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Torus actions on compact quotients

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Abstract. A Lefschetz formula for actions of noncompact tori on compact quotients of Lie groups is given.

Introduction

Let G denote a Lie group and Γ a uniform lattice in G. We fix a maximal torus T in G and consider the action of T on the compact quotient $\Gamma\backslash G$. Assuming T to be noncompact we will prove a Lefschetz formula relating compact orbits as local data to the action of the torus T on a global cohomology theory (tangential cohomology). Modulo homotopy, the compact orbits are parametrized by those conjugacy classes $[\gamma]$ in Γ whose G-conjugacy classes meet T in points which are regular in the split component. Having a bijection between homotopy classes and conjugacy classes in the discrete group we will identify these two. For a class $[\gamma]$ let X_γ be the union of all compact orbits in that class. Then it is known that X_γ is a smooth submanifold and with $\chi_r(X_\gamma)$ we denote its de-twisted Euler characteristic (see sect. 2.). Note that $\chi_r(X_\gamma)$ is local, i.e. it can be expressed as the integral over X_γ of a canonical differential form (generalized Euler form). On the other hand $\chi_r(X_\gamma)$ can be expressed as a simple linear combination of Betti numbers (see sect. 2.). Next, λ_γ will denote the volume of the orbit and P_s the stable part of the Poincaré map around the orbit. Then the number

$$L(\gamma) := \frac{\lambda_\gamma \chi_r(X_\gamma)}{\det(1-P_s)}$$

will be called the Lefschetz number of $[\gamma]$ (compare [8]). The class $[\gamma]$ defines a point a_γ in the split part A of the torus T modulo the action of the Weyl group. In the case when the Weyl group has maximal size (for example when T is maximally split) our Lefschetz formula is an equality of distributions:

$$\sum_{[\gamma]} L(\gamma)\delta_{a_\gamma} = \mathrm{tr}(.|H^*(\mathcal{F})),$$

where H^* is the tangential cohomology of the unstable/neutral foliation $\mathcal{F}$ induced by the torus action. In [6] a similar formula is proven to hold up to a smooth

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function in the case of a flow. The present paper extends results of Andreas Juhl [10], [13] in the real rank one case. See also [11], [12].

1. Euler-Poincaré functions

In this section and the next we list some technical results for the convenience of the reader. Let G denote a real reductive group of inner type [14] and fix a maximal compact subgroup K. Let (τ, V_τ) be a finite dimensional unitary representation of K and write $(\breve{\tau}, V_{\breve{\tau}})$ for the dual representation. Assume that G has a compact Cartan subgroup $T \subset K$. Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the polar decomposition of the real Lie algebra $\mathfrak{g}_0$ of G and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for its complexification. Choose an ordering of the roots $\Phi(\mathfrak{g}, \mathfrak{t})$ of the pair $(\mathfrak{g}, \mathfrak{t})$. This choice induces a decomposition $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{p}_+$.

Proposition 1.1. *For (τ, V_τ) a finite dimensional representation of K there is a compactly supported smooth function f_τ on G such that for every irreducible unitary representation (π, V_π) of G we have:*

$$\mathrm{tr}\ \pi(f_\tau) = \sum_{p=0}^{\dim(\mathfrak{p})} (-1)^p \dim(V_\pi \otimes \wedge^p \mathfrak{p} \otimes V_{\breve{\tau}})^K.$$

Proof. [5]. □

Proposition 1.2. *Let g be a semisimple element of the group G. If g is not elliptic, then the orbital integral $\mathcal{O}_g(f_\tau)$ vanishes. If g is elliptic we may assume $g \in T$, where T is a Cartan in K and then we have*

$$\mathcal{O}_g(f_\tau) = \frac{\mathrm{tr}\ \tau(g) |W(\mathfrak{t}, \mathfrak{g}_g)| \prod_{\alpha \in \Phi_g^+} (\rho_g, \alpha)}{[G_g : G_g^0] c_g},$$

where c_g is Harish-Chandra's constant, it does only depend on the centralizer G_g of g. Its value is given for example in [3].

Proof. [5]. □

Proposition 1.3. *For the function f_σ we have for any $\pi \in \hat{G}$:*

$$\mathrm{tr}\ \pi(f_\sigma) = \sum_{p=0}^{\dim\ \mathfrak{g}/\mathfrak{k}} (-1)^p \dim\ \mathrm{Ext}^p_{(\mathfrak{g},K)}(V_\sigma, V_\pi),$$

i.e. f_σ gives the Euler-Poincaré numbers of the $(\mathfrak{g}, K)$-modules (V_σ, V_π). This justifies the name Euler-Poincaré function.

Proof. [5]. □

2. De-twisted Euler characteristics

Let $\mathcal{C}^+$ denote the category of complexes of $\mathbf{C}$-vector spaces which are zero in negative indices and have degreewise finite dimensional cohomology, i.e. the dimension of $H^j(E)$ is finite for all j. Let $\mathcal{K}^+$ denote the weak Grothendieck group of $\mathcal{C}^+$, i.e. $\mathcal{K}^+$ is the abelian group generated by all isomorphism classes of objects modulo the relations $A = B + C$, whenever any object in A is isomorphic to the direct sum of an object in B and one in C. An element $E = E_+ - E_-$ of $\mathcal{K}^+$ is called a *virtual complex*. Define the *de-twist* of an element E of $\mathcal{K}^+$ as

$$E' = \sum_{k=0}^{\infty} E[-k],$$

where $E[k]_j = E_{k+j}$. Since the sum is degreewise finite this defines a new element of $\mathcal{K}^+$. The higher de-twists are defined inductively, so $E^{(0)} = E$ and $E^{(r+1)} = E^{(r)\prime}$.

We need to extend the notion of an *Euler characteristic* to infinite virtual complexes by

$$\chi(E) = \sum_{k=0}^{\infty} (-1)^k \dim H^k(E),$$

provided $\dim H^k(E) = \dim H^k(E_+) - \dim H^k(E_-)$ vanishes for almost all k.

Call a virtual complex *cohomologically finite* if $\dim H^j(E) = 0$ for large j, in other words, the total cohomology H(E) is finite dimensional.

Observation. *Let the virtual complex E be cohomologically finite and assume that the Euler characteristic $\chi(E)$ vanishes. Then the de-twist E' is cohomologically finite.*

So start with a cohomologically finite virtual complex E. If $E^{(1)}, \dots, E^{(r)}$ are cohomologically finite we have

$$\chi(E^{(0)}) = \dots = \chi(E^{(r-1)}) = 0$$

and

$$\chi(E^{(r)}) = (-1)^r \sum_{j=0}^{\infty} \binom{j}{r} (-1)^j \dim H^j(E).$$

This is easily proven by induction on r. This motivates the following definition: The *r-th de-twisted Euler characteristic* of a cohomologically finite virtual complex E is defined by

$$\chi_r(E) := (-1)^r \sum_{j=0}^{\infty} \binom{j}{r} (-1)^j \dim H^j(E).$$

To every compact manifold M we now can attach a sequence of Euler numbers

$$\chi_0(M), \dots, \chi_n(M),$$

where n is the dimension of M. The most significant of these is, as we shall see, the first nonvanishing one, so define the *generic Euler number* of M as

$$\chi_{gen}(M) = \chi_r(M), \quad \text{where } r \text{ is the least index with } \chi_r(M) \neq 0.$$

Proposition 2.1. *Let M,N be compact manifolds. We have*

$$\chi_{gen}(M \times N) = \chi_{gen}(M)\chi_{gen}(N).$$

Proof. See [4]. □

To give another example of a situation in which higher Euler characteristics occur we will describe a situation in Lie algebra cohomology which will show up later.

We consider a short exact sequence

$$o \to \mathfrak{n} \to l \to \mathfrak{a} \to 0$$

of finite dimensional complex Lie algebras where $\mathfrak{a}$ is abelian. In such a situation an l-module V is called *acceptable* , if the $\mathfrak{a}$-module $H^q(\mathfrak{n}, V)$ is finite dimensional. Note that V itself need not be finite dimensional.

Example 1. Any finite dimensional l-module will be acceptable.

Example 2. Let $\mathfrak{g}_0$ denote the Lie algebra of a semisimple Lie group G of the Harish-Chandra class, i.e. G is connected and has a finite center. Let K be a maximal compact subgroup of G and let $G = KAN$ be an Iwasawa decomposition of G. Write the corresponding decomposition of the complexified Lie algebra as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Now let $l = \mathfrak{a} \oplus \mathfrak{n}$ with the structure of a subalgebra of $\mathfrak{g}$. Consider an admissible $(\mathfrak{g}, K)$-module V. A theorem of [HeSchm] assures us that V then is an acceptable l-module.

Proposition 2.2. *Let*

$$o \to \mathfrak{n} \to l \to \mathfrak{a} \to 0$$

be an exact sequence of finite dimensional complex Lie algebras. Assume that the Lie algebra $\mathfrak{a}$ *is abelian. Let* V *be an acceptable* l*-module. Then with* $r = \dim(\mathfrak{a})$ *we have*

$$\chi_0(H^*(l, V)) = \ldots = \chi_{r-1}(H^*(l, V)) = 0,$$

and

$$\chi_r(H^*(l, V)) = \chi_0(H^*(\mathfrak{n}, V)^{\mathfrak{a}}),$$

where $H^*(\mathfrak{n}, V)^{\mathfrak{a}}$ *denotes the* $\mathfrak{a}$*-invariants in* $H^*(\mathfrak{n}, V)$.

Proof. [5]. □

3. The Lefschetz formula

Let G be a connected Lie group and $\Gamma \subset G$ a uniform lattice. Note that the existence of Γ forces G to be unimodular. Fix a Haar measure on G and consider the representation of G on the Hilbert space $L^2(\Gamma \backslash G)$ given by $R(g)\varphi(x) = \varphi(xg)$. For any smooth compactly supported function f on G define $R(f)\varphi(x) := \int_G f(y)\varphi(xy)dy$, then a calculation shows that $R(f)$ is an integral operator with smooth kernel $k(x, y) = \sum_{\gamma \in \Gamma} f(x^{-1}\gamma y)$. From this it follows that $R(f)$ is a trace class operator. Since this holds for any f we conclude that $L^2(\Gamma\backslash G)$ decomposes under G as a discrete sum of irreducibles with finite multiplicities:

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi\in\hat{G}} N_\Gamma(\pi)\pi.$$

It follows that $\operatorname{tr} R(f) = \sum_{\pi\in\hat{G}} N_\Gamma(\pi) \operatorname{tr}\pi(f)$. On the other hand, the trace of $R(f)$ equals the integral over the diagonal of the kernel, so

$$\begin{aligned} \operatorname{tr} R(f) &= \int_{\Gamma\backslash G} k(x,x)dx \\ &= \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f), \end{aligned}$$

where $\mathcal{O}_\gamma(f) := \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx$ is the orbital integral. Note that this expression depends on the choice of a Haar measure on G_γ. So we state the Selberg trace formula as

$$\sum_{\pi\in\hat{G}} N_\Gamma(\pi) \operatorname{tr}\pi(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma)\mathcal{O}_\gamma(f).$$

From now on we will *assume*:

(A1) G is a semidirect product:

$$G \cong H \ltimes R$$

of an abelian Lie group R and a semisimple connected Lie group H with finite center.

For the following fix a maximal torus T of H, write $T = AB$, where A is the split component and B is compact. Let $P = MAN$ a parabolic then $B \subset M$. Let A^{reg} be the set of regular elements of the split torus A. Since H acts on R it acts on the unitary dual $\hat{R}$. Our second assumption is

(A2) Any element of $A^{reg}M$ acts freely on $R - \{0\}$ and on $\hat{R} - \{\text{triv}\}$.

For any $\tau \in \hat{R}$ let H_τ be its stabilizer in H. For the trivial representation we clearly have $H_{\text{triv}} = H$. The condition (A2) says that for any nontrivial $\tau \in \hat{R}$ we have $H_\tau \cap A^{reg}M = \emptyset$.

Example. Clearly any semisimple connected G with finite center would give an example but there are also a lot of nonreductive examples such as the following: Let $R := \mathrm{Mat}_2(\mathbb{R})$ with the addition, $H := \mathrm{SL}_2(\mathbb{R})$ and let H act on R by matrix multiplication from the left. Let $G := H \ltimes N$ and $T := \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} \right\}$. It is easily seen that our assumptions are satisfied in this case.

We will only consider uniform lattices of the form $\Gamma = \Gamma_H \ltimes \Gamma_R$, where Γ_H and Γ_R are uniform lattices in H and R. We will further assume Γ_H to be *weakly neat*, this means, Γ_H is a cocompact torsion free discrete subgroup of H which is such that for any $\gamma \in \Gamma_H$ the adjoint $\mathrm{Ad}(\gamma)$, acting on the Lie algebra of H does not have a root of unity $\neq 1$ as an eigenvalue. Any arithmetic group has a weakly neat subgroup of finite index [1].

Example. Take up the above example and let D denote a quaternion division algebra over $\mathbb{Q}$ which splits over $\mathbb{R}$. So we have $D \hookrightarrow GL_2(\mathbb{R})$ and $D^1 \hookrightarrow SL_2(\mathbb{R})$, where D^1 is the set of elements of reduced norm 1. Let $\mathcal{O}$ denote an order in D and $\mathcal{O}^1 := \mathcal{O} \cap D^1$. Then $\Gamma := \mathcal{O}^1 \ltimes \mathcal{O}$ is a uniform lattice in $SL_2(\mathbb{R}) \ltimes \mathrm{Mat}_2(\mathbb{R})$.

Write the real Lie algebras of G, H, M, A, N, R as $\mathfrak{g}_0, \mathfrak{h}_0, \mathfrak{m}_0, \mathfrak{a}_0, \mathfrak{n}_0, \mathfrak{r}_0$ and their complexifications as $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}, \mathfrak{a}, \mathfrak{n}, \mathfrak{r}$. Let $\Phi(\mathfrak{h}, \mathfrak{a})$ denote the set of roots of the pair $(\mathfrak{h}, \mathfrak{a})$. The choice of the parabolic P amounts to the same as a choice of a set of positive roots $\Phi^+(\mathfrak{h}, \mathfrak{a})$. Let $A^- \subset A$ denote the negative Weyl chamber corresponding to that ordering, i.e. A^- consists of all $a \in A$ which act contractingly on the Lie algebra $\mathfrak{n}$. Further let $\overline{A^-}$ be the closure of A^- in G, this is a manifold with boundary. Let K_M be a maximal compact subgroup of M. We may suppose that K_M contains B. Fix an irreducible unitary representation (τ, V_τ) of K_M. Let K be a maximal compact subgroup of H. We may assume $K \supset K_M$.

Since Γ_H is the fundamental group of the Riemannian manifold

$$X_{\Gamma_H} = \Gamma_H \backslash X = \Gamma_H \backslash H / K$$

it follows that we have a canonical bijection of the homotopy classes of loops:

$$[S^1 : X_{\Gamma_H}] \to \Gamma_H/\text{conjugacy}.$$

For a given class $[\gamma]$ let X_γ denote the union of all closed geodesics in the corresponding class in $[S^1 : X_\Gamma]$. Then X_γ is a smooth submanifold of X_{Γ_H} [7]. Let $\chi_r(X_\gamma)$ denote the r-fold de-twisted Euler characteristic of X_γ, where $r = \dim A$.

Let $\mathcal{E}_P(\Gamma)$ denote the set of all conjugacy classes $[\gamma]$ in Γ such that γ_H is in H conjugate to an element $a_\gamma b_\gamma$ of $A^- B$.

Take a class $[\gamma]$ in $\mathcal{E}_P(\Gamma)$. Modulo conjugation assume $\gamma \in T = AB$, then the centralizer $\Gamma_{H,\gamma}$ projects to a lattice $\Gamma_{A,\gamma}$ in the split part A. Let λ_γ be the covolume of this lattice. Normalize the measure on R such that $\mathrm{vol}(\Gamma_R \backslash R) = 1$.

Theorem 3.1. (Lefschetz formula, first version) *Let φ be compactly supported on $\overline{A^-}$, $\dim G$-times continuously differentiable and suppose φ vanishes on the boundary to order $\dim G + 1$. Then we have that the expression*

$$\sum_{\substack{\pi \in \hat{G} \\ \pi|_R \equiv 1}} N_\Gamma(\pi) \sum_{p,q} (-1)^{p+q} \int_{A^-} \varphi(a) \operatorname{tr}(a | (H^q(\mathfrak{n}, \pi) \otimes \wedge^p \mathfrak{p}_M \otimes V_{\breve{\tau}})^{K_M}) da$$

equals

$$(-1)^{\dim(N)} \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \lambda_\gamma \chi_r(X_\gamma) \frac{\varphi(a_\gamma) \operatorname{tr} \tau(b_\gamma)}{\det(1 - a_\gamma b_\gamma | \mathfrak{n}) \det(1 - \gamma | \mathfrak{r})}.$$

Proof. Let H act on itself by conjugation, write $h.x = hxh^{-1}$, write $H.x$ for the orbit, so $H.x = \{hxh^{-1}|h \in H\}$ as well as $H.S = \{hsh^{-1}|s \in S, h \in H\}$ for any subset S of H. We are going to consider functions that are supported on the closure of the set $H.(MA^-)$. At first let f_τ be the Euler-Poincaré function defined on M attached to the representation (τ, V_τ) of K_M. Next fix a smooth function η on N which has compact support, is positive, invariant under K_M and satisfies $\int_N \eta(n)dn = 1$. Given these data let $\phi = \phi_{\eta,\tau,\varphi} : H \to \mathbf{C}$ be defined by

$$\phi(knma(kn)^{-1}) := \eta(n)f_\tau(m)\frac{\varphi(a)}{\det(1-(ma)|\mathfrak{n})},$$

for $k \in K, n \in N, m \in M, a \in \overline{A^-}$. Further $\phi(h) = 0$ if h is not in $H.(M\overline{A^-})$.

Next choose any compactly supported positive function ψ on R with $\int \psi = 1$. Let $\Phi(h,r) := \phi(h)\psi(r)$. We will plug Φ into the trace formula. For the geometric side let $\gamma = (\gamma_H, \gamma_R) \in \Gamma$. We have to calculate the orbital integral:

$$\mathcal{O}_\gamma(\Phi) = \int_{G_\gamma\backslash G} \Phi(x^{-1}\gamma x)dx.$$

Now let $x = (h,r) \in G$ and compute

$$x^{-1}\gamma x = (h^{-1}\gamma_H h, r + h^{-1}\gamma_R - h^{-1}\gamma_H hr).$$

So (h,r) lies in the centralizer G_γ iff $h \in H_{\gamma_H}$ and $r \in R$ satisfies

$$(1-h^{-1})\gamma_R = (1-\gamma_H^{-1})r.$$

Note that by (A2) to any γ such that γ_H is conjugate to an element of $A^{reg}M$, and to any $h \in H_{\gamma_H}$ such an r exists and is unique. But this condition on γ is satisfied if $\varphi(h^{-1}\gamma_H h) \neq 0$. So suppose γ_H is in $H.(A^{reg}M)$. In this case we have the integration rule

$$\int_{G_\gamma\backslash G} f(g)dg = \int_{H_{\gamma_H}\backslash H}\int_R f(h,r)drdh.$$

This is proven by showing that the right hand side is in fact G-invariant. We compute

$$\int_R \Phi((h,r)^{-1}\gamma(h,r))dr = \frac{\varphi(h^{-1}\gamma_H h)}{\det(1-\gamma_H|\mathfrak{r})},$$

from which we see that the geometric side of the trace formula coincides with our claim.

Now for the spectral side let $\pi \in \hat{G}$ then the restriction of π to R is a direct integral over $\hat{R}$. The irreducibility of π implies that the corresponding measure is supported on a single orbit o of the H-action on $\hat{R}$. So we have

$$\pi|_R = \int_o V_\pi(\tau)dm(\tau),$$

where m is a scalar valued measure and $V_\pi(\tau)$ is a multiple of τ. Fix $\tau_0 \in o$ then the stabilizer H_{τ_0} acts trivially on τ_0 and not only its class since by $\dim\tau_0 = 1$ these two notions coincide. It follows that as H_{τ_0}-modules we have $V_\pi(\tau) \cong \eta \otimes \tau$

for some representation η of H_{τ_0}. The measure m induces a measure on $G_{\tau_0}\backslash G$ also denoted m which is quasi-invariant. It follows that $\pi = \mathrm{ind}_{H_{\tau_0}\ltimes R}^G(\eta\otimes\tau)$ and hence η must be irreducible since π is. Let $\lambda(x,y)$ denote the Radon-Nikodym derivative of the translate m_x with respect to m. We conclude that $\pi(\Phi)$ is given as an integral operator on $G_{\tau_0}\backslash G$ with kernel

$$k(x,y) = \int_{G_{\tau_0}} \Phi(x^{-1}zy)\lambda(x^{-1}zy,x)^{\frac{1}{2}}(\eta\otimes\tau_o)(z)dz.$$

From this we get

$$\mathrm{tr}\,\pi(\Phi) = \int_{G_{\tau_0}} \mathrm{tr}(\int_{G_{\tau_0}} \Phi(x^{-1}zx)\lambda(x^{-1}zx,x)^{\frac{1}{2}}(\eta\otimes\tau_0)(z)dz)dx.$$

Consider the term $\Phi(x^{-1}zx) = \varphi(x_H^{-1}z_Hx_H)\psi(\ldots)$. By (A2) this expression vanishes unless τ_0 is the trivial character. In the case $\tau_0 = \mathrm{triv}$ it follows that $\pi(R)=1$, so π may be viewed as an element of $\hat{H}$.

To evaluate $\mathrm{tr}\,\pi(\Phi)$ further we will employ the Hecht-Schmid character formula [9]. For this let

$$(MA)^- = \text{interior in } MA \text{ of the set}$$

$$\left\{g\in MA | \det(1-ga|\mathfrak{n})\geq 0 \text{ for all } \mathfrak{a}\in A^-\right\}.$$

The character Θ_π^G of $\pi\in\hat{G}$ is a locally integrable function on G. In [9] it is shown that for any $\pi\in\hat{H}$, denoting by π^0 the underlying Harish-Chandra module we have that all Lie algebra cohomology groups $H^p(\mathfrak{n},\pi^0)$ are Harish-Chandra modules for MA. The main result of [9] is that for $ma\in(MA)^-\cap H^{reg}$, the regular set, we have

$$\Theta_\pi^H(ma) = \frac{\sum_{p=0}^{\dim\mathfrak{n}}(-1)^p\Theta_{H_p(\mathfrak{n},\pi^0)}^{MA}(ma)}{\det(1-ma|\mathfrak{n})}.$$

Let f be supported on $H.(MA^-)$, then the Weyl integration formula states that

$$\int_H f(x)dx = \int_{H/MA}\int_{MA^-} f(hmah^{-1})|\det(1-ma|\mathfrak{n}\oplus\bar{n})dadmdh.$$

So that for $\pi\in\hat{H}$:

$$\begin{aligned}\mathrm{tr}\,\pi(\phi) &= \int_H \Theta_\pi^H(x)\phi(x)dx\\ &= \int_{MA^-}\Theta_\pi^H(ma)f_\tau(m)\varphi(a)|\det(1-ma|\bar{\mathfrak{n}})|dadm\\ &= (-1)^{\dim N}\int_{MA^-} f_\tau(m)\Theta_{H^*(\mathfrak{n},\pi^0)}^{MA}(ma)\varphi(a)dadm,\end{aligned}$$

where we have used the isomorphism $H_p(\mathfrak{n},\pi^0)\cong H^{\dim N-p}(\mathfrak{n},\pi^o)\otimes\wedge^{top}\mathfrak{n}$. This gives the claim. $\square$

In the second version of the Lefschetz formula we want to substitute the character of the representation τ by an arbitrary central function on K_M. A smooth function f on K_M is called *central* if $f(kk_1k^{-1}) = f(k_1)$ for all $k, k_1 \in K_M$. Since B is a Cartan subgroup of the compact group K_M, any $k \in K_M$ is conjugate to some element of B so the restriction gives in isomorphism from the space of smooth central functions on K_M to the space of smooth functions on B, invariant under the Weyl group. Hence we are led to consider Weyl group invariant functions on T.

Let $\mathcal{A}$ denote the convolution algebra of all $W(H,T)$-invariant smooth functions on T with compact support. Let $S \subset T$ be the set of all ab with singular a-part.

For any $t = ab$ in T let $\mathfrak{n}_t$ be the space of all $X \in \mathrm{ad}(t)\mathfrak{g}$ on which t acts contractingly. Then $\mathfrak{n}_t$ is a nilpotent Lie subalgebra of $\mathfrak{g}$.

Theorem 3.2. (Lefschetz formula, second version) *Let $\varphi \in \mathcal{A}$ and suppose φ vanishes on the singular set to order* $\dim G + 1$ *then the expression*

$$\sum_{\substack{\pi \in \hat{G} \\ \pi|_R \equiv 1}} N_\Gamma(\pi) \sum_q (-1)^q \int_{T/W(H,T)} \varphi(t) \operatorname{tr}(t|H^q(\mathfrak{n}_t, \pi))dt$$

equals

$$(-1)^{\dim(N)} \sum_{[\gamma] \in \mathcal{E}_P(\Gamma)} \lambda_\gamma \chi_r(X_\gamma) \frac{\varphi(t_\gamma)}{\det(1 - t_\gamma|\mathfrak{p}_M \oplus \mathfrak{n}_{h_\gamma})\det(1-\gamma|\mathfrak{r})}.$$

Proof. Extend $b \mapsto \varphi(ab)$ to a central function on K_M. Then expand φ into K_M-types:

$$\varphi(ab) = \sum_{\tau \in \hat{K_M}} c_\tau \operatorname{tr} \tau(b) \varphi_\tau(a),$$

since φ is smooth the coefficients c_τ are rapidly decreasing so the expressions of Theorem 3.1 when plugging in $\varphi_\tau|_{A^-}$ converge to

$$\sum_{\substack{\pi \in \hat{G} \\ \pi|_R \equiv 1}} N_\Gamma(\pi) \sum_{p,q} (-1)^{p+q} \int_{T/W(H,T)} \varphi(t) \operatorname{tr}(t|H^q(\mathfrak{n}_t, \pi) \otimes \wedge^p \mathfrak{p}_M)dt,$$

which equals

$$(-1)^{\dim(N)} \sum_{[\gamma] \in \mathcal{E}_H(\Gamma)} \lambda_\gamma \chi_r(X_\gamma) \frac{\varphi(t_\gamma)}{\det(1 - t_\gamma|\mathfrak{n}_{h_\gamma})\det(1-\gamma|\mathfrak{r})}.$$

Now replace $\varphi(t)$ by $\varphi(t)/\det(1 - t|\mathfrak{p}_M)$ which gives the claim. □

At last we also mention a reformulation in terms of relative Lie algebra cohomology. Again, fix a parabolic $P = MAN$ and now fix also a finite dimensional irreducible representation (σ, V_σ) of M.

Theorem 3.3. (Lefschetz formula, third version) *Let φ be compactly supported on $\overline{A^-}$, $\dim G$-times continuously differentiable and suppose φ vanishes on the boundary to order $\dim G+1$. Then we have that the expression*

$$\sum_{\substack{\pi \in \hat{G} \\ \pi|_R \equiv 1}} N_\Gamma(\pi) \sum_q (-1)^q \int_{A^-} \varphi(a) \operatorname{tr}(a|H^q(\mathfrak{m}\oplus\mathfrak{n}, K_M, \pi\otimes V_{\breve{\sigma}})$$

equals

$$(-1)^{\dim(N)} \sum_{[\gamma]\in\mathcal{E}_P(\Gamma)} \lambda_\gamma \chi_r(X_\gamma) \frac{\varphi(a_\gamma)\operatorname{tr}\sigma(b_\gamma)}{\det(1-a_\gamma b_\gamma|\mathfrak{n})\det(1-\gamma|\mathfrak{r})}.$$

Proof. Extend $V_{\breve{\sigma}}$ to a $\mathfrak{m}\oplus\mathfrak{n}$-module by letting $\mathfrak{n}$ act trivially. We then get

$$H^p(\mathfrak{n},\pi^0)\otimes V_{\breve{\sigma}} \cong H^p(\mathfrak{n},\pi^0\otimes V_{\breve{\sigma}}).$$

The $(\mathfrak{m}, K_M)$-cohomology of the module $H^p(\mathfrak{n},\pi^0\otimes V_{\breve{\sigma}})$ is the cohomology of the complex (C^*) with

$$\begin{aligned} C^q &= \operatorname{Hom}_{K_M}(\wedge^q\mathfrak{p}_M, H^p(\mathfrak{n},\pi^0)\otimes V_{\breve{\sigma}}) \\ &= (\wedge^q\mathfrak{p}_M\otimes H^p(\mathfrak{n},\pi^0)\otimes V_{\breve{\sigma}})^{K_M}, \end{aligned}$$

since $\wedge^p\mathfrak{p}_M$ is a self-dual K_M-module. Therefore we have an isomorphism of virtual A-modules:

$$\sum_q (-1)^q (H^p(\mathfrak{n},\pi^0)\otimes\wedge^q\mathfrak{p}_M\otimes V_{\breve{\sigma}})^{K_M} \cong \sum_q (-1)^q H^q(\mathfrak{m}, K_M, H^p(\mathfrak{n},\pi^0\otimes V_{\breve{\sigma}})).$$

Now one considers the Hochschild-Serre spectral sequence in the relative case for the exact sequence of Lie algebras

$$0\to\mathfrak{n}\to\mathfrak{m}\oplus\mathfrak{n}\to\mathfrak{m}\to 0$$

and the $(\mathfrak{m}\oplus\mathfrak{n}, K_M)$-module $\pi\otimes V_{\breve{\sigma}}$. We have

$$E_2^{p,q} = H^q(\mathfrak{m}, K_M, H^p(\mathfrak{n},\pi^0\otimes V_{\breve{\sigma}}))$$

and

$$E_\infty^{p,q} = \operatorname{Gr}^q(H^{p+q}(\mathfrak{m}\oplus\mathfrak{n}, K_M, \pi^0\otimes V_{\breve{\sigma}})).$$

Now the module in question is just

$$\chi(E_2) = \sum_{p,q} (-1)^{p+q} E_2^{p,q}.$$

Since the differentials in the spectral sequence are A-homomorphisms this equals $\chi(E_\infty)$. So we get an A-module isomorphism of virtual A-modules

$$\sum_{p,q} (-1)^{p+q} (H^p(\mathfrak{n},\pi^0)\otimes\wedge^q\mathfrak{p}_M\otimes V_{\breve{\sigma}})^{K_M} \cong \sum_j (-1)^j H^j(\mathfrak{m}\oplus\mathfrak{n}, K_M, \pi^0\otimes V_{\breve{\sigma}}).$$

The claim follows. □

4. Geometric interpretation

Now consider the first version of the Lefschetz formula in the case $R = 0$. The representation τ defines a homogeneous vector bundle E_τ over G/K_M and by homogeneity this pushes down to a locally homogeneous bundle over $\Gamma\backslash G/K_M = {}_MX_\Gamma$. The tangent bundle $T({}_MX_\Gamma)$ can be described in this way as stemming from the representation of K_M on

$$\mathfrak{g}/\mathfrak{k}_M \cong \mathfrak{a} \oplus \mathfrak{p}_M \oplus \mathfrak{n} \oplus \bar{\mathfrak{n}}.$$

We get a splitting into subbundles

$$T({}_MX_\Gamma) = T_c \oplus T_n \oplus T_u \oplus T_s.$$

These bundles can be characterized by dynamical properties: the action of $A \cong \mathbb{R}^r$ is furnished with a positive time direction given by the positive Weyl chamber A^+. Then T_s, the *stable part* is characterized by the fact that A^+ acts contractingly on T_s. On the *unstable part* T_u the opposite chamber A^- acts contractingly. T_c, the *central part* is spanned by the "flow" A itself and T_n is an additive *neutral part*. Note that T_n vanishes if we choose H to be the maximal split torus. The bundle $T_n \oplus T_u$ is integrable, so it defines a foliation $\mathcal{F}$. To this foliation we have the tangential cohomology $H^*(\mathcal{F})$ and also for its τ-twist: $H^*(\mathcal{F}\otimes\tau)$. The flow A acts on the tangential cohomology whose alternating sum we will consider as a virtual A-module. For any $\varphi \in C_c^\infty(\overline{A^-})$ we define $L_\varphi = \int_{A^-} \varphi(a)(a|H^*(\mathcal{F}\otimes\tau))da$ as a virtual operator on $H^*(\mathcal{F}\otimes\tau)$. Then we have

Proposition 4.1. *Under the assumptions of* Theorem 3.1 *the virtual operator L_φ is of trace class and the RHS of Theorem 3.1 can be written as*

$$\sum_q (-1)^q \operatorname{tr}(L_\varphi | H^q(\mathcal{F}\otimes\tau)).$$

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