times \overline{Z_0}) \cup (\overline{Z_0} \times Z_0)$ if $p = q$.

Assume $(L_p, L'_q) \in \widehat{X_0}$. We first show that both are definite subspaces. If not, the one which is not must contain a null vector v. We can always find a flag in X_0 such that, if (ω_j) represents the flag, v is a representative of ω_1. This means either $\omega_q \sqcup L_p = 0$ or $\omega_p \sqcup L'_q = 0$ and we have a contradiction.

Next, we show that L_p and L'_q cannot have the same sign. We first consider the case when $q = 1$. The case $U(1,1)$ is well understood. Thus, we can assume that $p > 1$. If L_p and L'_1 have the same sign, since we know that L_p is positive, then L'_1 is also positive. Pick a point $a \in L'_1$. We can choose a base point in X_0 such that ω_p contains a. (See Equation 8). At this base point $D_0 = 0$ and we have a contradiction.

If $q > 1$ and both L_p and $L'q$ have the same sign, when $p > q$ we know L_p must be positive, and hence L'_q is positive also; when $p = q$, since the signs are symmetric, we can without loss of generality assume that both L_p, L'_q are positive.

Under this assumption, we have $a \in L_p, b \in L'_q$ such that the restriction of $\langle\ ,\ \rangle$ on the span of a, b has mixed signatures (or else, we can find a subspace of dimension $p+1$ on which $\langle\ ,\ \rangle$ is positive definite). We can find an orthonormal basis for this span, i.e., $\{u, v\}$ such that $\langle u, u\rangle = 1, \langle v, v\rangle = -1, \langle u, v\rangle = 0$. We can always find a flag in X_0 which can be represented by an ordered basis $(z_i, 1 \leq i \leq p+q)$ in which $z_{q-1} = u+v, z_{p+1} = u-v$ (see Equation 8).

In that case ω_{p+1} contains both a, b, hence $\omega_{p+1} \cap L_p$ contains a. We may as well assume that $\omega_{p+1} \sqcap L_p \neq 0$, or else, $D_{q-1} = 0$ and we already have a contradiction. So the span of $\omega_{p+1} \sqcap L_p$ and L'_q contains a, b and hence $u+v$, which is also contained in ω_{q-1}, hence $(\omega_{p+1} \sqcap L_p) \sqcup L'_q \sqcup \omega_{q-1} = 0$, i.e., $D_{q-1} = 0$, and we have a contradiction.

In other words L_p, L'_q must be definite of opposite signs, and we are done.

■

6. Linear Cycles

Next, we would like to relate our Stein extension $\widehat{X_0}$ with the one obtained by Wolf and Zierau. Their results can be briefly summarized as follows. Fix an open G_0 orbit D of the flag manifold. In D is a special maximally compact subvariety Γ. In details, if we express $D = G_0/T_0$, where T_0 is a compact Cartan subgroup, then $\Gamma = K_0/T_0$. Now consider the set of all those G translates of Γ which stay inside D, they are called linear cycles. It is known that the space of linear cycles admits the structure of a Stein manifold ([9]). It is now known that if G_0/K_0 admits a Hermitian symmetric structure, and if D is not of Hermitian type (i.e., it does not fibrate holomorphically over G_0/K_0 if the latter is given either of the two Hermitian symmetric structures), then the space of linear cycles is exactly $G_0/K_0 \times \overline{G_0/K_0}$.

In the case of $\mathrm{SU(p,q)}$, we can describe the space of linear cycles explicitly as follows. First of all, we need a way of parametrizing the flag domain D. It turns out that we can parametrize them by a non-decreasing sequence of integers $0 \leq a_1 \leq \ldots \leq a_q \leq p$. We have a related sequence $1 \leq b_1 < \ldots < b_q \leq (p+q)$, by putting $b_i := a_i + i$. There are various ways to specify the orbit once a sequence (a_i) or (b_i) is given. We can pin down the open orbit by specifying the signatures of the restrictions of $\langle\ ,\ \rangle$ to various subspaces in a flag belonging to an orbit. Given (b_i), let the orbit determined to be $\mathfrak{O}(b)$, then $\omega \in \mathfrak{O}(b)$ iff for all j, ω_j has signature $(\overbrace{+,\ldots,+}^{j-l},\overbrace{-,\ldots,-}^{l})$. Here we define $l := \max_i\{i | b_i \leq j\}$.

A G_not orbit ,S, and a K-orbit, Q, are dual to each other (in the sense of Matsuki) if the intersection $S \cup Q$ consists of exactly one K_not-orbit. The K-orbit dual to an open G_not-orbit is closed. We can pin down the Matsuki dual (closed) K orbit as follows. Recall the subspaces l_p, l'_q in Section 3. A flag ω lies in the K orbit dual to $\mathfrak{O}(b)$ iff $\dim(\omega_j \cap l_p) = j - l$ and $\dim(\omega_j \cap l'_q) = l$, where l retains the same meaning as just now. Finally, we can specify the orbit $\mathfrak{O}(b)$ by picking a "nice" base point. We can use the following ordered basis (v_j) to pin down the "nice" base point. Let σ be the permutation of $\{1, \ldots, p+q\}$ which sends $p+i$ to b_i for $1 \leq i \leq q$ and sends i to $i + n(i)$ for $1 \leq i \leq p$. Here $n(i) := \sharp\{j | a_j < i\}$. With this notation in hand, define $v_i = e_{\sigma^{-1}(i)}$.

Secondly, we view $Z_0 \times \overline{Z_0} = G_0/K_0 \times \overline{G_0/K_0}$ as the space:

$$\{(L_p, L'_q) \in \mathrm{G(p, p+q)} \times \mathrm{G(q, p+q)} | \quad \langle\ ,\ \rangle \text{ is positive (negative) definite on } L_p(L'_q)\}.$$

The space $Z_0 \times \overline{Z_0}$ parametrizes the space of linear cycles associated to D. Suppose D is pinned down by the sequence (a_i). Suppose we choose $(L_p, L'_q) \in Z_0 \times \overline{Z_0}$, then the linear cycle thus parametrized consists of flags (ω_j) where $\dim(\omega_j \cap L_p) = j - l, \dim(\omega_j \cap L'_q) = l$. (Keep the same notations as just now.)

For the sake of completeness, we consider the two flag domains of Hermitian type. The flag domains are either $(a_j = p; 1 \le j \le q)$ or $(a_j = 0; 1 \le j \le q)$. In the former case, the space of linear cycle is Z_0, and in the latter case $\overline{Z_0}$. In the former case, suppose we have $L_p \in Z_0 \subseteq \mathrm{G(p, p+q)}$, the linear cycle thus parametrized is given by $\omega_j \subset L_p; 1 \le j \le p$ and $L_p \subseteq \omega_j; p+1 \le j \le p+q$. Interchange p and q and replace Z_0 by $\overline{Z_0}$ in the previous sentence and we have the latter case.

Finally, we would like to discuss briefly how the various minimal K orbits sit in the boundaries of $\mathfrak{O}_j$. These minimal orbits are dual (via Matsuki duality) to the flag domains, indeed, we have described them already. We have the following.

Proposition 6.1. *Any minimal K orbit dual to a flag domain lies in $\mathfrak{O}_j$ for the following ranges of j. If it is not of Hermitian type, we have $0 \le j \le q$. If it is dual to the domain given by $a_k = p, 1 \le k \le q$, it is $1 \le j \le q$. If it is given by $a_k = 0, 1 \le k \le q$, then it is $0 \le j \le q-1$.*

Proof. As all $\mathfrak{O}_j$ are K invariant, it suffices to pick a nice base point of the orbit and check which $\mathfrak{O}_j$ it belongs to. To do that, we only have to consider which equations $D_j(\ ; l_p, l'_q) = 0$ it satisfies. The base point chosen earlier in the section suffices. We will only argue carefully that all minimal orbits lie in $\mathfrak{O}_j$ for $1 \le j \le q-1$. The rest can be similarly argued.

Observe that for any $1 \le j \le q-1$, ω_{p+q-j}, viewed as a linear subspace, must contain e_1, so $e_1 \in \omega_{p+q-1} \sqcap l_p$. On the other hand, either $\omega_j \sqcup l'_q = 0$, in that case $D_j(\omega; l_p, l'_q) = 0$, or $\omega_j \subseteq l_p$ (when viewed as a linear subspace) and $e_1 \in \omega_j$, in that case, we are forced to conclude $D_j = (\omega_{p+q-j} \sqcap l_p) \sqcup (\omega_j \sqcup l'_q) = 0$ also. ■

We know that higher codimensional orbits occur as certain strata of a K invariant stratification of the boundary of the configuration $\cup_{j=0}^{q} \mathfrak{O}_j$. Proposition 6.1 implies that this stratification is more than mere intersections of the boundaries of a subcollection of $\mathfrak{O}_j$.

Finally, we would like to prove a result which makes use of knowledge of the orbit structure. Recall the notions of polar and double polar introduced in Section 1.

Proposition 6.2. *For any subset $A \subset X$, we have $A \subseteq \widehat{\widehat{A}}$. When $A = X_0$, we have $\widehat{\widehat{X_0}} = X_0$.*

Proof. The fact that $A \subseteq \widehat{\widehat{A}}$ in general is basically a tautology. When $A = X_0$, since X_0 is G_0 invariant, so is $\widehat{\widehat{X_0}}$. So the latter is a G_0 invariant subset, hence a finite union of G_0 orbits. According to Matsuki's duality, a K orbit Γ and a G_0 orbit S are dual iff $\Gamma \cap S$ consists of exactly one K_0 orbit. Suppose $\widehat{\widehat{X_0}}$ contains the G_0 orbit S. Let Γ be the K orbit dual to S. Now $\Gamma \cap S \subset S \subset \widehat{\widehat{X_0}}$. As $(l_p, l'_q) \in \widehat{X_0}$, we conclude that $\Gamma \cap S$ is in the open K orbit. In othe words Γ intersects non-trivially with the open K orbit. Hence Γ is the open K orbit, and therefore $S = X_0$. ■

7. Appendix

Propositions 3.1 and 3.2 can be proved by using the well known correspondence between K orbits of X and the set of equivalence classes (under conjugation by G_0) of real data $\overline{\mathcal{D}}^{\mathbb{R}}$ as defined in [7, p. 107]. In this appendix we explain briefly this approach.

We recall that a set of real data for G_0 consists of a pair $(H_0, \Delta^+(\mathfrak{g}, \mathfrak{h}))$ where $H_0 \subset G_0$ is a Cartan subgroup, and Δ^+ a choice of positive system for $\mathfrak{h}$ in $\mathfrak{g}$. To each equivalence class corresponds a unique K orbit $\mathfrak{O}$ in X. Fix $\mathfrak{O}$ and denote by $(H_{0,r}, \Delta^+_r(\mathfrak{g}, \mathfrak{h}))$ a choice of representative for the corresponding equivalence class in $\overline{\mathcal{D}}^{\mathbb{R}}$. We may assume that $H_{0,r}$ is θ stable.

There is a formula which gives the dimension of $\mathfrak{O}$ in terms of the number of real, imaginary, and complex roots in Δ_r. Real, imaginary, and complex roots are defined in the obvious way. We denote by $\Delta_{r,\mathbb{R}}$, $\Delta_{r,I}$, $\Delta_{r,\mathbb{C}}$, and $\Delta_{r,CI}$ the set of real, imaginary, complex, and compact imaginary roots. If θ_r is the Cartan involution on Δ_r (chosen as in [6, pp. 147-148]), then set

$$D_+(\mathfrak{O}) = \{\alpha \in \Delta^+_r | \theta_r \alpha \in \Delta^+_r, \theta_r \alpha \neq \alpha\}$$

and set $d(\mathfrak{O}) = |D_+(\mathfrak{O})|$. We have (see [6, Lemma 5.6]):

Lemma 7.1. *Let $\mathfrak{O}$ be a K orbit in X, then*

$$\dim \mathfrak{O} = \frac{1}{2}(|\Delta_{r,CI}| + |\Delta_{r,\mathbb{R}}| + |\Delta_{r,\mathbb{C}}| - d(\mathfrak{O})).$$

The dimension of the K orbits associated to $H_{0,r}$ lies between ([6])

$$\frac{1}{2}(|\Delta_{r,CI}| + |\Delta_{r,\mathbb{R}}| + \frac{1}{2}|\Delta_{r,\mathbb{C}}|)$$

and

$$\frac{1}{2}(|\Delta_{r,CI}| + |\Delta_{r,\mathbb{R}}| + |\Delta_{r,\mathbb{C}}|).$$

The K orbit of maximal dimension associated to $H_{0,r}$ is called the Langlands orbit attached to $H_{0,r}$. The K orbit of minimal dimension associated to $H_{0,r}$ is called the Zuckerman orbit attached to $H_{0,r}$. It is well known that if X is connected, then there exists a unique open K orbit and it is attached to the maximally split Cartan. ([4])

When $G_0 = \mathrm{SU(p,q)}$, there is a convenient choice of representatives from the equivalence classes of $\overline{\mathcal{D}}^{\mathbb{R}}$. Here we follow [3, §1.3]. Let $\mathfrak{t}_0$ denote the maximally compact Cartan subalgebra of $\mathfrak{g}_0$. Set $\alpha_i^0 = e_i^0 - e_{i+1}^0, 1 \leq i \leq p+q-1$. The fixed choice of positive system for $\Delta(\mathfrak{g}, \mathfrak{t})$ for which α_i^0 are simple roots will be denoted by Δ_0^+. For $1 \leq r \leq q$, set

$$\beta_r^0 = \alpha_{p-r+1}^0 + \alpha_{p-r+2}^0 + \ldots + \alpha_{p+r-1}^0 = e_{p-r+1}^0 - e_{p+r}^0.$$

Let $\mathbf{c}_r$ be the Cayley transform through β_r^0. Set $\mathfrak{h}_0 = \mathfrak{t}$ and for $r > 0$, set inductively $\mathfrak{h}_r = \mathbf{c}_r(\mathfrak{h}_{r-1})$, and let $H_{0,r}$ be the Cartan subgroup corresponding to $\mathfrak{h}_r \cap \mathfrak{g}_0$. Write $W_r(\mathfrak{g}, \mathfrak{h}_r)$ for the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{h}_r)$. Define $\Delta_r^+, \alpha_i^r, \beta_i^r$ inductively.

Let $s_i^r \in W_r$ denote the simple reflection through α_i^r. Any Cartan subalgebra $\widetilde{\mathfrak{h}_0} \subset \mathfrak{g}_0$ is G_0 conjugated to one of the $\mathfrak{h}_{0,r}$, so we assume that the representatives of the equivalence classes in $\overline{\mathcal{D}}^{\mathbb{R}}$ are of the form $(H_{0,r}, \Delta_r^+)$.

Lemma 7.2. *Let $G_0 = \mathrm{SU(p,q)}$, with $p \geq q$.*

1. *If $p > q$, then the codimension one K orbits in X (if any) are associated to the maximally split Cartan $H_{0,q}$.*

2. *If $p = q$, then the codimension one K orbits in X (if any) are associated either to the maximally split Cartan or to $H_{0,q-1}$ (The Cartan subgroup with split rank $q-1$).*

Proof. We will only sketch the proof. Consider $H_{0,q-1}$, the Cartan subgroup with split rank $q-1$. It is not difficult to show that

$$\begin{aligned} |\Delta_{q-1,\mathbb{R}}| &= |\Delta_{q,\mathbb{R}}| - 2 \\ |\Delta_{q-1,CI}| &= |\Delta_{q,CI}| + 2(p-q) \\ |\Delta_{q-1,\mathbb{C}}| &= |\Delta_{q,\mathbb{C}}| - 4(p-q). \end{aligned}$$

By formula 7.1, the dimension of the Langlands orbit associated to $H_{0,q-1}$ is equal to $\dim \mathfrak{O}_{\text{open}} - (p-q) - 1$. Thus, if $p > q$, then the dimensions of the K orbits associated to $H_{0,q-1}$ are always smaller than $\dim \mathfrak{O}_{\text{open}} - 1$. We can treat the other Cartan subgroups $H_{0,i}$ in a similar way. ∎

Next, we use Vogan's parameter set $S(p+q,p,r)$, as explained in [3, §1.6], to parametrize K-orbits. A parameter set consists of pairs $(x,y) \in \{1,\ldots,p+q\} \times \{1,\ldots,p+q\}$ or $(x,y) \in \{1,\ldots,p+q\} \times \{+,-\}$ satisfying certain conditions. There exist bijections

$$\frac{|W(\mathfrak{g}, \mathfrak{h}_r)|}{|W(G_0, H_{0,r})|} \xrightarrow{g_1} S(p+q,p,r) \xrightarrow{g_2} \mathcal{D}_r = \{(H_{0,r}, \Delta^+)\}.$$

In particular, $S(p+q,p,r)$ classifies the K orbits attached to $H_{0,r}$.

Lemma 7.3. *The parameter set associated to the open K orbit in X is*

$$P_{\text{open}} = (p,+)(p-1,+)\dots(q+1,+)(1,p+q)(2,p+q-1)\dots(q,p+1). \quad (9)$$

Proof. We will only sketch the proof. The bijection g_1 associates to P_{open} (as in Formula 9) an equivalence class in $\frac{W(\mathfrak{g},\mathfrak{h}_q)}{W(G_0,H_{0,q})}$. Let $w \in W(\mathfrak{g},\mathfrak{h}_q)$ be a representative of such an equivalence class. Consider $w\Delta^+(\mathfrak{g},\mathfrak{h}_q)$ and use the dimension formula 7.1 to show that the corresponding K orbit is a Langlands orbit. ∎

There is a well defined action of the Weyl group elements s_i in $\mathcal{D}_r$ ([3, §1.8]). We have

$$s_i \times (H_r, w{\Delta^+}_r) = (H_r, ws_i^r\Delta_r^+).$$

On the other hand, there is a compatible definition for the action of s_i on the parameter set ([3, §1.9]). Indeed, if $P \in S(p+q,p,r)$ corresponds to $(H_r, w{\Delta^+}_r)$, then $s_i \times P$ corresponds to $(H_r, ws_i^r\Delta_r^+)$.

Lemma 7.4. *1. If $p > q$, then the K orbit corresponding to*

$$s_i \times P_{\text{open}}$$

are codimension one orbits.

2. If $p > q$, then there are exactly $q+1$ co-dimensional one K orbits of X.

3. If $p = q$, then the K orbits associated to

$$s_i \times P_{\text{open}}$$

are codimensional one orbits. There are exactly $q-1$ orbits of this type.

4. If $p = q$, then the Langlands orbits associated to $H_{0,q-1}$ are codimensional one orbits. There are two orbits of this type. If $p = q$, then the total number of codimension one orbits is $q+1$.

Proof. We will only sketch the proof. We use [3, §1.9] and the bijection between K orbits in X and the parameter set to show that the K orbits attached to the maximally split Cartan correspond to $wP_{\text{open}} \in S(p+q,p,q)$ with $w \in \langle s_p^q; s_i^q, 1 \le i \le q\rangle \subset W(\mathfrak{g},\mathfrak{h}_q)$ if $p > q$, and $w \in \langle s_i^p, 1 \le i \le q-1\rangle$ if $p = q$. Here, s_i^r is the reflection with respect to the root α_i^r.

If $(H_{0,q}, \Delta_{\text{open}}^+)$ denotes the pair that corresponds to P_{open} in the parameter set, then $(H_q, s_i^r\Delta_{\text{open}}^+)$ corresponds to $s_i \times P_{\text{open}}$. We observe that the new positive system differs from Δ_{open}^+ by one complex root. Thus, by the dimension formula 7.1, the K orbits associated to $(H_{0,r}, s_i^r\Delta_{\text{open}}^+)$ are co-dimensional one orbits. Any other orbit is attached to a pair $(H_{0,q}, w\Delta_{\text{open}}^+)$ with the length of w bigger than 1 and thus its dimension is lower.

If $p = q$, then

$$P_1 = (p,+)(p+1,-)(1,2p)\dots(p-1,p+2)$$

and

$$P_2 = (p+1,+)(p,-)(1,2p)\dots(p-1,p+2)$$

parametrize two Langlands orbits associated to $H_{0,q-1}$. All other orbits attached to $H_{0,q-1}$ are obtained by letting elements in $W(\mathfrak{g},\mathfrak{h}_{q-1})$ act on P_1 and P_2. It can be easily verified that, in this case, the same procedure will produce orbits of lower dimensions. ■

Remark 7.5. The method described in this section selects the elements in the parameter set that are associated to codimensional one orbits. Associated to that element is an equivalence class in $W(\mathfrak{g},\mathfrak{h}_r)/W(G_0,H_{0,r})$. Let w be a representative of the equivalence class. We can build the pair $(H_r, w\Delta^+_{\mathrm{open}})$ that is attached to the K orbit. The corresponding Borel subalgebra is then a base point of the orbit. This is an idea of Vogan that is explained in [3].

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