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On some causal and conformal groups

Wolfgang Bertram

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Abstract. We determine the causal transformations of a class of *causal symmetric spaces* (Th. 2.4.1). As a basic tool we use *causal imbeddings* of these spaces as open orbits in the conformal compactification of Euclidean Jordan algebras. In the first chapter we give elementary constructions of such imbeddings for the classical matrix-algebras. In the second chapter we generalize these constructions for arbitrary semi-simple Jordan algebras: we introduce *Makarevič spaces* which are open symmetric orbits in the conformal compactification of a semi-simple Jordan algebra. We describe examples and some general properties of these spaces which are the starting point of an algebraic and geometric theory we are going to develop in subsequent work [Be96b].

0. Introduction

0.1 The classical theorems of Liouville **and** Lie**.** A *conformal transformation* of the Euclidean space $V = \mathbb{R}^n$ is a locally defined diffeomorphism $\phi : V \supset V_1 \to V_2 \subset V$ such that for all $x \in V_1$, the differential $D\phi(x)$ of ϕ at x is a similarity (multiple of an orthogonal transformation), that is, $D\phi(x)$ belongs to the linear group $G = \mathrm{O}(n) \times \mathbb{R}^+$ generated by the orthogonal group and the multiples of the identity. The translations by vectors of V and the elements of G are trivial examples of such transformations. One can check that the inversion $x \mapsto \frac{x}{||x||^2}$ is conformal. A classical theorem of Liouville (1850) states that every conformal transformation of $\mathbb{R}^3$ (of class $\mathcal{C}^4$) is in fact rational and is a composition of the previously described ones. S. Lie has generalized this theorem for general $n > 2$ and general non-degenerate quadratic forms replacing the Euclidean norm; the inversion is still defined by the same formula. We would like to emphasize that the theorem contains a "local - global" statement: from a *local* property and $\mathcal{C}^4$-regularity we can deduce *rationality* and a *global* extension.

The case $n = 4$, with the Lorentz pseudo-metric of signature $(3,1)$, leads to the *causal group of the* Minkowski-*space*: let Ω be the associated forward light-cone and $G = G(\Omega) = \{g \in \mathrm{Gl}(V) | g \cdot \Omega = \Omega\}$ be the group

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of linear automorphisms of Ω. The condition $D\phi(x) \in G$ (for all x in some domain of V) is equivalent to $D\phi(x) \cdot \Omega = \Omega$, and we then say that ϕ is a local *automorphism of the flat causal structure defined on V by Ω.* Knowing that $G(\Omega) = \mathrm{O}(3,1)^+ \times \mathbb{R}^+$, we can apply a version of LIE's theorem in order to conclude that such local automorphisms are in fact rational and given by a composition of the trivial ones ($G(\Omega)$ and the translations) and the negative of the inversion, $x \mapsto -\frac{x}{\langle x,x \rangle}$. In [Be96a] we have generalized these results in the framework of Jordan algebras. Before discussing this general context, let us mention one other important and very typical example.

0.2 Causal transformations of the space of Hermitian matrices. Let V be the space $\mathrm{Herm}(r,\mathbb{C})$ of Hermitian $r \times r$-matrices, $\Omega \subset V$ the open cone of positive definite Hermitian matrices and $G := G(\Omega)$ the group of all linear invertible maps of V which map Ω onto itself. Then we define, as above, a (local) *causal automorphism of the flat causal structure defined on V by Ω* to be a locally defined diffeomorphism ϕ such that $D\phi(x) \cdot \Omega = \Omega$ (this is equivalent to $D\phi(x) \in G$) for all x where ϕ is defined. It is a special case of our generalized LIOUVILLE-theorem for Jordan algebras [Be96a, Th.2.3.1] that every such transformation is in fact rational and is a composition of a translation, of elements of G and of $-j$, where j is the matrix-inversion $j(X) = X^{-1}$, and these transformations form a *group of birational transformations* of V the identity component of which is isomorphic to $\mathrm{SU}(r,r)$. Now, it is known that the group $\mathrm{U}(r)$ is, via the CAYLEY-transform, locally causally isomorphic to V. Because our LIOUVILLE-theorem is of *local* nature, we can conclude that the causal group of $\mathrm{U}(r)$ is also isomorphic to $\mathrm{SU}(r,r)$, thus giving a positive answer to a conjecture by Segal [Se76, p.35]. Moreover, the groups $\mathrm{U}(p,q)$ ($p+q=r$) can be causally imbedded into the group $\mathrm{U}(r)$ by the *Potapov-Ginzburg transformation* (see [AI89]) which assigns to a graph Γ_g of $g \in \mathrm{U}(p,q)$ the graph $P \cdot \Gamma_g$, where P is the endomorphism of $(\mathbb{C}^p \times \mathbb{C}^q)^2$ defined by $P(x_1,x_2,y_1,y_2) = (y_1,x_2,x_1,y_2)$; one easily checks that $P \cdot \Gamma_g$ is in fact a graph belonging to an element $P(g)$ of $\mathrm{U}(r)$. There is a causal structure on $\mathrm{U}(p,q)$ for which the transformation P is a causal map, and we may conclude that the "causal pseudogroup" of $\mathrm{U}(p,q)$ (see below for a precise definition) is also isomorphic to $\mathrm{SU}(r,r)$.

The general problem we are interested in is the following: given a space having a *conformal* or *causal* structure (as $\mathrm{U}(p,q)$ in the example), we would like to determine all transformations of this space (even those which are only locally defined) preserving the causal or conformal structure. As the last example shows, this problem may in some cases be devided into two sub-problems: first determine whether the given causal or conformal space locally looks like a matrix space (or, more generally, a Jordan algebra) with a "constant" causal or conformal structure, then use the "local-global" statement of our LIOUVILLE-theorem in order to completely describe the conformal (or causal) transformations of the given space. The first problem is of geometrical, and the second one of analytical nature. Since the analytical problem is entirely resolved by our LIOUVILLE-problem, it is the geometrical problem we are interested in now. Generalizing the example of the group $\mathrm{U}(p,q)$, we will find and describe a fairly large class of spaces having a conformal or causal structure which is locally equivalent to a Jordan algebra with its flat (i.e. "constant") structure. It remains an open problem to give an intrinsic criterion permitting to decide when, in general, a conformal or causal structure is actually flat in this sense. In fact, it even is not obvious what the suitable general definition of "conformal structure" should be. We will briefly discuss this problem at the end of the introduction.

0.3 Causal symmetric spaces and their causal groups. The notion of "causal structure" is less problematic than the general notion of "conformal structure": it is given by a field of cones $(C_x)_{x\in M}$ on a manifold M, where C_x is a regular (i.e. open, convex and pointed) cone in the tangent space T_xM. A *local causal diffeomorphism* between two manifolds M and N with causal structures $(C_x)_{x\in M}$, $(C'_y)_{y\in N}$ is a locally defined diffeomorphsm ϕ such that $T_x\phi\cdot C_x = C'_{\phi(x)}$ for all x where ϕ is defined. If $M = N$, then these maps form an object called the *causal pseudogroup of* M; it is not a group because the maps are in general not defined everywhere.

We now describe some spaces X having a causal structure which is locally equivalent to the flat causal structure given by the symmetric cone Ω of some Euclidean Jordan algebra V (the cone Ω of positive definite Hermitian matrices introduced above is such a cone; for the general notion *cf.* [FK94].) The spaces X we are interested in are actually *symmetric spaces*, i.e. homogeneous spaces $X = G/H$ under the action of some Lie group G such that H is open in the fixed point group G^σ of some involution σ of G, and they are furthermore *causal symmetric spaces* in the sense that G preserves the given causal structure. The following table shows that the whole causal pseudogroup of X is actually much bigger than G. The table should be read as follows: the spaces $X = L/H$, respectively $X = L'/H$ are locally causally isomorphic to the Euclidean Jordan algebra V given to the right in the corresponding line; then, by the generalized LIOUVILLE-theorem, the corresponding causal pseudogroup can be identified with the group $\mathrm{Co}(V)_0$ given for each of the five types. The precise statement can be found in Theorem 2.4.1. (In the following $p+q=n$.)

0.3.1 Table of causal pseudogroups.

I. $V_n = \mathrm{Herm}(n,\mathbb{C})$, $n>1$.
Causal groups: $\mathrm{Co}(V_n)_0 = \mathrm{SU}(n,n)$, $\mathrm{Co}(V_n\times V_n)_0 = \mathrm{SU}(n,n)\times\mathrm{SU}(n,n)$.

L	L'	H	V
$\mathrm{SU}(n,n)$	$\mathrm{SU}(n,n)$	$\mathrm{Sl}(n,\mathbb{C})\times\mathbb{R}^+$	$V_n\times V_n$
$\mathrm{U}(p,q)\times\mathrm{U}(p,q)$	$\mathrm{Gl}(n,\mathbb{C})$	$\mathrm{U}(p,q)$	V_n
$\mathrm{SO}^*(2n)$	$\mathrm{SO}(n,n)$	$\mathrm{SO}(n,\mathbb{C})$	V_n
$\mathrm{Sp}(2n,\mathbb{R})$	$\mathrm{Sp}(n,n)$	$\mathrm{Sp}(n,\mathbb{C})$	V_{2n}

II. $V_n = \mathrm{Sym}(n,\mathbb{R})$, $n>1$.
Causal groups: $\mathrm{Co}(V_n)_0 = \mathrm{Sp}(n,\mathbb{R})$, $\mathrm{Co}(V_n\times V_n)_0 = \mathrm{Sp}(n,\mathbb{R})\times\mathrm{Sp}(n,\mathbb{R})$

$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{Sl}(n,\mathbb{R})\times\mathbb{R}^+$	$V_n\times V_n$
$\mathrm{U}(p,q)$	$\mathrm{Gl}(n,\mathbb{R})$	$\mathrm{O}(p,q)$	V_n
$\mathrm{Sp}(n,\mathbb{R})\times\mathrm{Sp}(n,\mathbb{R})$	$\mathrm{Sp}(n,\mathbb{C})$	$\mathrm{Sp}(n,\mathbb{R})$	V_{2n}

III. $V_n = \mathrm{Herm}(n,\mathbb{H})$,
Causal groups: $\mathrm{Co}(V_n)_0 = \mathrm{SO}^*(4n)$, $\mathrm{Co}(V_n\times V_n)_0 = \mathrm{SO}^*(4n)\times\mathrm{SO}^*(4n)$

$\mathrm{SO}^*(4n)$	$\mathrm{SO}^*(4n)$	$\mathrm{SO}^*(2n)\times\mathbb{R}^+$	$V_n\times V_n$
$\mathrm{U}(2p,2q)$	$U^*(2n)$	$\mathrm{Sp}(p,q)$	V_n
$\mathrm{SO}^*(2n)\times\mathrm{SO}^*(2n)$	$\mathrm{SO}(2n,\mathbb{C})$	$\mathrm{SO}^*(2n)$	V_{2n}

IV. $V_n=\mathbb{R}\times\mathbb{R}^{n-1},\ n>2$.
Causal groups: $\mathrm{Co}(V_n)_0=\mathrm{SO}(2,n)$, $\mathrm{Co}(V_n\times V_n)_0=\mathrm{SO}(2,n)\times\mathrm{SO}(2,n)$.

$\mathrm{SO}(2,n)$	$\mathrm{SO}(2,n)$	$\mathrm{SO}(1,n-1)\times\mathbb{R}^+$	$V_n\times V_n$
$\mathrm{SO}(n)\times S^1$	$\mathrm{SO}(1,n-1)\times\mathbb{R}^+$	$\mathrm{SO}(n-1)$	V_n
$\mathrm{SO}(2,n-1)$	$\mathrm{SO}(1,n)$	$\mathrm{SO}(1,n-1)$	V_n

V. $V_3=\mathrm{Herm}(3,\mathbb{O})$,
Causal groups: $\mathrm{Co}(V_3)_0=E_{7(-25)}$, $\mathrm{Co}(V_3\times V_3)_0=E_{7(-25)}\times E_{7(-25)}$

$E_{7(-25)}$	$E_{7(-25)}$	$E_{6(-25)}\times\mathbb{R}^*$	$V_3\times V_3$
$\mathrm{SU}(6,2)$	$\mathrm{SU}^*(8)$	$\mathrm{Sp}(3,1)$	V_3
$E_{6(-14)}\times\mathbb{R}^+$	$E_{6(-14)}\times\mathrm{U}(1)$	$F_{4(-20)}$	V_3

Let us make a few comments on this table: the example of the group $U:=\mathrm{U}(p,q)$ mentioned above appears in the second line for type I, where the group U is considered as symmetric space $U\times U/\,\mathrm{dia}(U\times U)$. Furthermore, in each of the cases I - V there appears exactly one *compact* symmetric space and one Riemannian symmetric space of *non-compact type* (in cases I-III it appears for $p=n,\ q=0$; we assume $p+q=n$); the latter is actually isomorphic to the *symmetric cone* associated to the Jordan algebra V. These spaces are *c-dual* to each other in the sense of duality of symmetric spaces. More generally, the spaces L/H and L'/H are c-dual to each other, the spaces L/H being *compactly causal* and the spaces L'/H *non-compactly causal.* The spaces appearing in the first line for each type are self-dual; they are known as causal symmetric spaces *of Cayley type.*

The *irreducible* causal symmetric spaces have been classified by G. Ólafsson (see [FÓ95]); "most" of them appear in our list. Those which do not appear here fall into two classes: first, the semi-simple parts of the reductive spaces appearing in our list - for example, the group $\mathrm{U}(p,q)$ is reductive, and its semi-simple part $\mathrm{SU}(p,q)$ is included in the classification by Ólafsson. Such spaces are hypersurfaces in a space appearing in our list. We conjecture that they are not "causally flat" and that their causal pseudogroups are small, possibly reduced to the affine group of the affine connection belonging to the underlying symmetric space. Secondly, there are two other series of causal symmetric spaces (the groups $\mathrm{SO}(n,2)$ and the spaces $\mathrm{SO}(2,p+q)/\,\mathrm{SO}(p,1)\times\mathrm{SO}(q,1)$ with $\min(p,q)>1$) as well as four exceptional spaces which do not appear in our list; it remains an intriguing question whether these spaces can be related to Jordan algebras or not.

In [Be96b] we will show that the "flat" realization of the spaces given in the table is very useful to study problems related to their geometry and harmonic analysis.

0.4. Makarevič spaces. The Jordan algebras V corresponding to types I, II and III in table 0.3.1 are spaces of *symmetric*, resp. *Hermitian* matrices. For these types, the local equivalence of the spaces L/H and L'/H with V can be established by very elementary methods. We do this in Chapter one. This chapter could be read by an undergraduate student having no knowledge in Jordan theory but a good understanding of linear algebra. The basic idea is to analyze a linear transformation by its graph, and in particular to understand the notion of *adjoint operator* in this way. This permits to interprete the realizations for the types I, II and III as natural analogues of the classical SIEGEL-space (Theorem 1.7.1).

However, for a deeper understanding one needs the general context of *Jordan algebras* which we introduce in Chapter two. The fact that a homogeneous space $X = G/H$ is locally conformally or causally equivalent to such an algebra V will be made precise in the following way: let $\mathrm{Co}(V)$ be the *conformal* or *Kantor-Koecher-Tits group* associated to V and V^c be the *conformal compactification* of V; this is an open dense and $\mathrm{Co}(V)$-equivariant imbedding of V into a compact space V^c. If G is a subgroup of $\mathrm{Co}(V)$ and $x \in V^c$ such that the orbit $G/H \cong G \cdot x$ is open in V^c, then the homogeneous space $X := G/H$ inherits by restriction from V a flat G-invariant conformal structure, and our generalized LIOUVILLE-theorem implies that $\mathrm{Co}(V)$ can be identified with the corresponding pseudogroup of conformal transformations.

Without loss of generality we can assume that the base point x is the origin 0 of $V \subset V^c$. If now we restrict our attention to *symmetric* spaces $X = G/H$, then work of A.A. Rivillis [Ri70] and B.O. Makarevič [Ma73] has shown that the space X can be realized in the form

$$X = X^{(\alpha)} := \mathrm{Co}(V)_0^{(j\alpha)_*} \cdot 0,$$

where $j(x) = x^{-1}$ is the inverse in the Jordan algebra V, α is an invertible linear map of V having the property that $(j\alpha)^2 = \mathrm{id}_V$ and $(j\alpha)_*$ is the involution of $\mathrm{Co}(V)$ given by conjugation with $j\alpha$. An automorphism as upper index of a group denotes as usual the fixed point group, and lower index 0 denotes the identity component. Clearly $j^2 = \mathrm{id}_V$ and $(-j)^2 = \mathrm{id}_V$, hence we can choose $\alpha = \mathrm{id}_V$ or $\alpha = -\mathrm{id}_V$, but we can also take any involutive automorphism of V or its negative. We will call the space $X^{(\alpha)}$ a *Makarevič space* since such spaces have been classified by B.O. Makarevič in [Ma73]. However, since the main interest of [Ma73] lay in the classification problem, the simplicity of the construction of the spaces $X^{(\alpha)}$ by the above formula is rather hidden there. The formula indeed defines an *open symmetric orbit* in V^c; we will give the simple proof in Proposition 2.2.1. Examples of Makarevič spaces, besides the causal symmetric spaces listed in table 0.3.1, are given by *orthogonal groups*, *general linear groups*, *symmetric cones* and their non-convex analogues, *Hermitian* and *pseudo-Hermitian symmetric spaces*. It seems very interesting that methods well-known from one of these classes can be adapted to others of these classes where they are less obvious. In particular, we will generalize the algebraic methods developed by Koecher and Loos for Hermitian symmetric spaces (see [Lo77]) in subsequent work [Be96b].

In this work we will only describe some basic features of Makarevič spaces. The first one gives a particularly nice description of *c-duality*: the spaces $X^{(\alpha)}$ and $X^{(-\alpha)}$ are *c-duals* of each other (Prop.2.3.2); we have observed this duality already at the example of table 0.3.1. Another basic feature is the existence of *Cayley-transformed* realizations in the case where α is an *involution* of V. It will carry the space $X^{(\alpha)}$ onto a *generalized tube domain* (Example 2.2.6). Furthermore, it seems remarkable that all Makarevič spaces appear as real forms of pseudo-Hermitian symmetric spaces (2.3.3). Finally, we prove that the classification by Makarevič is indeed complete in the Euclidean case, given by table 0.3.1; the general classification in [Ma73] is given without proof of completeness (actually one class of pseudo-Hermitian spaces of tube type is missing in [Ma73].)

0.5 The conformal group of a Makarevič space. Our main theorem on causal groups (Theorem 2.4.1) generalizes to the case of general Makarevič spaces: the group $\mathrm{Co}(V)$ can be characterized as the conformal pseudo-group of the space $X^{(\alpha)}$. But as mentioned in Section 0.2, already the definition of "conformal structure" is not obvious in this general case. There are different possibilities to define it:

a) in [Be96a] we used the notion of a *field of groups* to define a very general kind of conformal structure, closely related to the so called *G-structures.* When the field of groups is "constant", given by the structure group of a semi-simple Jordan algebra, we obtain the "flat conformal structure of a semi-simple Jordan algebra". This notion is not very geometrical but convenient for proving the LIOUVILLE-theorem.

b) We can make the previously mentioned notion more geometric by using the following characterization of the structure group of a semi-simple Jordan algebra: if Δ is the *norm-polynomial* of V (in the case of matrix algebras this is the usual determinant, in the case of the Lorentz cone this is the Lorentz quadric), then $\mathrm{Str}(V)$ is the group of invertible linear mappings preserving Δ up to a factor (see [FK94, p.161]). If we define a *conformal structure* to be a field of polynomials, up to equivalence by nowhere vanishing functions, then the structure given by Δ turns V and V^c into a conformal space having $\mathrm{Co}(V)$ as its conformal group. This notion is very close to the classical notion of the conformal structure of a Riemannian manifold: the metric tensor field is just replaced by a symmetric tensor field of a higher degree, and our LIOUVILLE theorem generalizes exactly the classical one.

c) In [GK95] Gindikin and Kaneyuki propose a definition which has the advantage to apply also to some *Jordan triple systems* and the disadvantage not to cover the classical case of Riemannian conformal structure: they essentially define a *conformal structure* to be a distribution of conical subvarieties. In the Jordan algebra case, this is the set of zeros of the norm-polynomial Δ.

d) In the case of the classical matrix spaces we consider the "generalized line structure" of a Grassmann manifold given by incidence relations of subspaces (Section 1.8). This can be seen as a "global" version of the conformal structure introduced by Gindikin and Kaneyuki. The conformal group can then be characterized as the group preserving this "generalized line structure" (Theorem 1.8.1). In this context, our LIOUVILLE theorem thus shows a remarkable similarity with the *fundamental theorem of projective geometry* characterizing the projective group as the group preserving the line-structure of projective space.

See [Wey23] for an interesting discussion of this theorem, seeing it in the same context as the classical LIOUVILLE theorem.

e) It is in fact possible, following the viewpoint of WEYL, to prove the fundamental theorem of projective geometry by Jordan-methods. However, none of the previously mentioned notions of conformal structure does apply to this case.

Let us remark that in [Gi92] some special cases of conformal structures are studied, and the author adds that 'it would be interesting to develop a general theory of such structures' - we hope that this work might be a step in this direction.

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1. Elementary construction of causal and conformal imbeddings

1.0 Basic notions related to Jordan algebras.— In this chapter we will be concerned with *special Jordan algebras*; these are subspaces of some associative endomorphism-algebra $\mathrm{End}(E)$ of a vector space E which are stable under the *Jordan-product* $AB := \frac{1}{2}(A \circ B + B \circ A)$. Thus $V := \mathrm{End}(E)$ becomes a *commutative* algebra which is not associative, but satisfies the so-called *Jordan identity*: $A(A^2B) = A^2(AB)$. We will always assume that a Jordan algebra V contains a *unit element* $e \in V$; if $V = \mathrm{End}(E)$, then e is the identity operator I. There is a notion of *inverse* in a Jordan algebra. We will write $j(x) = x^{-1}$ for the inverse of $x \in V$; then j is a *birational* map of V. If $V = \mathrm{End}(E)$, then j is just the ordinary inverse which is clearly rational. For any birational map ϕ of V we denote by $j_*(\phi)$ the birational map $j \circ \phi \circ j$. The *structure group* $\mathrm{Str}(V)$ of V is defined as the group of invertible linear transformations g of V for which $j_*(g)$ is again linear. Then j_* defines an *involution* (an automorphism of order 2) of $\mathrm{Str}(V)$. The orbit $\Omega := \mathrm{Str}(V)_0 \cdot e$ (the subscript zero denoting the identity component) is then open in V and is a *symmetric space*, $\Omega \cong \mathrm{Str}(V)_0 / \mathrm{Str}(V)_0^{j_*}$.

The *conformal* or *Kantor-Koecher-Tits group* of the (semi-simple) Jordan algebra V is the group $\mathrm{Co}(V)$ of birational transformations generated by the translations τ_v, $v \in V$ (where $\tau_v(x) = x + v$), the elements of $\mathrm{Str}(V)$ and j. We identify V with the subgroup of translations τ_v, $v \in V$, and let P be the subgroup of $\mathrm{Co}(V)$ generated by $\mathrm{Str}(V)$ and jVj. Then the map

$$V \mapsto V^c := \mathrm{Co}(V)/P, \quad v \mapsto \tau_v P$$

is an open dense imbedding into a compact space, *cf.* [Be96a, Th.2.4.1]; we call it the *conformal compactification of* V. It is clear from the definition that the conformal group acts on V^c; in particular, addition by vectors of V and multiplication by scalars is defined on V^c, and maps like $-\operatorname{id}_V$ will always be understood as defined on V^c in this sense.

Besides Jordan algebras we will also use (implicitly) some special cases of *Jordan triple systems*, namely spaces of *skew-Hermitian matrices*. These spaces are usually considered as Lie algebras (of the corresponding unitary groups), but it will be rather important to keep in mind that they are considered here as Jordan triple systems and not as Lie algebras. Let us explain briefly what we mean by this: the *Jordan triple product* of a Jordan algebra V is given by the formula $\{a,b,c\} := a(bc) - b(ac) + (ab)c$; it is symmetric in a and c and satisfies an additional identity which is used to define abstract Jordan triple-systems (see [Sa80], [Lo77]). If α is an automorphism of a Jordan algebra, then the $+1$-eigenspace V^α is a Jordan subalgebra of V and the -1-eigenspace $V^{-\alpha}$ is stable for the triple product and is thus a sub-triple system of V. The triple product in a space of skew-Hermitian matrices arises in this way from the space of all matrices. (We conjecture that in fact all Jordan triple systems arise as -1-eigenspaces of a Jordan algebra-involution; see [Be96b].) There is also a notion of a *conformal group* related to an abstract triple system (see [Sa80], [Lo77]), but its description is more difficult than for Jordan algebras because there is no notion of inverse in a triple system.

1.1 The matrix-algebras $M(n,\mathbb{F})$. Let $\mathbb{F}$ be the field of real or complex numbers or the skew-field of quaternions and E be an n-dimensional vector space over $\mathbb{F}$ (acting from the left on E). The space $V := \operatorname{End}_{\mathbb{F}}(E)$ of $\mathbb{F}$-linear endomorphisms of E is a semi-simple Jordan algebra with Jordan-product $A\cdot B = \frac{1}{2}(AB+BA)$. (If $\mathbb{F}=\mathbb{H}$, then this is just an $\mathbb{R}$-algebra.) Using a basis of E, we can identify E with $\mathbb{F}^n$ and $\operatorname{End}(E)$ with the matrix-algebra $M(n,\mathbb{F})$. (The matrix of $A\in\operatorname{End}(\mathbb{F}^n)$ is defined by $Ae_i = \sum_j A_{ji}e_j$, and two matrices are multiplied by the rule $(A_{ij})\cdot(B_{jl}) = (C_{il})$, $C_{il} = \sum_j B_{jl}A_{ij}$.)

The (identity component of the) *structure group* of V is given by the action of the group $\operatorname{Gl}(E)\times\operatorname{Gl}(E)$ on V by $(g,h)\cdot A = g\circ A\circ h^{-1}$. The involution j_* is given by $j_*(g,h) = (h,g)$, where $j(X) = X^{-1}$ is the Jordan inverse. The open orbit $\Omega = \operatorname{Str}(V)_0\cdot e$ (here $e = \operatorname{id}_E$) is the group $\operatorname{Gl}(E)\subset\operatorname{End}(E)$ (or, if one prefers, $\operatorname{Gl}(n,\mathbb{F})\subset M(n,\mathbb{F})$), viewed as symmetric space with geodesic symmetry j at the origin I.

The graph-imbedding, and the conformal group. The *conformal compactification* V^c of $V = \operatorname{End}(E)$ is the Grassmannian $G_{2n,n}(\mathbb{F})$ of n-dimensional subspaces of $E\oplus E$, where the imbedding $\Gamma: V\to V^c$ is given by identifying $A\in\operatorname{End}(E)$ with its *graph* $\Gamma_A := \{(x,Ax)|\, x\in E\}\in G_{2n,n}$. The image of this imbedding is dense in $G_{2n,n}$ because $W\in G_{2n,n}$ is a graph if and only if the projection $pr_1^W : W\mapsto E\times 0$ onto the first factor is a bijection; the set of "non-graphs" is hence an algebraic subvariety of $G_{2n,n}$ of strictly lower dimension, defined by $\operatorname{Det}(pr_1^W) = 0$. (The determinant may be defined by choosing an auxiliary Euclidean metric on $E\oplus E$.)

The group $\operatorname{Gl}(2n,\mathbb{F}) = \operatorname{Gl}(E\oplus E)$ acts on the Grassmannian $G_{2n,n}$ in the natural way, and one easily verifies that $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\Gamma_A = \Gamma_{(aA+b)(cA+d)^{-1}}$ for $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\operatorname{Gl}(2n,\mathbb{F})$ and $A\in M(n,\mathbb{F})$. Thus the rational action of $\operatorname{Gl}(2n,\mathbb{F})$ on

$V \subset V^c$ is given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot X = (aX+b)(cX+d)^{-1}.$$

The effective group of this action is $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})$. It is clear that the translations are given by the matrices of the form $\left(\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right)$ (where $I = I_n$ denotes the $n \times n$-identity matrix), the elements of (the identity component of) the structure group by the invertible matrices of the form $\left(\begin{smallmatrix} g & 0 \\ 0 & h \end{smallmatrix}\right)$, and the inversion $j \in \mathrm{Co}(V)$ is induced by $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$. These matrices generate $\mathbb{P}\,\mathrm{Gl}(2n,\mathbb{F})$, which is thus the identity component of the conformal group $\mathrm{Co}(V)$. Remark that $-\mathrm{id}_V \in \mathrm{Co}(V)$ is induced by the matrix $\left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)$, which is conjugated to the matrix of j by the *real Cayley transform* ("rotation of angle $\frac{\pi}{2}$") $R = \left(\begin{smallmatrix} I & -I \\ I & I \end{smallmatrix}\right)$.

1.2. The matrix spaces $\mathrm{Sym}(A,\mathbb{F})$ and $\mathrm{Herm}(A,\mathbb{F})$, the orthogonal and unitary groups. Notation being as above, let $\langle \cdot , \cdot \rangle$ be a non-degenerate bilinear or sesquilinear form on E. We recall that sesquilinear forms are defined by bi-additivity and the property $\langle \lambda x, \mu y \rangle = \lambda \langle x, y \rangle \varepsilon(\mu)$, where ε denotes an anti-automorphism of the field $\mathbb{F}$. The field $\mathbb{R}$ has no non-trivial anti-automorphism, and $\mathbb{C}$ has only one, complex conjugation. The field $\mathbb{H}$ has many anti-automorphisms, but there is one among them which is canonical, namely the conjugation which equals one on the center $Z(\mathbb{H}) = \mathbb{R}$ and minus one on the pure quaternions $\Im(\mathbb{H})$. Usually, sesquilinearity is defined with respect to this conjugation, and if we use another anti-automorphism ε, we will call the form more precisely ε*-sesquilinear*. Non-trivial bilinear forms exist if and only if $\varepsilon = \mathrm{id}_{\mathbb{F}}$ is an anti-automorphism, i.e. if $\mathbb{F}$ is commutative.

On $E = \mathbb{F}^n$ the *standard* $\mathbb{R}$*-bilinear form* is given by $(x|y) = x^t y = \sum_i x_i y_i$. Then any ε-sesquilinear form can be written as $\langle x, y \rangle = (x|A\varepsilon(y)) = x^t A \varepsilon(y)$ with $A \in M(n,\mathbb{F})$ and $(\varepsilon(y))_i := \varepsilon((y)_i)$. With respect to the standard basis of $\mathbb{F}^n$, A is given by the matrix $(A_{ij}) = (\langle e_i, e_j \rangle)$. The form is non-degenerate if and only if the associated matrix is non-singular. The *adjoint operator* X^* *of* $X \in \mathrm{End}(E)$ (with respect to the given form) is then defined by the relation

$$(*) \qquad \forall u, v \in E: \quad \langle Xu, v \rangle = \langle u, X^* v \rangle$$

and is given in matrix representation by the formula

$$X^* = \varepsilon^{-1} A^{-1} X^t A \varepsilon,$$

where Y^t is the transpose of a matrix Y, and the matrix $\varepsilon(Y) := \varepsilon Y \varepsilon^{-1}$ is obtained by applying ε to every coefficient of the matrix Y (if ε is complex conjugation, this matrix is usually denoted by $\overline{Y}$.) We write

$$\mathrm{Herm}(A,\varepsilon,\mathbb{F}) := \{X \in M(n,\mathbb{F}) |\, A^{-1} X^t A = \varepsilon X \varepsilon^{-1}\},$$

$$\mathrm{Aherm}(A,\varepsilon,\mathbb{F}) := \{X \in M(n,\mathbb{F}) |\, A^{-1} X^t A = -\varepsilon X \varepsilon^{-1}\},$$

for the spaces of Hermitian, resp. skew-Hermitian operators (with respect to to the ε-sesquilinear form given by the matrix A.) If the form is bilinear (i.e.

$\varepsilon = id$), then we denote these spaces also by $\mathrm{Sym}(A,\mathbb{F})$, resp. $\mathrm{Asym}(A,\mathbb{F})$. If we just write $\mathrm{Herm}(A,\mathbb{C})$ or $\mathrm{Herm}(A,\mathbb{H})$, we always assume that ε is the canonical conjugation of the base field introduced above.

It should be remarked that in general there are *two* possibilities to define the adjoint operator, but they coincide if the given form satisfies one of the symmetry conditions $\langle u,v\rangle = \varepsilon(\langle v,u\rangle)$ or $\langle u,v\rangle = -\varepsilon(\langle v,u\rangle)$ for all $u,v\in E$.

1.2.1 Some special isomorphisms. If $\mathbb{F}=\mathbb{C}$, then it is easily seen that multiplication by i defines an $\mathbb{R}$-isomorphism $\mathrm{Herm}(A,\mathbb{C})\to\mathrm{Aherm}(A,\mathbb{C})$. For $\mathbb{F}=\mathbb{H}$, a similar statement is true: for any invertible element $u\in\mathbb{H}$ let u_* be the conjugation by u in $\mathbb{H}$; then for any anti-automorphism ε of $\mathbb{H}$, $u_*\circ\varepsilon$ is again an anti-automorphism of $\mathbb{H}$. The right multiplication by u is defined by $r_u:\mathbb{H}^n\to\mathbb{H}^n,(x_i)\mapsto(x_i\cdot u)$; this is a $\mathbb{H}$-linear map (recall that $\mathbb{H}$ acts from the left) and is represented by the matrix $(u\delta_{ij})=uI$. For $X\in M(n,\mathbb{H})$, the $\mathbb{H}$-linear map $X\circ r_u$ is represented by the matrix $(X\cdot uI)_{ij}=(u\cdot X_{ij})$. Let us now fix $u\in\mathbb{H}$, an anti-automorphism ε of $\mathbb{H}$ such that $\varepsilon(u)=-u$ and $A\in M(n,\mathbb{H})$ such that $A\cdot uI = uI\cdot A$, then

$$\mathrm{Herm}(A,\varepsilon,\mathbb{H})\to\mathrm{Aherm}(A,u_*\circ\varepsilon,\mathbb{H}),\quad X\mapsto X\cdot uI$$

is an $\mathbb{R}$-isomorphism. (Using the assumptions, one verifies the following equivalence: $(A^{-1}XA)_{ij}=\varepsilon(X_{ji}) \Leftrightarrow (A^{-1}\cdot X\cdot uI\cdot A)_{ij} = -u\varepsilon(X_{ji})u^{-1}$.) The above conditions are satisfied, for example if $u=j$ (defined by $\mathbb{H}=\mathbb{C}\oplus j\mathbb{C}$), ε the canonical involution of $\mathbb{H}$ or its composition with j_* and A any matrix with real coefficients.

1.2.2 The structure group, the orthogonal and unitary groups. Keeping notation, let us write $a(X):=X^*$. Then the map a "adjoint operator" is an anti-automorphism of the *associative* algebra $\mathrm{End}(E)$ and hence an automorphism of the *Jordan* algebra $V=\mathrm{End}(E)$. Its fixed point set (i.e. $\mathrm{Herm}(A)$) is a Jordan sub-algebra, and its -1-eigenspace (i.e. $\mathrm{Aherm}(A)$) is a Jordan sub-triple-system of V (see Section 1.0). We remark that a is an *involution* if the given form is *Hermitian* or *skew-Hermitian* (i.e. $\varepsilon(A)=\pm A^t$); we will mainly be interested in this case. The (identity component of) the structure group of $\mathrm{Herm}(A,\varepsilon,\mathbb{F})$ is obtained from the action of the elements of $\mathrm{Gl}(n,\mathbb{F})\times\mathrm{Gl}(n,\mathbb{F})$ acting on $M(n,\mathbb{F})$ and commuting with a; it is given by

$$\mathrm{Gl}(n,\mathbb{F})\times\mathrm{Herm}(A,\varepsilon,\mathbb{F})\to\mathrm{Herm}(A,\varepsilon,\mathbb{F}),\quad (g,X)\mapsto gXg^*.$$

It is clear from this formula that the orbit of I under this action is the open set $\{gg^*|g\in\mathrm{Gl}(n,\mathbb{F})\}\subset\mathrm{Herm}(A,\varepsilon,\mathbb{F})$ which can be considered as the symmetric space $\mathrm{Gl}(n,\mathbb{F})/\mathrm{U}(A,\varepsilon,\mathbb{F})$. Here we use the notation

$$\mathrm{O}(A,\mathbb{F})=\{g\in\mathrm{Gl}(n,\mathbb{F})|\,A^{-1}g^tA=g^{-1}\}$$

and

$$\mathrm{U}(A,\varepsilon,\mathbb{F}):=\{g\in\mathrm{Gl}(n,\mathbb{F})|\,A^{-1}g^tA=\varepsilon g^{-1}\varepsilon^{-1}\}$$

for the orthogonal, resp. ε-unitary groups of the form given by A. We will just write $\mathrm{U}(A,\mathbb{C})$ or $\mathrm{U}(A,\mathbb{H})$ if ε is the canonical conjugation of the base field, and we will just write $\mathrm{U}(A)$ and $\mathrm{O}(A)$ if the specification of the base field is not important. The symbol $\mathbb{P}$ denotes the quotient with respect to the central subgroup of multiples of the identity matrix.

1.2.3 Classical notation and special isomorphisms. In the classical notation, for $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$ we have $\mathrm{O}(I_{p,q},\mathbb{R})=\mathrm{O}(p,q)$ (where $I_{p,q}=\begin{pmatrix} I_p & 0 \\ 0 & -I_q\end{pmatrix}$), $\mathrm{O}(\begin{pmatrix} 0 & I_n \\ -I_n & 0\end{pmatrix},\mathbb{R})=\mathrm{Sp}(n,\mathbb{R})$, etc. For $\mathbb{F}=\mathbb{H}$, let us recall that the identification $\mathbb{H}=\mathbb{C}\oplus\mathbb{C}\cdot j$ induces an inclusion of $M(n,\mathbb{H})$ in $M(2n,\mathbb{C})$ as the set of matrices X such that $FXF^{-1}=\overline{X}$, where $F=\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix}$. Under this identification, the canonical conjugation ε of $\mathbb{H}$ induces the conjugation $X\mapsto \overline{X}^t$ of $M(n,\mathbb{H})$, and the conjugation $\varphi:=j_*\circ\varepsilon$ induces the conjugation $X\mapsto X^t$ of $M(n,\mathbb{H})$. Then we get from the definition of the unitary groups

$$\begin{aligned}\mathrm{SU}(I_n,\varepsilon,\mathbb{H}) &= \{g\in \mathrm{Sl}(2n,\mathbb{C})|\ FgF^{-1}=\overline{g},\ \overline{g}^t=g^{-1}\}\\ &= \mathrm{SU}(2n)\cap M(n,\mathbb{H}) \quad =: \mathrm{Sp}(n),\\ \mathrm{SU}(I_n,\varphi,\mathbb{H}) &= \{g\in \mathrm{Sl}(2n,\mathbb{C})|\ FgF^{-1}=\overline{g},\ g^t=g^{-1}\}\\ &= \mathrm{SO}(2n,\mathbb{C})\cap M(n,\mathbb{H}) \quad =: \mathrm{SO}^*(2n).\end{aligned}$$

Furthermore, we can replace the system of two conditions defining the above groups by an equivalent system and thus obtain

$$\mathrm{SU}(I_n,\varepsilon,\mathbb{H})=\mathrm{SO}(F,\mathbb{C})\cap M(n,\mathbb{H})=\mathrm{SO}(F,\mathbb{C})\cap \mathrm{SU}(2n),$$
$$\mathrm{SU}(I_n,\varphi,\mathbb{H})=\mathrm{SU}(F,\mathbb{C})\cap M(n,\mathbb{H})=\mathrm{SU}(F,\mathbb{C})\cap \mathrm{SO}(2n,\mathbb{C}).$$

We remark that similar notation and isomorphisms have already been introduced in [KN64], but the notation introduced here indicates in addition the imbedding of the corresponding orthogonal or unitary groups into the general linear group; this will be important in the next section.

1.3 Conformal compactification and conformal groups of the algebras $\mathrm{Sym}(A,\mathbb{F})$ **and** $\mathrm{Herm}(A,\mathbb{F})$**.** The conformal compactification of the algebra $V=M(n,\mathbb{F})$ has been constructed by the graph-imbedding into $G_{2n,n}$. We will now show that the automorphism a "adjoint operator" is transformed by this imbedding into an automorphism p "orthocomplement", and the conformal compactification of V^a will then be a connected component of the fixed point set of p. For this purpose, given a form $\langle\cdot,\cdot\rangle$ on $E=\mathbb{F}^n$ (with associated matrix A), we equip $E\oplus E$ with four non-degenerate bi- or sesquilinear forms:

(1) $\langle(x_1,x_2),(y_1,y_2)\rangle_1 := \langle x_1,y_1\rangle+\langle x_2,y_2\rangle$, given by the matrix $\begin{pmatrix} A & 0 \\ 0 & A\end{pmatrix}$,

(2) $\langle(x_1,x_2),(y_1,y_2)\rangle_2 := \langle x_1,y_1\rangle-\langle x_2,y_2\rangle$, given by $\begin{pmatrix} A & 0 \\ 0 & -A\end{pmatrix}$,

(3) $\langle(x_1,x_2),(y_1,y_2)\rangle_3 := \langle x_1,y_2\rangle-\langle x_2,y_1\rangle$, given by $\begin{pmatrix} 0 & A \\ -A & 0\end{pmatrix}$,

(4) $\langle(x_1,x_2),(y_1,y_2)\rangle_4 := \langle x_1,y_2\rangle+\langle x_2,y_1\rangle$, given by $\begin{pmatrix} 0 & A \\ A & 0\end{pmatrix}$,

and we will write $W^{\perp_j}:=\{v\in\mathbb{F}^{2n}|\ \langle W,v\rangle_j=0\}$ for the orthocomplement of $W\in G_{2n,n}$ with respect to the form (j), $j=1,...,4$. By the usual dimension formulas, $W^{\perp_j}$ is again an element of $G_{2n,n}$; hence we have maps (for $j=1,...,4$)

$$p=p_j: G_{2n,n}\to G_{2n,n},\quad W\mapsto W^{\perp_j}.$$

Remark, as for the definition of the adjoint operator, that there are in general *two* possibilities to define "orthocomplement", and that they coincide if (and only if) the map "orthocomplement" is an involution.

Lemma 1.3.1. *The graph of the adjoint g^* of $g \in \mathrm{End}(E)$ is given by the formulas*

(i) $\Gamma_{g^*} = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)\Gamma_g^{\perp_1}, \quad \Gamma_{-(g^{-1})^*} = \Gamma_g^{\perp_1},$

(ii) $\Gamma_{g^*} = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)\Gamma_g^{\perp_2}, \quad \Gamma_{(g^{-1})^*} = \Gamma_g^{\perp_2},$

(iii) $\Gamma_{g^*} = \Gamma_g^{\perp_3},$

(iv) $\Gamma_{g^*} = \Gamma_{-g}^{\perp_4} = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)\Gamma_g^{\perp_4},$

where the second identity in (i) resp. (ii) holds for all invertible g. These identities can also be written

$$\text{(i)} \ -aj = p_1, \qquad \text{(ii)} \ aj = p_2, \qquad \text{(iii)} \ a = p_3, \qquad \text{(iv)} \ -a = p_4,$$

with the maps a "adjoint" and p_k, $k = 1, ..., 4$ "orthocomplement" defined above.

Proof. Writing the defining relation 1.1 $(*)$ of the adjoint operator in the form

$$\forall v, w \in E: \quad 0 = \langle gv, w\rangle - \langle v, g^* w\rangle = -\langle (v, gv), (w, g^* w)\rangle_3,$$

we get (iii). The other relations are obtained similarly. ■

Proposition 1.3.2. *Let the notation be as above. By the graph-imbedding we get the following inclusions as open and dense subsets:*

$$\mathrm{Herm}(A, \varepsilon, \mathbb{F}) \hookrightarrow \{W \in G_{2n,n}(\mathbb{F}) |\, W = W^{\perp_3}\}_0,$$

$$\mathrm{Aherm}(A, \varepsilon, \mathbb{F}) \hookrightarrow \{W \in G_{2n,n}(\mathbb{F}) |\, W = W^{\perp_4}\}_0,$$

where the subscript 0 *denotes "connected component of* Γ_0 *". If A is Hermitian or anti-Hermitian, these inclusions describe the conformal compactification of the corresponding matrix spaces, considered as Jordan algebras, resp. - triple systems. The natural action of the corresponding unitary group on the right-hand side spaces gives the action of the corresponding conformal groups; i.e.*

$$\mathrm{Co}(\mathrm{Sym}(A, \mathbb{F}))_0 = \mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \mathbb{F})_0 \subset \mathbb{P}\,\mathrm{Gl}(2n, \mathbb{F}),$$

$$\mathrm{Co}(\mathrm{Herm}(A, \varepsilon, \mathbb{F}))_0 = \mathbb{P}\mathrm{U}(\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \varepsilon, \mathbb{F})_0 \subset \mathbb{P}\,\mathrm{Gl}(2n, \mathbb{F}),$$

$$\mathrm{Co}(\mathrm{Asym}(A, \mathbb{F}))_0 = \mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, \mathbb{F})_0 \subset \mathbb{P}\,\mathrm{Gl}(2n, \mathbb{F}),$$

$$\mathrm{Co}(\mathrm{Aherm}(A, \varepsilon, \mathbb{F}))_0 = \mathbb{P}\mathrm{U}(\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}, \varepsilon, \mathbb{F})_0 \subset \mathbb{P}\,\mathrm{Gl}(2n, \mathbb{F}).$$

Proof. a) Let us show that the first imbedding is well-defined: the condition that g belongs to $\mathrm{Herm}(A)$ can be written as $a(g) = g$ where a is the map "adjoint w.r.t A". By Lemma 1.3.1, case (iii), $a = p_3$, hence $\Gamma_g = \Gamma_g^{\perp_3}$, which was to be shown. Similarly for $-a$.

b) We will now show that the first imbedding has open and dense image. It is clear that an element of the right-hand side set is in the image of the imbedding if and only if it is a graph, because then the above reasoning can be reversed, showing that the graph necessarily belongs to a Hermitian endomorphism. Recall that the set of graphs in $G_{2n,n}$ is dense, its complement being given by the "non-graphs" $N := \{W \in G_{2n,n} | \det(pr_1^W) = 0\}$ (see 1.1). Hence the image of the imbedding is the set $\{W \in G_{2n,n} | W = W^{\perp_3}\}_0 \setminus N$ which is open and dense in $\{W \in G_{2n,n} | W = W^{\perp_3}\}_0$ (otherwise it would be empty, leading to a contradiction). Similar for the second imbedding.

c) The unitary group of a form acts naturally on the space of Lagrangian subspaces corresponding to this form. By the graph-imbedding just described, we have hence a rational action of the group $\mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right))_0$ on the space $\mathrm{Herm}(A)$. We want to show that this is the action of the conformal group, $\mathrm{Co}(\mathrm{Herm}(A))_0 = \mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right))_0$.

The map a "adjoint", being an automorphism of the Jordan algebra $V = M(n, \mathbb{F})$, belongs to the conformal group $\mathrm{Co}(V)$ of V. Recall that for any $\phi \in \mathrm{Co}(V)$ we denote by $\phi_*(g) = \phi \circ g \circ \phi^{-1}$ the conjugation by ϕ. The following lemma describes a_*:

Lemma 1.3.3. *For all* $g \in \mathrm{Gl}(2n, \mathbb{F})$,

(1) $(-ja)_* g = (g^*)^{-1}$,
(2) $(ja)_* g = (g^*)^{-1}$,
(3) $a_* g = (g^*)^{-1}$,
(4) $(-a)_* g = (g^*)^{-1}$,

where the adjoint $g^* \in \mathrm{Gl}(2n, \mathbb{F})$ *in formula* (k) *is taken with respect to the form* (k) *on* $E \oplus E$ *defined at the beginning of this section.*

Proof. It is an immediate consequence of the definition of the adjoint operator that

$$g \cdot W^{\perp} = ((g^*)^{-1} \cdot W)^{\perp}$$

for all $W \in G_{2n,n}$. This can be written as $p_*(g) = (g^*)^{-1}$, where p is the map "orthocomplement". The relations between p and a given by 1.3.1 now imply the lemma. ■

Equation (3) of the lemma shows that $\mathrm{Co}(V)^{a_*} = \mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right))$. In order to prove the claim we now only have to show that $\mathrm{Co}(V^a) = \mathrm{Co}(V)^{a_*}$. But this is easily verified since the translations by elements of V^a, the group $\mathrm{Str}(V^a)_0 = \mathrm{Str}(V)_0^{a_*}$ and j are in $\mathrm{Co}(V)^{a_*}$ and they generate its identity component. In a similar way, we have $\mathrm{Co}(V^{-a})_0 = \mathrm{Co}(V)_0^{(-a)_*} = \mathrm{U}(\left(\begin{smallmatrix} 0 & A \\ A & 0 \end{smallmatrix}\right))_0$; but as we will not use in this work the formal definition of the conformal group of a Jordan triple-system, one may take this equality here simply as definition of $\mathrm{Co}(V^{-a})$.

d) It remains to show that the imbedding of V^a into the corresponding set of Lagrangian subspaces is a conformal compactificaton as defined in [Be96a, Th.2.4.1]. Because we already know that the conformal group acts on it by continuing the corresponding rational action on V^a, it is now enough to show that this action is transitive. This is the contents of Witt's Theorem, see [Bou59, 4, no.3, Cor 2], the hypothesis of which are verified under our assumption on A. ■

Remarks. 1. When A is neither Hermitian nor anti-Hermitian, the Witt theorem cannot always be applied. Furthermore, $\mathrm{Herm}(A)$ will then in general not be semi-simple.

2. The whole set of Lagrangian subspaces is in general not connected. as shows the example $\mathbb{F}=\mathbb{R}$, $A=I_{2n+1}$.

1.4. Normal forms of the algebras $\mathrm{Sym}(A,\mathbb{F})$ and $\mathrm{Herm}(A,\mathbb{F})$ and of their conformal groups. We specialize Proposition 1.3.2 to the matrices of some standard bilinear forms. In some cases we obtain block-matrices which can be diagonalized: let the following endomorphisms of $E\oplus E$ be defined by $2n\times 2n$ block-matrices: $R=\begin{pmatrix} I & -I \\ I & I\end{pmatrix}$ (the real Cayley transform), $C=\begin{pmatrix} -I & -iI \\ iI & I\end{pmatrix}$ (the Cayley transform). We then have for all $A\in M(n,\mathbb{F})$,

(1) $R\begin{pmatrix} 0 & A \\ A & 0\end{pmatrix}R^{-1}=\begin{pmatrix} -A & 0 \\ 0 & A\end{pmatrix}$, $R^2=-2\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}$, $2R^{-1}=R^t$,

(2) $R\begin{pmatrix} 0 & A \\ -A & 0\end{pmatrix}R^{-1}=\begin{pmatrix} 0 & A \\ -A & 0\end{pmatrix}$,

(3) $C\begin{pmatrix} 0 & A \\ A & 0\end{pmatrix}C^{-1}=\begin{pmatrix} 0 & -A \\ -A & 0\end{pmatrix}$, $C^2=2I$, $\overline{C}^t=C$, $2C^{-1}=\overline{C}^t$.

(4) $C\begin{pmatrix} 0 & A \\ -A & 0\end{pmatrix}C^{-1}=\begin{pmatrix} -A & 0 \\ 0 & A\end{pmatrix}$.

The conformal group then also takes a simpler form: we apply the relations

$$\mathrm{O}(g^tAg)=g^{-1}\mathrm{O}(A)g,\quad \mathrm{U}(g^tA\overline{g}^{-1})=g^{-1}\mathrm{U}(A)g,$$

which follow easily from the definitions (the last identity should, more conceptually but less familiar, be written $\mathrm{U}(g^tA\varepsilon(g)^{-1},\varepsilon)=g^{-1}\mathrm{U}(A,\varepsilon)g$.) Now Proposition 1.3.2 gives the following table:

$$\mathbb{F}=\mathbb{R}:\quad \mathrm{Co}(\mathrm{Sym}(I_n,\mathbb{R}))_0=\mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix},\mathbb{R})=\mathrm{Sp}(n,\mathbb{R})$$

$$\mathrm{Co}(\mathrm{Asym}(I_n,\mathbb{R}))_0=\mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix},\mathbb{R})$$

$$=R\,\mathbb{P}\mathrm{O}(\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix},\mathbb{R})\,R^{-1}=R\ \mathrm{SO}(n,n)\,R^{-1}$$

$$\mathbb{F}=\mathbb{C}:\quad \mathrm{Co}(\mathrm{Sym}(I_n,\mathbb{C}))_0=\mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix},\mathbb{C})=\mathrm{Sp}(n,\mathbb{C})$$

$$\mathrm{Co}(\mathrm{Asym}(I_n,\mathbb{C}))_0=\mathbb{P}\mathrm{O}(\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix},\mathbb{C})$$

$$=R\,\mathbb{P}\mathrm{O}(\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix},\mathbb{C})\,R^{-1}=R\ \mathrm{SO}(n,n;\mathbb{C})\,R^{-1}$$

$$\mathrm{Co}(\mathrm{Herm}(I_n,\mathbb{C}))_0=\mathbb{P}\mathrm{U}(\begin{pmatrix} 0 & I \\ -I & 0\end{pmatrix},\mathbb{C})=C\ \mathrm{SU}(n,n)\,C^{-1}$$

$$\mathrm{Co}(\mathrm{Aherm}(I_n,\mathbb{C}))_0=\mathbb{P}\mathrm{U}(\begin{pmatrix} 0 & I \\ I & 0\end{pmatrix},\mathbb{C})$$

$$=R\,\mathbb{P}\mathrm{U}(\begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix},\mathbb{C})\,R^{-1}=R\ \mathrm{SU}(n,n)\,R^{-1}$$

$$\mathbb{F}=\mathbb{H}: \quad \mathrm{Co}(\mathrm{Herm}(I_n,\mathbb{H}))_0 = \mathbb{P}\mathrm{U}(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \mathbb{H})$$
$$= r_j C\, \mathbb{P}\mathrm{U}(I_{2n}, \varepsilon \circ j_*, \mathbb{H})\, C^{-1} r_j^{-1} \cong \mathrm{SO}^*(4n)$$
$$\mathrm{Co}(\mathrm{Aherm}(I_n,\mathbb{H}))_0 = \mathbb{P}\mathrm{U}(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \mathbb{H})$$
$$= R\, \mathbb{P}\mathrm{U}(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \mathbb{H})\, R^{-1} = R\, \mathrm{Sp}(n,n)\, R^{-1}$$

(Here j is the element of the canonical basis of $\mathbb{H}$ and r_j is right multiplication by j.) By the following lemmas we reduce the description of the conformal groups of the spaces $\mathrm{Herm}(\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right))$ and $\mathrm{Herm}(I_{p,q})$ to the above listed spaces.

Lemma 1.4.1. *Let $F := F_m := \left(\begin{smallmatrix} 0 & I_m \\ -I_m & 0 \end{smallmatrix}\right)$ and $l_F : M(n,\mathbb{F}) \to M(n,\mathbb{F})$, $X \mapsto FX$. The following restrictions of l_F:*

$$\mathrm{Herm}(I_{2m},\varepsilon,\mathbb{F}) \to \mathrm{Aherm}(F,\varepsilon,\mathbb{F}), \quad X \mapsto FX$$

$$\mathrm{Aherm}(I_{2m},\varepsilon,\mathbb{F}) \to \mathrm{Herm}(F,\varepsilon,\mathbb{F}), \quad X \mapsto FX$$

are bijections which induce isomorphisms of the corresponding conformal groups, given by conjugation with $\left(\begin{smallmatrix} F & 0 \\ 0 & I \end{smallmatrix}\right)$.

Proof. The condition $X^t = X$ is equivalent to $F^{-1}(FX)^t F = -FX$, which implies that the maps are well-defined. They are bijective, the inverse given by $X \mapsto -FX$. More conceptually: define involutions $\alpha(X) := X^t$ and $\beta(X) := F^{-1}X^tF$ of $V = M(2m,\mathbb{F})$; then α and $-\beta$ are conjugate: $-\beta = l_F \circ \alpha \circ l_F^{-1}$. Therefore $l_F \cdot V^\alpha = V^{-\beta}$ and

$$\mathrm{Co}(V^{-\beta}) = \mathrm{Co}(V)^{(-\beta)_*} = (l_F)_* \,\mathrm{Co}(V)^{\alpha_*} = (l_F)_* \,\mathrm{Co}(V^\alpha),$$

where $(l_F)_*$ is nothing but conjugation with $\left(\begin{smallmatrix} F & 0 \\ 0 & I \end{smallmatrix}\right)$ in $\mathrm{Co}(V) = \mathbb{P}\,\mathrm{Gl}(4m)$, thus proving the statement about the conformal groups for the first case; similar for the other. ∎

Lemma 1.4.2. *The map*

$$l_{I_{p,q}} : \mathrm{Herm}(I_{p,q},\varepsilon,\mathbb{F}) \to \mathrm{Herm}(I_n,\varepsilon,\mathbb{F}), \quad X \mapsto I_{p,q}X$$

(where $n = p+q$) is a bijection, and $(l_{I_{p,q}})_*$ (= conjugation by $\left(\begin{smallmatrix} I_{p,q} & 0 \\ 0 & I \end{smallmatrix}\right)$ in $\mathbb{P}\,\mathrm{Gl}(2n)$) defines an isomorphism of the corresponding conformal groups. A similar statement holds for $\mathrm{Aherm}(A,\varepsilon,\mathbb{F})$.

Proof. Remark that the condition $X = X^t$ is equivalent to $(I_{p,q}X)^t I_{p,q} = I_{p,q}I_{p,q}X$; thus the map is well-defined. It is clear now how to adapt the proof of the previous lemma to the given situation. ∎

Remark. We used in 1.4.1 and 1.4.2 implicitly a notion of "conformal isomorphism" as an isomorphism inducing an isomorphism of conformal groups. It is therefore not very important that l_F and $l_{I_{p,q}}$ are actually *linear*. There are other conformal isomorphisms not having this property. For example, the *Potapov-Ginzburg transformation* (see Section 0.2) can be used; it is given by a rational conformal map which is not linear when restricted to a matrix space. The isomorphisms introduced here hence preserve some additional structure (which we do not need at this stage); namely they are *isomorphisms of Jordan triple-systems*: we equip all matrix spaces in question with a Jordan triple product given by $\{X,Y,Z\} = \frac{1}{2}(XY^tZ + ZY^tX)$. Then, for all matrices A and B, $F(AB^tA) = FA(FB)^tFA$, which implies that l_F is a homomorphism of triple systems; similarly for $l_{I_{p,q}}$.

Open orbits in the conformal compactification of $\mathrm{Herm}(A,\varepsilon,\mathbb{F})$**.** We now come to the main topic of this chapter: find subgroups $G \subset \mathrm{Co}(\mathrm{Herm}(A))$ such that there exists an open orbit $G \cdot x \subset \mathrm{Herm}(A)^c$ for some $x \in \mathrm{Herm}(A)^c$. Remark that we have two distinguished base points: $x = 0$ and $x = I$. These two base points are related by the real Cayley-transform $R = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$.

A. Open orbits of type $\mathrm{Gl}(n,\mathbb{F})/\mathrm{O}(A,\mathbb{F})$ **or** $\mathrm{Gl}(n,\mathbb{F})/\mathrm{U}(A,\mathbb{F})$**.** The open cone $\Omega = \mathrm{Str}(\mathrm{Herm}(A))_0 \cdot I \subset \mathrm{Herm}(A)$ is isomorphic to $\mathrm{Gl}(n,\mathbb{F})/\mathrm{U}(A,\mathbb{F})$, see Section 1.2.2. There is also a Cayley-transformed realization having 0 as base point, *cf.* Example 2.2.5.

B. Open orbits of group type. The second type of open orbits is of the form $G \times G/\operatorname{dia}(G \times G)$ which is nothing but the group G considered as a homogeneous space under the action $(G \times G) \times G \to G$, $((g,h),x) \mapsto gxh^{-1}$. Let us consider $G = \mathrm{U}(A) = \mathrm{U}(A,\varepsilon,\mathbb{F})$. Using Lemma 1.3.1 in the same way as in the proof of Proposition 1.3.2 we get the following graph-imbedding:

$$\mathrm{U}(A)_0 \overset{\Gamma}{\hookrightarrow} \{W \in G_{2n,n} \,|\, W = W^{\perp_2}\}_I.$$

Composing with the real Cayley-transform R and using $R\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}R^{-1} = \begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$, we obtain:

$$\mathrm{U}(A)_0 \overset{\Gamma}{\hookrightarrow} \{W \in G_{2n,n} \,|\, W = W^{\perp_2}\}_I \overset{R}{\to} \{W \in G_{2n,n} \,|\, W = W^{\perp_4}\}_0 = \mathrm{Aherm}(A)^c.$$

We will write this shorter as

$$R \cdot \mathrm{U}(A)_0 \subset \mathrm{Aherm}(A)^c,$$

which can be interpreted as a rational relation: if $g \in \mathrm{U}(A)_0$, then $(g-I)(g+I)^{-1} \in \mathrm{Aherm}(A)$ whenever $g+I$ is invertible. Conversely, whenever $X \in \mathrm{Aherm}(A)$ and $X-I$ and $X+I$ are invertible, then $R^{-1}X = (X+I)(X-I)^{-1} \in \mathrm{U}(A)$. This implies that the intersection of $R \cdot \mathrm{U}(A)$ with $\mathrm{Aherm}(A)^c$ is dense in $\mathrm{Aherm}(A)^c$ and described by the condition $\mathrm{Det}(X^2 - I) \neq 0$. (In [Wey39, II.10] this realization is called "CAYLEY's rational parametrization of the orthogonal group".) The example of $\mathrm{O}(2n+1,\mathbb{R})$ shows that $R \cdot \mathrm{U}(A)$ may be bigger than $\mathrm{Aherm}(A)^c$. We will now see that $R \cdot \mathrm{U}(A)_0 \subset \mathrm{Aherm}(A)^c$ is actually an orbit of the form we are looking for.

Proposition 1.5.1. *For all non-singular matrices A,*

$$R\cdot \mathrm{U}(A,\varepsilon,\mathbb{F})_0 = \mathrm{Co}(\mathrm{Aherm}(A,\varepsilon,\mathbb{F}))_0^{j_*}\cdot 0 \subset \mathrm{Aherm}(A,\varepsilon,\mathbb{F})^c$$

are open and symmetric orbits.

Proof. Only the stated equality remains to be shown. First,

$$\mathrm{U}(\begin{pmatrix} A & 0 \\ 0 & -A\end{pmatrix})_0^{(-\,\mathrm{id})_*}\cdot I = \left(\mathrm{U}(\begin{pmatrix} A & 0 \\ 0 & -A\end{pmatrix})\cap\begin{pmatrix}\mathrm{Gl}(n) & 0 \\ 0 & \mathrm{Gl}(n)\end{pmatrix}\right)_0\cdot I$$
$$= \{gh^{-1}|\, g,h\in \mathrm{U}(A)_0\} = \mathrm{U}(A)_0.$$

The last equality just describes the usual realization of $U=\mathrm{U}(A)_0$ as a symmetric space $U\times U/\,\mathrm{dia}(U\times U)$ with base point I. Transforming by R,

$$R(\mathrm{U}(A)_0) = R(\mathrm{U}(\begin{pmatrix} A & 0 \\ 0 & -A\end{pmatrix})_0^{(-\,\mathrm{id})_*}\cdot I)$$
$$= \mathrm{U}(\begin{pmatrix} 0 & A \\ A & 0\end{pmatrix})_0^{(R(-\,\mathrm{id})R^{-1})_*}\cdot 0 = \mathrm{Co}(\mathrm{Aherm}(A))_0^{j_*}\cdot 0.$$

■

We can now use the isomorphisms 1.4.1 and 1.4.2 in order to get an imbedding of some of the groups $\mathrm{U}(A)$ into spaces of *Hermitian* matrices (which thus appear less natural, but have the advantage that the latter spaces are Jordan-*algebras*, not only triple-systems). We get the following table (where $p+q$ is even in the first line and n even in the third line, and the notation for the quaternionic case has been introduced in Section 1.2):

$$\mathbb{F}=\mathbb{R}:\quad \mathrm{O}(p,q)=\mathrm{O}(I_{p,q},\mathbb{R})\overset{R}{\hookrightarrow}\mathrm{Asym}(I_{p,q},\mathbb{R})^c\xrightarrow{\left(\begin{smallmatrix} I_{p,q}F & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Sym}(F,\mathbb{R})^c$$

$$\mathrm{Sp}(n,\mathbb{R})=\mathrm{O}(F,\mathbb{R})\overset{R}{\hookrightarrow}\mathrm{Asym}(F,\mathbb{R})^c\xrightarrow{\left(\begin{smallmatrix} F & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Sym}(I_{2n},\mathbb{R})^c$$

$$\mathbb{F}=\mathbb{C}:\quad \mathrm{O}(I,\mathbb{C})\overset{R}{\hookrightarrow}\mathrm{Asym}(I,\mathbb{C})^c\xrightarrow{\left(\begin{smallmatrix} F & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Sym}(F,\mathbb{C})^c$$

$$\mathrm{Sp}(n,\mathbb{C})=\mathrm{O}(F,\mathbb{C})\overset{R}{\hookrightarrow}\mathrm{Asym}(F,\mathbb{C})^c\xrightarrow{\left(\begin{smallmatrix} F & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Sym}(I_{2n},\mathbb{C})^c$$

$$\mathrm{U}(p,q)=\mathrm{U}(I_{p,q},\mathbb{C})\overset{R}{\hookrightarrow}\mathrm{Aherm}(I_{p,q},\mathbb{C})^c\xrightarrow{\left(\begin{smallmatrix} iI_{p,q} & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Herm}(I_n,\mathbb{C})^c$$

$$C\,\mathrm{U}(n,n)\,C^{-1}=\mathrm{U}(F,\mathbb{C})\overset{R}{\hookrightarrow}\mathrm{Aherm}(F,\mathbb{C})^c\xrightarrow{\left(\begin{smallmatrix} F & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Herm}(I_{2n},\mathbb{C})^c$$

$$\mathbb{F}=\mathbb{H}:\quad \mathrm{Sp}(p,q)=\mathrm{U}(I_{p,q},\mathbb{H})\overset{R}{\hookrightarrow}\mathrm{Aherm}(I_{p,q},\mathbb{H})^c\xrightarrow{\left(\begin{smallmatrix} I_{p,q}r_j & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Herm}(I,j_*\circ\varepsilon,\mathbb{H})^c$$

$$\mathrm{SO}^*(4n)\cong\mathrm{U}(F,\mathbb{H})\overset{R}{\hookrightarrow}\mathrm{Aherm}(F,\mathbb{H})^c\xrightarrow{\left(\begin{smallmatrix} F & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Herm}(I_{2n},\mathbb{H})^c$$

$$\mathrm{SO}^*(2n)=\mathrm{U}(I,j_*\circ\varepsilon,\mathbb{H})\overset{R}{\hookrightarrow}\mathrm{Aherm}(I,j_*\circ\varepsilon,\mathbb{H})^c\xrightarrow{\left(\begin{smallmatrix} r_j & 0 \\ 0 & I\end{smallmatrix}\right)}\mathrm{Herm}(I,\mathbb{H})^c$$

(The list [Ma73, p.415] contains an error: the image of the fourth imbedding for $\mathbb{F} = \mathbb{C}$ described above is there denoted by $U^*(2m)$; but $\mathrm{U}(F, \mathbb{C})$ is not isomorphic to this group.) We can keep track of the group $\mathrm{Co}(\mathrm{Aherm}(A))^{j_*} \cong \mathrm{U}(A) \times \mathrm{U}(A)$ acting on $\mathrm{U}(A)$ as described in Prop.1.5.1 by conjugating it by the isomorphisms given above. For the case of the Euclidean Jordan algebras we write down the result:

$$\mathrm{Sp}(n, \mathbb{R}) \times \mathrm{Sp}(n, \mathbb{R}) = \mathrm{Co}(\mathrm{Sym}(I_{2n}, \mathbb{R}))^{(\alpha j)_*}; \quad \alpha(X) = -FXF^{-1}.$$

$$\mathrm{U}(p, q) \times \mathrm{U}(p, q) = \mathrm{Co}(\mathrm{Herm}(I_n, \mathbb{C}))^{(\alpha j)_*}; \quad \alpha(X) = -I_{p,q} X I_{p,q}.$$

$$\mathrm{SO}^*(2n) \times \mathrm{SO}^*(2n) = \mathrm{Co}(\mathrm{Herm}(I_n, \mathbb{H}))^{(\alpha j)_*}; \quad \alpha(X) = -\varphi X \varphi^{-1},$$

where in the last case φ is the conjugation of $\mathbb{H}^n$ defined in Section 1.2 (we remarked there that $\alpha(X) = X^t$ under the usual imbedding $M(n, \mathbb{H}) \subset M(2n, \mathbb{C})$.)

C. Orbits of type $\mathrm{Co}(V)/\,\mathrm{Str}(V)$**.** The homogeneous space $\mathrm{Co}(V)/\,\mathrm{Str}(V)$ can be imbedded as open orbit into the conformal compactification $V^c \times V^c$ of $V \times V$. This is best described in the general context of Jordan algebras, see Example 2.2.10. In the cases of Hermitian matrices we get imbeddings

$$\mathrm{SU}(\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \mathbb{F})/(\mathrm{Sl}(n, \mathbb{F}) \times R^*) \hookrightarrow \mathrm{Herm}(A, \mathbb{F})^c \times \mathrm{Herm}(A, \mathbb{F})^c.$$

D. Orbits of other type. In Section 2.4 we will see that open imbeddings of symmetric spaces into compactifications of Euclidean Jordan algebras are related to involutions of these algebras. It is known that the Euclidean matrix algebra $\mathrm{Herm}(I, \mathbb{C})$ has "more" involutions than the other Euclidean matrix algebras; for this reason we still have to discuss two classes of orbits related to this algebra:

Proposition 1.5.2. (i) *Let* $\alpha(Z) = Z^t$, $Z \in \mathrm{Herm}(I_n, \mathbb{C})$. *Then the orbits*

$$X^{(\pm\alpha)} := \mathrm{Co}(\mathrm{Herm}(I_n, \mathbb{C}))_0^{(\pm\alpha j)_*} \cdot 0 \subset \mathrm{Herm}(I_n, \mathbb{C})^c$$

are open. For $-\alpha$, *the orbit is isomorphic to* $\mathrm{SO}^*(2n)/\,\mathrm{SO}(n, \mathbb{C})$, *and for* $+\alpha$ *it is isomorphic to* $\mathrm{SO}(n, n)/\,\mathrm{SO}(n, \mathbb{C})$ *as a symmetric space.*

(ii) *Let* $\beta(Z) = FZ^tF^{-1}$, $Z \in \mathrm{Herm}(I_{2n}, \mathbb{C})$ *with* $F = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. *Then the orbits*

$$X^{(\pm\beta)} := \mathrm{Co}(\mathrm{Herm}(I_{2n}, \mathbb{C}))_0^{(\pm\beta j)_*} \cdot 0 \subset \mathrm{Herm}(I_{2n}, \mathbb{C})^c$$

are open. For $-\beta$, *the orbit is isomorphic to* $\mathrm{Sp}(2n, \mathbb{R})/\,\mathrm{Sp}(n, \mathbb{C})$, *and for* $+\beta$ *it is isomorphic to* $\mathrm{Sp}(n, n)/\,\mathrm{Sp}(n, \mathbb{C})$ *as a symmetric space.*

Proof. The orbits in question are of the form $\mathrm{Co}(\mathrm{Herm}(A))^{(\pm bj)_*} \cdot 0$, where b is the involution defined by $b(Z) = BZ^tB^{-1}$ with some symmetric or anti-symmetric matrix B commuting with A. A simple computation in the Lie algebra of $\mathrm{Co}(\mathrm{Herm}(A))$ (see Prop.2.2.1) shows that such orbits are open. The involution $(\pm bj)_*$ of $\mathrm{Gl}(2n)$ is given by taking the adjoint with respect to the form given by $\left(\begin{smallmatrix} B & 0 \\ 0 & \mp B \end{smallmatrix}\right)$, see 1.3.1. Hence

$$\mathrm{Co}(\mathrm{Herm}(A))^{(\pm bj)_*} = \mathbb{P}(\mathrm{U}(\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}) \cap \mathrm{O}(\begin{pmatrix} B & 0 \\ 0 & \mp B \end{pmatrix})).$$

A matrix of this form fixes the base point 0 if it is in the the group $\left(\begin{smallmatrix} \mathrm{Gl}(n) & 0 \\ 0 & \mathrm{Gl}(n) \end{smallmatrix}\right)$ fixed by $(-\mathrm{id}_V)_*$. One easily verifies that these are the matrices of the form $\left(\begin{smallmatrix} a & 0 \\ 0 & A^{-1}a^tA \end{smallmatrix}\right)$ with $a \in \mathrm{O}(B)$; hence the stabilizer of the base point is isomorphic to $\mathrm{O}(B)$. We now specialize to the cases (i) and (ii) given above:

In case (i), $A = I_n$, $B = I_n$. For $-\alpha$, note that $\mathrm{SU}(\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)) \cap \mathrm{SO}(2n, \mathbb{C})$ equals $\mathrm{SO}^*(2n)$ (see Section 1.2). For $+\alpha$, the Cayley transform maps $\mathrm{SU}(\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)) \cap \mathrm{SO}(n,n;\mathbb{C})$ onto $\mathrm{SO}(n,n;\mathbb{R})$.

In case (ii), $A = I_{2n}$, $B = F$. For $-\beta$, observe that $\mathrm{SU}(\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)) \cap \mathrm{SO}(\left(\begin{smallmatrix} F & 0 \\ 0 & F \end{smallmatrix}\right), \mathbb{C})$ is mapped by the real Cayley transform R onto the group $\mathrm{Sp}(n,n)$ (see Section 1.2). For $+\beta$, one uses that $\mathrm{SU}(\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)) \cap \mathrm{SO}(\left(\begin{smallmatrix} F & 0 \\ 0 & -F \end{smallmatrix}\right), \mathbb{C})$ is mapped by the complex Cayley transform C onto the group $\mathrm{Sp}(2n, \mathbb{R})$. ■

1.6 Open orbits in the conformal compactification of $V = M(n, \mathbb{F})$**.** As the constructions in these cases are similar to the preceding ones and because these algebras are never Euclidean, we will give less details than in the preceding section.

A. Orbits of type $\mathrm{Gl}(n, \mathbb{F})$**.** For $V = M(n, \mathbb{F})$ the open cone $\Omega = \mathrm{Str}(V)_0 \cdot I$ coincides with the group-type orbits and is isomorphic to $\mathrm{Gl}(n, \mathbb{F}) \subset M(n, \mathbb{F})$, see Section 1.1. There is also a Cayley-transformed realization having 0 as base point.

B. Orbits of type $\mathrm{U}(\left(\begin{smallmatrix} A & 0 \\ 0 & \pm A \end{smallmatrix}\right), \mathbb{F})/\mathrm{U}(A) \times \mathrm{U}(A)$**.** Let A be a non-degenerate Hermitian or skew-Hermitian matrix and define an involution of $V = M(n, \mathbb{F})$ by $\alpha(X) = A^{-1}\overline{X}^t A$. Then the orbits $X^{(\pm\alpha)} := \mathrm{Co}(V)_0^{(\pm\alpha j)_*} \cdot 0$ are open in V^c (Prop.2.2.1), and a calculation similar to the one given in 1.5 D shows that $X^{(\pm\alpha)} \cong \mathrm{U}(\left(\begin{smallmatrix} A & 0 \\ 0 & \mp A \end{smallmatrix}\right), \mathbb{F})/\mathrm{U}(A) \times \mathrm{U}(A)$.

C. Orbits related to complex conjugation. This is a special type of orbits having no analogue for Hermitian matrices because no unitary groups will be involved: for $V = M(n, \mathbb{C})$, $\tau(X) = \overline{X}$ defines a conjugation with respect to the real form $M(n, \mathbb{R})$ of V. The orbits $X^{(\pm\tau)} := \mathrm{Co}(V)_0^{(\pm\tau j)_*} \cdot 0$ are open in V^c (Prop. 2.2.1), and $C(X^{(\tau)}) = \mathrm{Co}(V)^{\tau_*} \cdot ie \cong \mathrm{Gl}(2n, \mathbb{R})/\mathrm{Gl}(n, \mathbb{C})$. We have $\mathrm{Co}(V)^{(-\tau j)_*} = \{g \in \mathrm{Gl}(2n, \mathbb{C}) | F\overline{g}F^{-1} = g\} = \mathrm{Gl}(n, \mathbb{H})$, hence $X^{(-\tau)} \cong \mathrm{Gl}(n, \mathbb{H})/\mathrm{Gl}(n, \mathbb{C})$.

1.7 A generalization of the SIEGEL**-space.** We will give now a geometric description of the orbits constructed in Section 1.5. Generalizing the arguments used in the proof of Proposition 1.5.2, we obtain

Theorem 1.7.1. *Let A and B be non-singular $n \times n$- $\mathbb{F}$-matrices and ε_1, ε_2 involutions of $\mathbb{F}$, canonically extended to $E := \mathbb{F}^n$, such that $\varepsilon_1 A \circ \varepsilon_2 B =$*

$\varepsilon_2 B \circ \varepsilon_1 A$. Assume that A is ε_1-Hermitian or -skew-Hermitian and B is ε_2-Hermitian or -skew-Hermitian. Then $\beta(Z) = B^{-1}\varepsilon_2(Z)^t B$ defines an involution of $V = \mathrm{Herm}(A, \varepsilon_1, \mathbb{F})$, and the orbits

$$\begin{aligned} X^{(\pm\beta)} &:= \mathrm{Co}(V)_0^{(\pm j\beta)_*} \cdot 0 \\ &= (U(\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}, \varepsilon_1, \mathbb{F}) \cap U(\begin{pmatrix} B & 0 \\ 0 & \mp B \end{pmatrix}, \varepsilon_2, \mathbb{F}))_0 \cdot 0 \end{aligned}$$

are open in the conformal compactification V^c of V. The stabilizer of the base point $0 \in V$ is isomorphic to $\mathrm{U}(B, \varepsilon_2, \mathbb{F})$. The orbit $X^{(\pm\beta)}$ is a connected component of the set of n-dimensional subspaces W of $E \oplus E = \mathbb{F}^n \oplus \mathbb{F}^n$ such that

(i)*: $W = W^\perp$ w.r.t. the ε_1-sesquilinear form $\begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix}$,*

(ii)*: the ε_2-sesquilinear form $\begin{pmatrix} B & 0 \\ 0 & \mp B \end{pmatrix}$ is non-degenerate on W.*

The intersection $X^{(\pm\beta)} \cap V$ is a union of connected components of the set of matrices

$$\{Z \in V \mid \mathrm{Det}(B \mp \varepsilon_2(Z)^t BZ) \neq 0\}.$$

Similar statements hold for the spaces $V = M(n, \mathbb{F})$ and $V = \mathrm{Aherm}(A, \varepsilon_1, \mathbb{F})$. In the latter case, the choice $B = I$ and $\varepsilon_2 = \mathrm{id}_{\mathbb{F}}$ yields orbits of group type.

Proof. The compatibility conditions imply that $V = \mathrm{Herm}(A)$ is stable under the involution β. We are going to show in Prop.2.2.1 that the orbit $X^{(\pm\beta)} = \mathrm{Co}(V)_0^{(\pm\beta j)_*} \cdot 0$ is open in V^c. Lemma 1.3.3 gives the description of $\mathrm{Co}(V)^{(\pm\beta j)_*}$ in terms of unitary groups, and the stabilizer of the base point is calculated as in 1.5.2.

The condition (i) in the description of $X^{(\pm\beta)}$ is nothing but the description of V^c, see 1.3.2. For the second condition, recall that the base point 0 is identyfied with its graph $\Gamma_0 = E \oplus 0 \in G_{2n,n}$. The restriction of the form $\begin{pmatrix} B & 0 \\ 0 & \pm B \end{pmatrix}$ to this space is given simply by B which is non-degenerate by assumption. But then the whole $\mathrm{U}(\begin{pmatrix} B & 0 \\ 0 & \pm B \end{pmatrix})$-orbit of Γ_0 consists of spaces on which this form is non-degenerate. If $W = \Gamma_X$ is a graph of $X \in V = \mathrm{Herm}(A)$, then the non-degeneracy condition is seen to be equivalent to the condition that $\det(B \mp \varepsilon_2(X)^t BX) \neq 0$. All that remains to show in order to conclude that the connected component of Γ_0 of the set of graphs satisfying the non-degenerate condition coincides with the orbit $X^{(\pm\beta)}$, is: *if the form $\begin{pmatrix} B & 0 \\ 0 & \pm B \end{pmatrix}$ is non-degenerate on some n-dimensional subspace W, then the orbit of W is open.* In [Be96b] we will give a general algebraic and very natural proof of this fact using the Jordan-theoretic idea of "mutation". In the more special situation here one may use the following arguments: first consider the case $\mathbb{F} = \mathbb{C}$ and $\varepsilon_1 = \varepsilon_2 = \mathrm{id}_{\mathbb{C}}$. If the restriction of the $\mathbb{C}$-bilinear form $\begin{pmatrix} B & 0 \\ 0 & \mp B \end{pmatrix}$ to W is non-degenerate, then this restriction has normal form B. The stabilizer of W can thus be considered as a subgroup of $\mathrm{O}(B)$; but then the dimension of the orbit of W is bigger or equal than the dimension of the orbit of Γ_0. Because the latter is of maximal dimension, we must have equality, and the orbit of W is open in V^c (and one sees also that in the complex case there is just one open orbit). To prove the desired statement in the general "real" set-up of the theorem, we complexify first all the structures involved, use than the special case just discussed and then restrict again to the real form we are interested in. ■

An example: the classical SIEGEL*space.* Let $\mathbb{F} = \mathbb{C}$, $\varepsilon_1 = \mathrm{id}_{\mathbb{C}}$, ε_2 is complex conjugation, and $A = B = I_n$. Then B is positive definite on the base point, and

$$X^{(\beta)} \cap V = \{Z \in \mathrm{Sym}(n, \mathbb{C}) |\, I - \overline{Z}^t Z >> 0\}.$$

Since this set is bounded and open in $\mathrm{Sym}(n, \mathbb{C})$, we may conclude from the last condition stated in the theorem that $X^{(\beta)}$ is isomorphic to this space; in this case our theorem describes thus the classical SIEGEL-space, see [Sa80, chap. II.7]. Similarly, the condition of the theorem implies that $X^{(-\beta)} \cap V = V$ because the Hermitian form $\left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$ is non-degenerate on *any* subspace. We will see in chapter 2 that actually $X^{(-\beta)} = V^c$; this is a compact Hermitian symmetric space.

Let us also mention that the preceding theorem permits to get hold of the $\mathrm{Co}(V)^{(\pm\beta j)_*}$-orbit structure of V^c. In fact, orbits are characterized by the rank (and other invariants, such as signature) of the resctriction of $\left(\begin{smallmatrix} B & 0 \\ 0 & \pm B \end{smallmatrix}\right)$ to subspaces W. An orbit is in the closure of another only if the rank corresponding to this orbit is strictly lower than the rank corresponding to the other. In particular, there is one orbit (or finite union of orbits) lying in the closure of every other orbit, namely the set of n-dimensional subspaces W such that the forms $\left(\begin{smallmatrix} 0 & A \\ -A & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} B & 0 \\ 0 & \pm B \end{smallmatrix}\right)$ vanish simultaneously on $W \times W$. When the orbit is a bounded symmetric domain (as the SIEGEL-space), then this space is known to be the *Shilov-boundary* of the orbit.

1.8 The LIOUVILLE**-theorem for the matrix algebras.** We will specialize our LIOUVILLE-theorem [Be96a, Th.2.3.1] to the case of the special Jordan algebras $M(n, \mathbb{F})$ and $\mathrm{Herm}(A, \mathbb{F})$. As explained in the introduction, the case of a *Euclidean* Jordan algebra is related to *causal* groups. In the general non-Euclidean situation, the result will show strong analogies with the *fundamental theorem of projective geometry* stating that every transformation of projective space $\mathbb{P}\mathbb{C}^{n+1}$ (whith $n > 1$) preserving collinearity is induced by an element of the group generated by $\mathbb{P}\,\mathrm{Gl}(n+1, \mathbb{C})$ and complex conjugation (see [L85, p.158]).

Let us define for the Grassmannian $G_{2n,n}(\mathbb{F})$ a notion similar to collinearity in projective space: let $F \subset \mathbb{F}^{2n}$ be a subspace of dimension $n + k$, $-n \leq k \leq n$. If $k \geq 0$, we will call the set

$$[F] := \{W \in G_{2n,n} |\, W \subset F\}$$

the *k-pencil in* $G_{2n,n}$ *defined by* F. In other terms, elements $W_1, ..., W_m$ of $G_{2n,n}$ lie on the same k-pencil iff the subspace $< W_1, \ldots, W_m >$ generated by them is of dimension less or equal than $n + k$. If $k \leq 0$, then the *k-pencil in* $G_{2n,n}$ *defined by* F is by definition the set

$$[F] := \{W \in G_{2n,n} |\, F \subset W\}.$$

In other terms, elements $W_1, \ldots, W_m$ of $G_{2n,n}$ lie in this case on the same k-pencil iff their intersection is of dimension greater or equal than $n + k$. It is clear from the usual dimension formulas that k-pencils and $-k$-pencils are in bijection by the map $W \mapsto W^\perp$ where the orthocomplement is taken with respect to some non-degenerate form on $\mathbb{F}^{2n}$; hence k-pencils and $-k$-pencils are the same objects. In contrast to the situation of lines in projective space, there is in general no k-pencil joining two different points, and already 1-pencils are higher dimensional objects. A *k-pencil in* $\mathrm{Herm}(A,\mathbb{F})^c$ is by definition the intersection of a k-pencil in $G_{2n,n}(\mathbb{F})$ with $\mathrm{Herm}(A,\mathbb{F})^c$. The following theorem is a translation of our LIOUVILLE-theorem to this geometrical setting:

Theorem 1.8.1. *Let V be the Jordan algebra $M(n,\mathbb{F})$ $(n>1)$ or one of its subalgebras* $\mathrm{Herm}(A,\mathbb{F})$ *(assumed to be non-isomorphic with $\mathbb{R}$ or $\mathbb{C}$) with conformal compactification V^c given by $G_{2n,n}(\mathbb{F})$, resp. by the set of Lagrangian subspaces described in* Proposition 1.3.1. *Then* $\mathrm{Co}(V)$ *is the group of transformations preserving k-pencils. More precisely, if $\phi : V^c \supset V_1 \to V_2 \subset V^c$ is a locally defined (on some domain V_1 of V^c) diffeomorphism of class $\mathcal{C}^4$ such that pieces of k-pencils contained in V_1 are transformed into pieces of k-pencils contained in V_2 $(k=1,\ldots,n-1)$, then ϕ is rational and has a rational continuation onto V^c, given by an element of* $\mathrm{Co}(V)$.

Proof. We first check that $\mathrm{Co}(V)$ does indeed preserve k-pencils: this is clear for the action of $\mathrm{Gl}(2n,\mathbb{F})$, for the maps induced by conjugations of the base field (i.e. complex conjugation if $\mathbb{F}=\mathbb{C}$) and for maps of the form $W \mapsto W^{\perp}$ (due to fact that k-pencils and $-k$-pencils are the same objects as explained above). As these elements generate $\mathrm{Co}(V)$, this group preserves k-pencils.

Conversely, let ϕ as in the theorem be given. We may assume that $V_1, V_2 \subset V$. In order to conclude using [Be96a, Th.2.3.1] we have to show that for all $x \in V_1$, $D\phi(x) \in \mathrm{Str}(V)$. For this purpose, let us describe the affine picture of k-pencils: two points $X, Y \in V = \mathrm{End}(\mathbb{F}^n)$ lie on the same k-pencil iff $\dim(\Gamma_X \cap \Gamma_Y) \geq k$, that is iff $\mathrm{rk}(X-Y) \leq n-k$. Any k-pencil through X is obtained by forming the sum of X with a k-pencil through 0. The set $\{X | \mathrm{rk}(X) \leq k\}$ is the union of k-pencils through 0, and the set $\{X | \mathrm{Det}(X)=0\}$ is the union of $n-1$-pencils through 0. Let $[V]$ be a k-pencil through 0 defined by F with $\dim F = n+k > n$. We write F_i, $i=1,2$ for the projections of F onto the first, resp. second factor of $\mathbb{F}^n \oplus \mathbb{F}^n$. Because $\Gamma_0 = \mathbb{F}^n \oplus 0 \subset F$, we have $\dim F_1 = n$, hence $\dim F_2 = k$. The condition $\Gamma_X \subset [F]$ is now seen to be equivalent to $im(X) \subset F_2$. This description shows that sums and scalar multiples of elements of $[F]$ still lie on $[F]$; i.e. the affine picture $[F] \cap V$ of $[F]$ is a linear subspace of V. By translation, the affine picture $[F] \cap V$ of a k-pencil through X is an affine subspace $[F_0]+X$ of V where $[F_0] = [F] - X$ is a k-pencil through 0. Coming back to our k-pencil preserving transformation ϕ, it is now clear from the very definition of $D\phi(x)$ being the linearization of ϕ at x that $D\phi(x)$ preserves k-pencils passing through the origin. In formulas,

$$D\phi(X) \cdot [F] = (\phi([F]+X) - \phi(X))$$

for all k-pencils $[F]$ passing through the origin.

Now it remains to show that $\mathrm{Str}(V)$ is the group of linear transformations of V preserving k-pencils passing through the origin. In particular, such a transformation g stabilizes the set $\{X \in V | \mathrm{Det}(X)=0\}$ which is the union of $n-1$-pencils passing through the origin. But the polynomial Det being irreducible, we may conclude that $\mathrm{Det} \circ g$ is a multiple of the polynomial Det, which in turn implies that $g \in \mathrm{Str}(V)$ ([FK94, p.161]). ∎

The Jordan-theoretic construction

In this chapter we construct open and symmetric orbits in the conformal compactification of general semi-simple Jordan algebras (Prop. 2.2.1); we call them *Makarevič spaces* since they have been classified by B.O. Makarevič [Ma73]. We first recall some basic properties of the conformal group of a Jordan algebra.

2.1 The three canonical involutions of the conformal group. We have defined in Section 1.0 the *conformal* or *Kantor-Koecher-Tits group* of a (semi-simple) Jordan algebra V as the group of birational mappings of V generated by the translations τ_v, $v \in V$ ($\tau_v(x) = x + v$), the structure group $\mathrm{Str}(V)$ and the Jordan inversion $j(x) = x^{-1}$. The Lie algebra $\mathfrak{co}(V)$ of $\mathrm{Co}(V)$ is the graded Lie algebra of polynomial vector fields ξ on V which we write as

$$\xi(x) = v + H(x) + P(x)w, \quad v, w \in V, H \in \mathfrak{str}(V) = Lie(\mathrm{Str}(V)),$$

where $P(x)w = 2x^2 w - x(xw)$ is the *quadratic representation* of V. We write

$$\mathfrak{co}(V) = V \oplus \mathfrak{str}(V) \oplus W,$$

for the decomposition of $\mathfrak{co}(V)$ in spaces of homogeneous polynomial vector fields of degree 0, 1 and 2.

Consider the following three involutive conformal mappings of V: $-\mathrm{id}_V$, j and $-j = (-\mathrm{id}) \circ j = j \circ (-\mathrm{id})$. The conjugation in $\mathrm{Co}(V)$ by each of these elements defines an involutive automorphism of $\mathrm{Co}(V)$. Let us describe these involutions. We will, for any $\phi \in \mathrm{Co}(V)$, denote by ϕ_* the induced automorphism on the group level (i.e. the conjugation) as well as on the Lie algebra level (i.e. the adjoint representation). The latter is described by the formula for the action of diffeomorphisms on vector fields living on a vector space,

$$(\phi_* \xi)(x) = ((D\phi)(\phi^{-1}(x))) \cdot \xi(\phi^{-1}(x)).$$

For $\phi = -\mathrm{id}_V$, $((-\mathrm{id})_* \xi)(x) = -\xi(-x)$, and we get the decomposition of $\mathfrak{co}(V)$ in ± 1-eigenspaces of the involution $(-\mathrm{id})_*$:

$$\mathfrak{co}(V) = \mathfrak{str}(V) \oplus (V \oplus W).$$

We know that $(Dj)(x) = -P(x)^{-1}$ (see [FK94, Prop.II.3.3]); hence $(j_* \xi)(x) = -P(x) \cdot \xi(x^{-1})$, in particular $(j_* v)(x) = -P(x)v$ for constant vector fields v. The decomposition of $\mathfrak{str}(V)$ in ± 1-eigenspaces of j_* is given by

$$\mathfrak{str}(V) = \mathrm{Der}(V) \oplus L(V),$$

where $\mathrm{Der}(V)$ is the Lie algebra of derivations of the Jordan algebra V and $L(V) = \{L(v) | v \in V\}$ where $L(v)x = vx$. We thus obtain the following decompositions of $\mathfrak{co}(V)$:

$$\text{w.r.t. } j_* : \quad \mathfrak{co}(V) = (\mathrm{Der}(V) \oplus \mathfrak{q}^{(+)}) \oplus (\mathfrak{q}^{(-)} \oplus L(V)),$$
$$\text{w.r.t.} (-j)_* : \quad \mathfrak{co}(V) = (\mathrm{Der}(V) \oplus \mathfrak{q}^{(-)}) \oplus (\mathfrak{q}^{(+)} \oplus L(V)),$$

where $\mathfrak{q}^{(\pm)} := \{v \pm j_* v | \ v \in V\}$.

Lemma 2.1.1. (i) *The conformal mappings* $-\mathrm{id}_V$ *and* j *are conjugate in* $\mathrm{Co}(V)$, *namely*

$$R(x) := (x-e)(x+e)^{-1}$$

defines an element (the "real Cayley transform") of $\mathrm{Co}(V)$ *such that* $R \circ j \circ R^{-1} = -\mathrm{id}_V$ *and* $R^{-1} \circ j \circ R = -\mathrm{id}_V$. *The inverse of* R *is given by*

$$R^{-1}(x) = -(x+e)(x-e)^{-1} = R \circ (-j)(x) = (-j) \circ R(x).$$

(ii) *If* V *is in addition a complex Jordan algebra, then also* $-\mathrm{id}_V$ *and* $-j$ *are conjugate in* $\mathrm{Co}(V)$, *namely* $C \circ (-j) \circ C^{-1} = -\mathrm{id}_V$ *with* $C := i \circ R \circ i$ *("the Cayley transform"), where* i *is the multiplication by* $i = \sqrt{-1}$ *in* V. *The inverse of* C *is* C.

Proof. (i) Writing $R(x) = e - 2(x+e)^{-1} = \tau_e \circ 2\,\mathrm{id}_V \circ (-j) \circ \tau_e(x)$, we see that $R \in \mathrm{Co}(V)$. Using that V is power-associative, we easily verify that $(e + x^{-1})^{-1} = e - (e+x)^{-1}$. ¿From this we obtain

$$R \circ j(x) = (x^{-1} - e)(x^{-1} + e)^{-1} = (x^{-1} - e)(e - (e+x)^{-1})$$

$$= (x^{-1} - e)x(e+x)^{-1} = (e-x)(e+x)^{-1} = (-\mathrm{id}_V) \circ R(x).$$

For the second relation we have similarly

$$j(R(x)) = ((x-e)(x+e)^{-1})^{-1} = (x+e)(x-e)^{-1} = R(-x).$$

From this we get $R \circ (-j) = j \circ R \circ j = (-j) \circ R$, and one now verifies by similar calculations the formula for the inverse of R.

(ii) $C^{-1}(-\mathrm{id}_V)C = i^{-1}R^{-1}(-\mathrm{id}_V)Ri = -j$ using (i) and $j \circ i = -i \circ j$. Furthermore, $C^2 = iR(-\mathrm{id}_V)Ri = ijR^2 i = ij(-j)i = \mathrm{id}_V$. ■

Remark 2.1.2. If α is an automorphism of V, then $\alpha((x-e)(x+e)^{-1}) = (\alpha(x) - e)(\alpha(x) + e)^{-1}$, i.e. $\alpha R = R\alpha$. From this we get $R(j\alpha)R^{-1} = -\alpha$. (This is a generalization of 1.4 (1).) If V is complex and α a $\mathbb{C}$-conjugate linear ($i\alpha i^{-1} = -\alpha$) automorphism, $C(j\alpha)C^{-1} = \alpha$.

2.2 Makarevič spaces: definition and examples. Consider $\alpha \in \mathrm{Gl}(V)$ with the property that $(j\alpha)^2 = \mathrm{id}_V$—by this we mean that $(\alpha(\alpha x)^{-1})^{-1} = x$ for all x where this expression is defined. Then α is actually in the structure group since $j\alpha j = \alpha^{-1}$ is a linear transformation. Consequently, the conjugation by $j\alpha$ defines an involutive automorphism $j_*\alpha_* = (j\alpha)_*$ of the conformal group $\mathrm{Co}(V)$. We write $\mathrm{Co}(V)^{j_*\alpha_*}$ for its fixed point group. Certainly there exist α such that $(j\alpha)^2 = \mathrm{id}_V$: one may choose α or $-\alpha$ to be an involutive automorphism.

Proposition 2.2.1. *If* V *is a semi-simple Jordan algebra and* $\alpha \in \mathrm{Gl}(V)$ *is such that* $(j\alpha)^2 = \mathrm{id}_V$, *then the orbit*

$$X^{(\alpha)} := \mathrm{Co}(V)_0^{j_*\alpha_*} \cdot 0 \subset V^c$$

is open in the conformal compactification V^c *and is a symmetric space with symmetry* $-\mathrm{id}_V$ *at the origin. The symmetric pair associated to* $X^{(\alpha)}$ *is*

$$(\mathfrak{co}(V)^{j_*\alpha_*}, \mathfrak{str}(V)^{j_*\alpha_*}).$$

Proof. Because α and j commute with $-\mathrm{id}_V$, $\mathrm{Co}(V)^{j_*\alpha_*}$ is stable under the conjugation $(-\mathrm{id})_*$. Therefore $(-\mathrm{id}_V)_*$ induces an involution on this group and on its Lie algebra $\mathfrak{co}(V)^{j_*\alpha_*}$. Recalling that $\mathfrak{co}(V) = \mathfrak{str}(V) \oplus (V \oplus W)$ is the decomposition in ± 1-eigenspaces, we see that

$$\mathfrak{co}(V)^{j_*\alpha_*} = \mathfrak{h}^{(\alpha)} \oplus \mathfrak{q}^{(\alpha)}; \quad \mathfrak{h}^{(\alpha)} = \mathfrak{str}(V)^{j_*\alpha_*},\ \mathfrak{q}^{(\alpha)} = \{v + j_*\alpha v |\, v \in V\};$$

is the decomposition in ± 1-eigenspaces with respect to $(-\mathrm{id}_V)_*$. Let us show now that $\kappa : \mathrm{Co}(V)_0^{j_*\alpha_*} \to V^c, g \mapsto g \cdot 0$ is a submersion, so its image will be open in V^c. By equivariance, it is enough to show that the differential at the origin

$$\dot{\kappa} : \mathfrak{co}(V)^{j_*\alpha_*} \to T_0 V^c = V, \quad \xi \mapsto \xi(0)$$

is surjective. But this is clear because for $v + j_*\alpha v \in \mathfrak{q}^{(\alpha)}$, $(v + j_*\alpha v)(0) = v$ (recall that $j_*\alpha v$ is homogeneous quadratic und thus zero at the origin). Furthermore, $\ker \dot{\kappa} = \mathfrak{h}^{(\alpha)}$ because $\xi(0) = 0$ for all $\xi \in \mathfrak{str}(V)$. This means that the stabilizer of the base point 0 in $\mathrm{Co}(V)^{j_*\alpha_*}$ has Lie algebra $\mathfrak{h}^{(\alpha)}$, and so is open in the subgroup fixed by the involution $(-\mathrm{id}_V)_*$. Thus $X^{(\alpha)}$ is a symmetric space with the associated decomposition of the Lie algebra given above. Let us calculate its symmetry σ at the origin: if $x = g \cdot 0$ with $g \in \mathrm{Co}(V)^{j_*\alpha_*}$, then $\sigma(x) = ((-\mathrm{id}_V)_* g) \cdot 0 = -g \cdot 0 = -x$. ■

We will call a symmetric space, realized as an open orbit $X^{(\alpha)}$ as in the preceding proposition, a *Makarevič space*. Work of Rivillis and Makarevič shows that actually any reductive symmetric space G/H which can be realized as an open orbit in V^c (in such a way that G acts as a subgroup of $\mathrm{Co}(V)$) is isomorphic to a space $X^{(\alpha)}$ ([Ri70, Th.3], [Ma73, Th.3]). Spaces of the form $X^{(\alpha)}$ have been classified by B.O.Makarevič in [Ma73] (but no proof of completeness of this classification is given there). The most important examples are:

2.2.2 A class of causal symmetric spaces. All spaces listed in table 0.3.1 are of the form $X^{(\alpha)}$; see Theorem 2.4.1.

2.2.3 General linear groups $\mathrm{Gl}(n, \mathbb{F})$**.** See 1.6.A.

2.2.4 Unitary groups $\mathrm{U}(A, \varepsilon, \mathbb{F})$**.** See 1.5.B. We have remarked there that only some of these groups can be realized as open orbits in Jordan *algebras*, but all of them in a Jordan *triple system* of skew-Hermitian matrices. In [Be96b] we will call such spaces Makarevič spaces *of the second kind.*

2.2.5 Symmetric cones and their non-convex analogues. See 1.5.A and 1.6.A. We consider here a *Cayley-transformed realization*: we use the relation $\alpha(G^\beta) = G^{\alpha\beta\alpha^{-1}}$ holding for any group G and automorphisms α and β of G. By this relation, for all $\phi \in \mathrm{Co}(V)$,

$$\phi(\mathrm{Co}(V)^{j_*\alpha_*} \cdot 0) = \mathrm{Co}(V)^{\phi_* j_* \alpha_* \phi_*^{-1}} \cdot \phi(0).$$

Taking for ϕ the real Cayley-transform R given in Lemma 2.1.1 we obtain, using that $RjR^{-1} = -\mathrm{id}_V$,

$$R^{-1}(X^{(\mathrm{id}_V)}) = \mathrm{Co}(V)_0^{(R^{-1}jR)_*} \cdot R^{-1}(0) = \mathrm{Co}(V)_0^{(-\mathrm{id}_V)_*} \cdot e = \mathrm{Str}(V)_0 \cdot e,$$

which is by definition the open cone Ω associated to the Jordan algebra V.

2.2.6 Tubes over convex or non-convex cones. Generalizing the preceeding calculation, if α is an *involutive automorphism* of V, then (by remark 2.1.2)

$$R^{-1}(X^{(\alpha)}) = \mathrm{Co}(V)_0^{(-\alpha)_*} \cdot e.$$

This domain contains as an open subset the *tube* $V^- \oplus \Omega^+$, where $V^\pm = \{v \in V | \alpha(v) = \pm v\}$ and Ω^+ is the cone $\Omega^+ = \mathrm{Str}(V^+)_0 \cdot e$ associated to the Jordan algebra V^+. This is an immediate consequence of the fact that $\mathrm{Co}(V)^{(-\alpha)_*}$ contains the translations by elements of V^- and and the group $\mathrm{Str}(V)^{\alpha_*}$ which acts transitively on Ω^+.

2.2.7 Hermitian and pseudo-Hermitian symmetric spaces. If V is a complex Jordan algebra, then $\mathrm{Co}(V)$ is a complex Lie group acting $\mathbb{C}$-rationally on V; hence V^c is a complex manifold on which $\mathrm{Co}(V)$ acts holomorphically. Clearly any Makarevič space $X^{(\alpha)} \subset V^c$ inherits the invariant complex structure from V^c. If α is complex-linear, then $X^{(\alpha)}$ will be a *complex symmetric space*, i.e. a quotient of complex Lie groups. If α is conjugate-linear, then $X^{(\alpha)}$ will be a *pseudo-Hermitian symmetric space* in the proper sense, i.e. have an invariant complex structure without being a quotient of complex Lie groups.

2.2.8 Hermitian and pseudo-Hermitian symmetric spaces of tube type. Let V be a complex Jordan algebra and τ be a conjugation, i.e. a conjugate-linear involution of V. Then V^τ is a real form of V. A similar calculation as in 2.2.5, using the Cayley transform C from Lemma 2.1.1, yields

$$C(X^{(\tau)}) = \mathrm{Co}(V)_0^{(Cj\tau C^{-1})_*} \cdot (-ie) = \mathrm{Co}(V)_0^{\tau_*} \cdot (-ie).$$

This orbit contains as an open subset the *tube* $T_\Omega := V^\tau - i\Omega^\tau$ where Ω^τ is the open symmetric orbit associated to the real form V^τ. In fact, this is a special case of 2.2.6. Here we have the additional feature that multiplication by $i = \sqrt{-1}$ yields an isomorphism of V^τ and $V^{-\tau}$, permitting us to realize the tube over V^τ instead of $V^{-\tau}$. Spaces of this type are called *pseudo-Hermitian symmetric spaces of tube type*; they are studied in [FG95]. If V^τ is Euclidean, then $X^{(\tau)}$ is the well-known tube domain in its disc realization. In [Be96b] we will explain how to consider bounded symmetric domains which are not of tube type as Makarevič spaces *of the second kind.*

2.2.9 c-duals of the preceding spaces. We will prove that $X^{(-\alpha)}$ is the c-dual of $X^{(\alpha)}$ (Proposition 2.3.2). By duality, we get from 2.2.3 and 2.2.4 the spaces $G_\mathbb{C}/G$ where G is one of the above mentioned groups, from the symmetric cones we get compact causal symmetric spaces, and from the bounded symmetric domains we get compact Hermitian symmetric spaces.

2.2.10 Orbits of type $\mathrm{Co}(V)/\mathrm{Str}(V)$. If $V = W \times W$ is the product of two copies of the semi-simple Jordan algebra W, then $\mathrm{Co}(V)_0 = \mathrm{Co}(W)_0 \times \mathrm{Co}(W)_0$ and $(W \times W)^c = W^c \times W^c$ (see [Be96a, Th.2.3.1]). We define an involutive automorphism of V by $\alpha((x,y)) = (y,x)$. The induced involution of $\mathrm{Co}(W) \times \mathrm{Co}(W)$ is given by $\alpha_*(g,h) = (h,g)$, and because $j_V = j_W \times j_W$ we get $(\alpha j)_*(g,h) = (j_* h, j_* g)$. The fixed point group of this involution is $\{(g, j_* g) | g \in \mathrm{Co}(W)\}$, and an element of this group stabilizes $0_V = (0_W, 0_W)$ iff

$g \in \mathrm{Str}(W)$. Hence

$$X^{(\alpha)} = \mathrm{Co}(W \times W)_0^{(\alpha j)_*} \cdot 0 \cong \mathrm{Co}(W)_0 / \mathrm{Str}(W)_0.$$

If we replace α by $-\alpha$, we get an isomorphic orbit: let $J(x,y) = (-y,x)$; then $jJj = -J$, hence $J \in \mathrm{Str}(V)$, and the lemma to be stated next implies then that $X^{(-\alpha)} = JX^{\alpha}$.

Back to the general set-up. In order to get hold of the spaces $X^{(\alpha)}$, we write

$$\mathrm{Str}(V)^{Jj_*} := \{\alpha \in \mathrm{Str}(V) |\, j_*(\alpha) = \alpha^{-1}\}.$$

This is the fixed-point set of the involutive anti-automorphism Jj_* of $\mathrm{Str}(V)$, where $J(g) = g^{-1}$ is the inversion in $\mathrm{Str}(V)$ and $j_*(\alpha) = j\alpha j$ is the canonical involution of $\mathrm{Str}(V)$. The formula $g \cdot \alpha := j_* g \circ \alpha \circ g^{-1}$ defines an action of $\mathrm{Str}(V)$ on $\mathrm{Str}(V)^{Jj_*}$ which is in general not transitive. The next lemma states that $X^{(\alpha)}$ essentially only depends on the connected components of $\mathrm{Str}(V)^{Jj_*}$.

Lemma 2.2.11. (i) *The action of* $\mathrm{Str}(V)$ *is transitive on every connected component of* $\mathrm{Str}(V)^{Jj_*}$.

(ii) *For all* $g \in \mathrm{Str}(V)$ *and* $\alpha \in \mathrm{Str}(V)^{Jj_*}$, $X^{(g\cdot\alpha)} = g(X^{(\alpha)})$ *(the space translated by the map* g*).*

Proof. (i) This is a general fact about any Lie group G with involution σ, acting on the set $G^{J\sigma} = \{g \in G | \sigma(g) = g^{-1}\}$ by $g \cdot \alpha = \sigma(g)\alpha g^{-1}$: fix $\alpha \in G^{J\sigma}$; its stabiliser in G is the fixed point group $G^{\alpha_*^{-1}\circ\sigma}$, and the submanifold $\alpha G^{\alpha_*^{-1}\circ\sigma}$ of G intersects $G^{J\sigma}$ transversally at α (the condition $\alpha g \in G^{\sigma J}$ with $g \in G^{\alpha_*^{-1}\circ\sigma}$ implies that $g = g^{-1}$ which is only trivially solvable in a neighbourhood of the origin of G). Hence $G \cdot \alpha$ has the same dimension as $G^{J\sigma}$.

(ii) For $g \in \mathrm{Str}(V)$ and $\alpha \in G^{Jj_*}$, $j \circ (g \cdot \alpha) = j \circ jgj \circ \alpha \circ g^{-1} = g \circ j\alpha \circ g^{-1}$, so

$$\mathrm{Co}(V)^{j_*(g\cdot\alpha)_*} \cdot 0 = \mathrm{Co}(V)^{g_* j_* \alpha_* g_*^{-1}} \cdot 0 = (g_*(\mathrm{Co}(V)^{j_*\alpha_*})) \cdot 0 = g(\mathrm{Co}(V)^{j_*\alpha_*} \cdot 0).$$

■

Corollary 2.2.12. *Let* V *be a complex Jordan algebra, and denote by* $i : V^c \to V^c$ *multiplication by* $i = \sqrt{-1}$.

(i) *For all* $\mathbb{C}$*-linear* $\alpha \in \mathrm{Str}(V)^{Jj_*}$,

$$iX^{(\alpha)} = X^{(-\alpha)}.$$

(ii) *For all* $\mathbb{C}$*-conjugate-linear* $\alpha \in \mathrm{Str}(V)^{Jj_*}$ *and* $t \in \mathbb{R}$,

$$e^{it} X^{(\alpha)} = X^{(\alpha)}$$

(i.e. the pseudo-Hermitian spaces from example 2.2.7 are circled.)

Proof. For any scalar $\lambda \in \mathbb{C}$, $\lambda \cdot \alpha = \lambda^{-1} \circ \alpha \circ \lambda^{-1}$.

(i) By $\mathbb{C}$-linearity of α, $i \cdot \alpha = i \circ \alpha \circ i = -\alpha$. Now use the preceeding lemma, part (ii).

(ii) By conjugate-linearity of α, $e^{it} \cdot \alpha = \alpha$, and the claim follows as above. ■

2.3 Complexifications and c-duality. Any symmetric space $X = G/H$ admits, at least locally, a *complexification* $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$. We first show that the spaces $X^{(\alpha)}$ always admit *globally* a complexification, and we will not have to make assumptions of the kind "$G_{\mathbb{C}}$ simply connected". This is due to the fact that our set-up is essentially algebraic. There are two interesting features related to complexification: first, the spaces $X^{(\alpha)}$ and $X^{(-\alpha)}$ are *c-dual*, and second, the spaces $X^{(\alpha)}$ admit yet another kind of complexification which we call *Hermitian complexification*: namely, they are real forms of *pseudo-Hermitian spaces* introduced in Example 2.2.7. For general symmetric spaces, no such complexification is known; hence it seems that this is a quite specific feature of Makarevič spaces. It turns out that all is a straightforward consequence of the fact that a real Jordan algebra V can be complexified, just as a Lie algebra, in a natural way (see [FK94, ch. VIII]); we will denote by $V_{\mathbb{C}}$ the complexification of V.

Lemma 2.3.1. *Let V be a semi-simple Jordan algebra, $V_{\mathbb{C}}$ its complexification and τ the conjugation of $V_{\mathbb{C}}$ such that $V_{\mathbb{C}}^{\tau} = V$. Then* $\mathrm{Co}(V_{\mathbb{C}})$ *(respectively, its Lie-algebra $\mathfrak{co}(V_{\mathbb{C}})$) is stable under conjugation with τ (resp. under its differential at the origin), and if we define*

(i) $$\mathrm{Co}(V) \to \mathrm{Co}(V_{\mathbb{C}})^{\tau_*}, \quad \phi \mapsto \phi_{\mathbb{C}},$$

(ii) $$\mathfrak{co}(V) \to \mathfrak{co}(V_{\mathbb{C}})^{\tau_*}, \quad \xi \mapsto \xi_{\mathbb{C}},$$

where $\phi_{\mathbb{C}}$ (resp. $\xi_{\mathbb{C}}$) is the unique $\mathbb{C}$-rational (resp. -polynomial) continuation of ϕ (resp. ξ), then (i) *is an injection as an open subgroup and* (ii) *is an isomorphism.*

Proof. We first check, using that τ commutes with the Jordan-inverse j, that the structure group $\mathrm{Str}(V_{\mathbb{C}}) \subset \mathrm{Gl}(V_{\mathbb{C}})$ (and hence also its Lie algebra) is stable by the conjugation τ_*. Then the chain rule implies immediately that, if ϕ is $\mathrm{Str}(V_{\mathbb{C}})$-conformal, this is also the case for $\tau_*(\phi) = \tau\phi\tau$, so $\mathrm{Co}(V_{\mathbb{C}})$ is τ_*-stable, and taking differentials at the origin we get the analogous statement for the Lie algebra. It is clear that (i) and (ii) are injective. To show that (ii) is also surjective, one just shows that $\xi \mapsto \xi|_V$ is a well-defined inverse of (ii): in fact, if ξ is τ_*-fixed, then, if $x \in V = V_{\mathbb{C}}^{\tau}$ also $\xi(x) \in V$, and it is easily seen that then $\xi|_V$ is $\mathfrak{str}(V)$-conformal. So (ii) is an isomorphism, and that the image of (i) is open is an immediate consequence. ∎

We will from now on consider the map (i) of the preceding lemma as an inclusion and thus get an inclusion, compatible with the inclusion $V \subset V_{\mathbb{C}}$, of the corresponding conformal compactifications:

$$V^c = \mathrm{Co}(V_{\mathbb{C}})_0^{\tau_*} \cdot 0 \subset (V_{\mathbb{C}})^c = \mathrm{Co}(V_{\mathbb{C}})/P_{\mathbb{C}}.$$

If $X^{(\alpha)}$ is a conformally flat symmetric space, we can now define its inclusion in its complexified space $X_{\mathbb{C}}^{(\alpha)}$ by

$$X^{(\alpha)} = \mathrm{Co}(V)_0^{j_*\alpha_*} \cdot 0 \subset X_{\mathbb{C}}^{(\alpha)} := \mathrm{Co}(V_{\mathbb{C}})_0^{j_*\alpha_*} \cdot 0 \subset (V_{\mathbb{C}})^c,$$

where we use the same letters j and α for the corresponding conformal map of V as well as for its $\mathbb{C}$-rational continuations. The conjugation of $X_{\mathbb{C}}^{(\alpha)}$ with respect to $X^{(\alpha)}$ is τ. Recall that the *c-dual* of a symmetric space $X = G/H$ is a symmetric space $Y = L/H$ such that the associated eigenspace decomposition of the Lie algebra $\mathfrak{l}$ of L is $\mathfrak{l} = \mathfrak{h} \oplus i\mathfrak{q}$ if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the decomposition associated to G/H. It is well-known that Riemannian symmetric spaces of compact and non-compact type are c-dual in this sense.

Proposition 2.3.2. *For all $\alpha \in \mathrm{Str}(V)^{Jj_*}$, the symmetric spaces $X^{(\alpha)}$ and $X^{(-\alpha)}$ are c-duals of each other.*

Proof. By Corollary 2.2.12 (i) we have $iX_{\mathbb{C}}^{(-\alpha)} = X_{\mathbb{C}}^{(\alpha)}$. Recall that $-\mathrm{id}_V$ is the geodesic symmetry of a conformally flat symmetric space with respect to the origin. Hence the c-dual real form of $X_{\mathbb{C}}^{(\alpha)}$ is given by

$$(X_{\mathbb{C}}^{(\alpha)})_0^{-\tau} = (iX_{\mathbb{C}}^{(-\alpha)})_0^{-\tau} = i(X_{\mathbb{C}}^{(-\alpha)})_0^{\tau} = iX^{(-\alpha)},$$

i.e. $X^{(-\alpha)}$ is isomorphic to the real form of $X_{\mathbb{C}}$ with respect to the conjugation $-\tau$. This means that $X^{(-\alpha)}$ is (globally) c-dual to $X^{(\alpha)}$, and it implies the weaker, infinitesimal notion of c-duality introduced above. ∎

Example: the Borel-imbedding. a) Let V be Euclidean. Then $X^{(\mathrm{id}_V)}$ is a Riemannian symmetric space of non-compact type, isomorphic to the symmetric cone Ω (ex. 2.2.5). Hence $X^{(-\mathrm{id}_V)}$ is compact. Because it is open in the connected space V^c, we have $X^{(-\mathrm{id})} = V^c$.

b) Let V be complex and τ be a conjugation with respect to a Euclidean real form. Then $D = X^{(\tau)}$ is a bounded symmetric domain and hence of non-compact type (ex. 2.2.8), and it follows as above that $X^{(-\tau)} = V^c$. The imbedding $X^{(\tau)} = D \subset V^c = X^{(-\tau)}$ is the well-known *Borel-imbedding* of the disc D into its compact dual. We can thus consider the preceeding proposition as a generalization of the Borel-imbedding where in general the situation will be much more complicated because we won't have inclusion of one space in the other but only open intersections.

2.3.3 The Hermitian complexification of $X^{(\alpha)}$. It is defined by by

$$X_{\overline{\mathbb{C}}}^{(\alpha)} := \mathrm{Co}(V_{\mathbb{C}})_0^{(j\tau\alpha)_*} \cdot 0 \subset (V_{\mathbb{C}})^c.$$

Here $\tau\alpha = \alpha\tau$ is nothing but the conjugate-linear continuation of α onto $V_{\mathbb{C}}$. As explained in example 2.2.7, $X_{\overline{\mathbb{C}}}^{(\alpha)}$ is a *pseudo-hermitian symmetric space*, and $X^{(\alpha)}$ appears as its real form with respect to the conjugation τ. Remark that, by Cor. 2.2.12 (ii), the real form with respect to the conjugation $-\tau$ will be isomorphic to $X^{(\alpha)}$, so there is no notion of "hermitian c-dual". This is related to the following remarkable property: *Multiplication by i in any tangent space of $X_{\overline{\mathbb{C}}}^{(\alpha)}$ extends to a globally biholomorphic map of $X_{\overline{\mathbb{C}}}^{(\alpha)}$.* We finally remark that, if V is already a complex Jordan-algebra, then $V_{\mathbb{C}} \cong V \times V$, and then the Hermitian complexification will also be a complex symmetric space in the ordinary sense; one may check that in this case $X_{\overline{\mathbb{C}}}^{(\alpha)} \cong \mathrm{Co}(V)/\mathrm{Str}(V)$, *cf.* Ex. 2.2.10.

2.4 The Euclidean case (causal symmetric spaces). Recall that the cone $\Omega = \mathrm{Str}(V)_0 \cdot e$ associated to a Jordan-algebra is convex if and only if V is *Euclidean*. Thus the Makarevič spaces $X^{(\alpha)}$ have an invariant causal (flat) structure given by Ω if and only if the Jordan-algebra used in the construction is Euclidean. Recall that $G(\Omega)$ is the group of all linear automorphisms of Ω; it is an open subgroup of $\mathrm{Str}(V)$ ([FK94, p.150]).

Theorem 2.4.1. *Let V be a Euclidean Jordan algebra (having no ideal isomorphic to $\mathbb{R}$) and Ω the associated symmetric cone.*

(i) *Every locally defined causal transformation (class $\mathcal{C}^4$) of the flat causal structure defined by Ω is birational, and its rational extension is given by an element of the group* $\mathrm{Co}(G(\Omega))$ *generated by the translations, the group $G(\Omega)$ and $-j$, where j is the Jordan inversion. These transformations extend to globally defined causal automorphisms of the conformal compactification V^c of V. The identity component of the causal group* $\mathrm{Co}(G(\Omega))$ *is* $\mathrm{Co}(V)_0$.

(ii) *If $X^{(\alpha)} \subset V^c$ is a Makarevič space, then $X^{(\alpha)}$ inherits a (flat) causal structure from V^c. Every element of the causal pseudogroup of this structure on $X^{(\alpha)}$ is rational and can, by* (i)*, be identified with an element of* $\mathrm{Co}(G(\Omega))$.

(iii) *The symmetric spaces $X = L/H$ and $X' = L'/H$ given in table 0.3.1 can be realized as Makarevič spaces in V^c where V is given in the column to the right, and their causal pseudogroup can, by* (ii)*, be identified with a group of birational transformations the identity component of which is the group* $\mathrm{Co}(V)_0$ *given in the table.*

(iv) *Table* 0.3.1 *gives a complete list of Makarevič spaces associated to simple Euclidean Jordan algebras.*

Proof. The first claim of (i) is a restatement of [Be96a, Th.2.3.1 (ii)], observing that a local diffeomorphism ϕ is causal if and only if $D\phi(x) \in G(\Omega)$ for all x where ϕ is defined. By the same theorem, $\mathrm{Co}(V)$ and $\mathrm{Co}(G(\Omega))$ have the same identity component. Its action on V^c is transitive by definition of V^c, and hence the causal structure of V can, by forward transport, be extended to an invariant causal structure on V^c. This proves (i). Now (ii) is just the specialization of (i) to maps having domain and range in $X^{(\alpha)}$.

(iii) The spaces $X = L/H$ and $X' = L'/H$ are c-duals of each other, and by Prop. 2.3.2 one of these two spaces admits a causal imbedding into V^c if and only if the other does. For each line corresponding to the cases I - III we constructed in Section 1.5 explicitely the imbedding of one of the spaces L/H or L'/H into V^c; in fact, the first line in each case contains the so-called *Cayley-type spaces* which arise from ex. 2.2.10 (see 1.5.C); the second line contains the open $\mathrm{Str}(V)_0$-orbits in V, in particular the symmetric cone and its compact dual (ex. 2.2.5, see 1.5.A), and the following lines are from 1.5.D (case I) and 1.5.B (case II and III). The list for cases IV and V is taken from [Ma73, p.416]; for case IV see also [Ri69]. By (ii), the causal pseudogroup is given by $\mathrm{Co}(V)_0$ which we have described, for cases I–III, in Section 1.4; for the other cases *cf.* [Ma73].

The proof of part (iv) will be prepared by somme lemmas.

Lemma 2.4.2. *If V is Euclidean, then j_* is a Cartan-involution of* $\mathrm{Str}(V)$, *and* $\mathrm{Str}(V)^{j_*} = \{\pm 1\} \cdot \mathrm{Aut}(V)$.

Proof. If V is Euclidean, then $\mathrm{Str}(V) = \pm 1 \cdot G(\Omega)$ (see [FK94, p.150]). Because an element of the structure group is an automorphism if and only if it fixes the unit element e (see [FK94, p.148]), we can write $\Omega \cong G(\Omega)/\operatorname{Aut}(V)$, and the involution of this symmetric space is j. As is well known, Ω is a symmetric space of the non-compact type, and thus j_* is a Cartan-involution of $G(\Omega)$ and also of $\mathrm{Str}(V) = \pm 1 \cdot G(\Omega)$, and $\pm G(\Omega)^{j_*} = \pm 1 \cdot \operatorname{Aut}(V)$. ■

One can also show that, if V is Euclidean, using that $\mathrm{Co}(V)/\mathrm{Co}(V)^{(-j)_*}$ is a symmetric space of the non-compact type (the tube domain), that $(-j)_*$ is a Cartan-involution of $\mathrm{Co}(V)$.

Lemma 2.4.3. *Let G be a reductive Lie group, $\theta : G \to G$ be a Cartan-involution with associated decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ of the Lie algebra of G and Cartan decomposition $G = K \exp \mathfrak{p}$, and let $G^{J\theta} = \{g \in G | \theta(g) = g^{-1}\}$. Then the Cartan decomposition induces a diffeomorphism*

$$G^{J\theta} \cong \{(k, X) |\, k \in K, k = k^{-1}; X \in \mathfrak{p}, \mathrm{Ad}(k)X = X\}.$$

Proof. Writing $g \in G^{J\theta}$ as $g = k \exp X$ with $k \in K = G^\theta$ and $X \in \mathfrak{p} = \mathfrak{g}^{-\theta}$, an easy calculation shows that the condition $\theta(g) = g^{-1}$ is equivalent to $k = k^{-1}$ and $X = \mathrm{Ad}(k)X$. ■

Corollary 2.4.4. *If V is Euclidean, then any Makarevič space $X^{(\alpha)}$ associated to V is isomorphic to a space $X^{(\pm\beta)}$ arising from an involutive automorphism β or from its negative $-\beta$.*

Proof. As j_* is a Cartan-involution of $\mathrm{Str}(V)$, every connected component of $\mathrm{Str}(V)^{Jj_*}$ contains by the previous lemma an element $k \in K = \pm \operatorname{Aut}(V)$ with $k^2 = id$, which is thus either an involutive automorphism of V or its negative. By Lemma 2.2.12, the space associated to any element of the connected component of $\mathrm{Str}(V)^{Jj_*}$ containing k is isomorphic to the space associated to k. ■

The involutive automorphisms of a Euclidean Jordan algebra have been classified, see [Kay94] or [H67]. Using Corollary 2.4.4 and this classification, one can check that table 0.3.1 gives indeed a complete list of all Makarevič spaces associated to simple Euclidean Jordan algebras. ■

References

[AI89] Azizov, T. Y, and I.S. Iokhvidov, “Linear operators in spaces with an indefinite metric,” Wiley, 1989.

[Be96a] Bertram, W., *Un théorème de Liouville pour les algèbres de Jordan*, Bulletin Soc. Math. Française **124** (1996), 299–327.

[Be96b] —, *Algebraic structures of Makarevič spaces*, in preparation.

[Bou59] Bourbaki, N., “Algèbre, chapitre 9 - Formes sesquilinéaires et formes quadratiques,” Hermann, Paris 1959.

[FG95] Faraut, J., and S. Gindikin, *Pseudo-Hermitian Spaces of Tube Type*, Preprint 1995.

[FK94] Faraut, J. and A. Koranyi, “Analysis on symmetric Cones,” Oxford University Press, Oxford 1994.

[FÓ95] Faraut J., and G. Ólafsson, *Causal semi-simple symmetric spaces, the geometry and harmonic analysis*, in: Semigroups in Algebra, Geometry and Analysis, Eds. K. H. Hofmann, J. D. Lawson, and E. B. Vinberg; de Gruyter, Berlin 1995.

[Gi92] Gindikin, S., *Generalized conformal structures on classical real Lie groups and related problems on the theory of representations*, Comptes Rendus de l’Acad. des Sciences de Paris **315** (1992), 675–679.

[GK95] Gindikin, S., and S. Kaneyuki, *On the automorphism group of the generalized conformal structure of a symmetric* R*-space*, Preprint 1995.

[H67] Helwig, K., “Halbeinfache reelle Jordan-Algebren,” Habilitationsschrift, München 1967.

[Kay94] Kayoya, J. B., “Analyse sur les algèbres de Jordan simples,” Thése, Université de Paris 6, 1994.

[KN64] Kobayashi, S., and T.Nagano, *On filtered Lie Algebras and Geometric Structures, I*, J. Math. Mech **13** (1964), 875–908.

[L85] Lelong-Ferrand, J., “Les fondements de la géométrie,” PUF, Paris 1985.

[Lo77] Loos, O., “Bounded symmetric domains and Jordan pairs,” Lecture Notes, Irvine 1977.

[Ma73] Makarević, B. O., *Open symmetric orbits of reductive groups in symmetric R-spaces*, Math. USSR Sbornik **20**(3) (1973), 406–418.

[Ri69] Rivillis, A. A., *Homogeneous locally symmetric regions in a conformal space*, Soviet. Math. Dokl. **10** (1969), 147–150.

[Ri70] —, *Homogeneous locally symmetric regions in homogeneous spaces associated with semisimple Jordan-algebras*, Math. USSR Sbornik **11**(3) (1970), 377–389.

[Sa80] Satake, I., “Algebraic Structures of symmetric domains,” Iwanami Shoten, Princeton Univ. Press 1980.

[Se76] Segal, I. E., “Mathematical cosmology and extragalactic astronomy,” Academic Press 1976.

[Wey23] Weyl, H., “Mathematische Analyse des Raumproblems,” Springer, Berlin 1923.

[Wey39] —, “The classical groups,” Princeton Univ. Press 1939.

Technische Universität Clausthal
Institut für Mathematik
Erzstraße 1
D–38678 Clausthal-Zellerfeld

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