Journal of Lie Theory
Volume **6** (1996) 249–269

On invariants of a set of matrices

E. B. Vinberg

Communicated by K. H. Hofmann

Abstract. For any connected reductive linear group $H \subset SL_n(k)$ it is proved that the algebra of invariants of a set of matrices $X_1,\dots,X_p \in H$ $(p \geqslant 2)$ under simultaneous conjugations by matrices of $N(H)$ is the normalization of the algebra generated by the traces $\operatorname{tr} X_{i_1} \dots X_{i_k}$.

It is a classical fact that the algebra of invariants of a set of matrices with respect to simultaneous conjugations is generated by the traces of their products.

Let now $H \subset SL_n(k)$ be a reductive algebraic linear group. Consider the algebra of invariants of a set of matrices

$$(X_1, \dots, X_p) \in H^p = \underbrace{H \times \dots \times H}_{p}$$

with respect to simultaneous conjugations by the matrices of the normalizer $N(H)$ of H in $\mathrm{SL}_n(k)$. Is it true that this algebra is generated by the traces of products of $X_1, \dots, X_p$?

Making use of the classical invariant theory, one can easily show that it is true for any classical linear group

$$H = \mathrm{SL}_n(k), \quad \mathrm{SO}_n(k), \quad \mathrm{Sp}_n(k).$$

(Except for the case $\mathrm{SO}_n(k)$, n even, the action of $N(H)$ on H reduces to that of H. In the exceptional case it reduces to the action of $\mathrm{O}_n(k)$.)

An analogous question can be asked for the field of rational invariants. It was proved in my preprint [5] that the answer to this question is positive for any connected reductive linear group H and $p \geqslant 2$. For $p = 1$, the answer is in general negative (see a counterexample in Section 4).

As for the original question, the answer is in general negative, even for H connected and $p \geqslant 2$ (see a counterexample in Section 4). It was conjectured in [5] that under these assumptions the algebra of invariants is the normalization of the algebra generated by the traces. The main purpose of this paper is to prove this conjecture. In fact, we prove it in greater generality.

ISSN 0949–5932 / \$2.50

A portion of this work was done during my visit to the Tata Institute of Fundamental Research in January–February of 1993. I am grateful to this institute for its hospitality. The completion of this research was made possible by the grant MQZ300 from the International Science Foundation and and the Russian Government, and by the grant 95-01-00783a from the Russian Foundation of Fundamental Research.

I am grateful to E. A. Tevelev, who suggested a simpler proof of Proposition 8 in the case when G_0 is a torus.

1. Statement of the main results.

The ground field k is supposed to be algebraically closed and of characteristic zero. For an action of a reductive algebraic group H on an affine algebraic variety X the algebra of G-invariant polynomials (respectively, the field of G-invariant rational functions) on X is denoted by $k[X]^G$ (respectively, by $k(X)^G$). The categorical quotient of the action $G : X$, i.e., the spectrum of $k[X]^G$, is denoted by $X/\!\!/G$, and the canonical morphism $X \to X/\!\!/G$ defined by the embedding $k[X]^G \subset k[X]$ is denoted by π_G.

We use some standard facts of invariant theory, which can be found, for example, in [2]. In particular, it is a standard fact that each fiber of π_G contains exactly one closed G-orbit.

Let a reductive algebraic group G act on G^p by simultaneous conjugations. It is proved in [3] that the orbit of a p-tuple

$$\mathbf{g} = (g_1, \ldots, g_p) \in G^p$$

is closed iff the algebraic subgroup (topologically) generated by $\mathbf{g}$ is reductive.

For any reductive algebraic subgroup $H \subset G$ we have the following commutative diagram of natural morphisms:

$$\begin{array}{ccccc} H^p & \xrightarrow{\pi_H} & H^p/\!\!/H & \xrightarrow{\pi_{N(H)/H}} & H^p/\!\!/N(H) \\ \downarrow & & \Big\downarrow{\scriptstyle\psi} & & \Big\downarrow{\scriptstyle\varphi} \\ G^p & \xrightarrow{\pi_G} & G^p/\!\!/G & \hookleftarrow & H^p/\!\!/G \end{array} \tag{1}$$

where $N(H)$ is the normalizer of H in G and

$$H^p/\!\!/G \doteqdot \overline{\pi_G(H^p)}(= \overline{\varphi(H^p/\!\!/N(H))} = \overline{\psi(H^p/\!\!/H)}). \tag{2}$$

Since the group of automorphisms of H defined by $N(H)$ contains the group of inner automorphisms of H as a subgroup of finite index (see Corollary 2 to Proposition 3), the morphism $\pi_{N(H)/H}$ is finite.

Now we formulate the main results of the paper.

Theorem 1. *The morphism ψ is finite (and, in particular, $\pi_G(H^p)$ is closed in $G^p/\!\!/G$).*

An interpretation of this theorem in terms of varieties of characters of finitely generated groups will be given in Section 16.

Theorem 2. *If H is connected and $p \geqslant 2$, the morhism φ is birational.*

Very often, but not always, the morphism φ is birational for $p = 1$ as well.

Corollary 2a. *If H is connected and $p \geqslant 2$, the morphism φ is the normalization.*

In other words, the algebra $k[H^p]^{N(H)}$ is the normalization of the restriction to H^p of the algebra $k[G^p]^G$. Applying this to $G = \mathrm{SL}_n(k)$, we obtain that for any connected reductive group $H \subset \mathrm{SL}_n(k)$ and any $p \geqslant 2$ the algebra $k[H^p]^{N(H)}$ is the normalization of the algebra generated by the traces of products.

In the case $G = \mathrm{SL}_n(k)$ Theorem 2 was proved in the preprint [5]. In the general case the proof is the same. It will be given in the next section.

Theorem 2 can be generalized to non-connected subgroups. To this end, we need some modification of H^p.

Denote by $\widetilde{H}^p$ the union of connected (=irreducible) components of H^p containing a generating set of H. Note that any connected component of H^p has the form $H_1 \times \cdots \times H_p$, where $H_1, \ldots, H_p$ are some connected components of H. As we shall prove (see Corollary 1 of Proposition 8), for $p \geqslant 2$ $H_1 \times \cdots \times H_p \subset \widetilde{H}^p$ iff $H_1, \ldots, H_p$ generate the group H/H_0. Obviously, $\widetilde{H}^p$ is invariant under $N(H)$.

The diagram (1) can be re-written with replacing H^p by $\widetilde{H}^p$. Denote by $\tilde{\varphi}$ the morphism replacing φ in the modified diagram.

Theorem 2′. *The morphism $\tilde{\varphi}$ is birational.*

Corollary 2′a. *The morphism $\tilde{\varphi}$ is the normalization.*

2. Proof of Theorem 2

To prove the theorem, we need some preparatory results on generating sets of reductive groups.

Unless stated otherwise, we understand the term "generate" in the topological sense. So "an algebraic group G is generated by a p-tuple $\mathbf{g} \in G^p$" means that the subgroup algebraically generated by $\mathbf{g}$ is (Zariski) dense in G.

Proposition 1. *Any algebraic torus T is generated by one element. Moreover, the set of all generating elements is dense in T.*

Proof. An element t generates T iff there is no nontrivial character of T vanishing at t. For such t, one can take any element, whose coordinates are different primes. Such elements constitute a dense subset in T. ∎

Corollary 1a. *For any $p \geqslant 1$ the set of p-tuples generating T is dense in T^p.* ∎

Proposition 2. *Any connected reductive algebraic group G is generated by two elements. Moreover, the set of all generating pairs is dense in G^2.*

Proof. Let $t \in G$ be a semisimple element generating a maximal torus T of G. It follows from the root decomposition of the tangent algebra of G with respect to T that there exist only finitely many algebraic subgroups of G containing T. If an element h does not belong to any of maximal algebraic subgroups containing T, then the pair (t, h) generates G. The set of pairs of such type is dense in G^2. ∎

Corollary 2a. *For any $p \geqslant 2$ the set of p-tuples generating G is dense in G^p.* ∎

Proof of Theorem 2. Let us note that if $\mathbf{h} \in H^p$ generates H and, for some $g \in G$, we have $g\mathbf{h}g^{-1} \in H^p$, then $gHg^{-1} \subset H$, which implies $g \in N(H)$.

Denote by H^p_{gen} the set of all p-tuples generating H. Under the hypotheses of Theorem 2, H^p_{gen} is dense in H^p. It follows that $\pi_G(H^p_{\text{gen}})$ is dense in $H^p /\!\!/ G$. For any p-tuple $\mathbf{h} \in H^p$ generating a reductive subgroup, the fiber $\varphi^{-1}(\pi_G(\mathbf{h}))$ consists of the points $\pi_{N(H)}(\mathbf{h}')$, where $\mathbf{h}' \in H^p$ is G-equivalent to $\mathbf{h}$. It follows from what was proved above that if $\mathbf{h} \in H^p_{\text{gen}}$, then $\varphi^{-1}(\pi_G(\mathbf{h}))$ consists of a single point. Now Theorem 2 is implied by the following

Lemma 1. *Let $\varphi\colon X \to Y$ be a dominant morphism of irreducible algebraic varieties. If there exists a dense subset $Z \subset Y$ such that for any $y \in Z$ the fiber $\varphi^{-1}(y)$ consists of a single point, then φ is birational.*

Proof. If φ is not birational, then there exists an open subset $Y_0 \subset Y$ such that for any $y \in Y_0$ the fiber $\varphi^{-1}(y)$ contains more than one point. Since $Y_0 \cap Z \neq \varnothing$, this contradicts our condition. ∎

The proof of Theorem $2'$ is similar. It will be given in Section 8.

3. An intermediate result

Let us now prove that the morphism ψ is quasi-finite, i.e., that all of its fibers are finite. This means that, for any closed G-orbit O in G^p, the intersection $O \cap H^p$ decomposes into finitely many (closed) H-orbits.

To prove this, we apply the following lemma, going back to R. W. Richardson. Its proof can be found in [1].

Lemma 2. *Let an algebraic group G act on an algebraic variety X. Suppose $H \subset G$ is an algebraic subgroup and $Y \subset X$ is an H-invariant subvariety. If for any $y \in Y$*

$$\mathfrak{g}(y) \cap T_y(Y) = \mathfrak{h}(y), \tag{3}$$

then, for any G-orbit O, the intersection $O \cap Y$ decomposes into finitely many H-orbits, each of them being closed in $O \cap Y$.
(Here $\mathfrak{g}$ and $\mathfrak{h}$ are the tangent algebras of G and H, respectively, and $T_y(Y)$ is the tangent space of Y at y.)

Proposition 3. *Let G be a reductive group, and H a reductive subgroup of G. Then for any G-orbit O in G^p, the intersection $O \cap H^p$ decomposes into finitely many H-orbits, each of them being closed in $O \cap H^p$.*

Proof. We apply Lemma 2 to the action $G{:}\,G^p$ and the subvariety $H^p \subset G^p$. For a p-tuple $\mathbf{h} = (h_1, \ldots, h_p) \in H^p$, we have

$$T_{\mathbf{h}}(G^p) = \mathfrak{g}h_1 \oplus \ldots \mathfrak{g}h_p,$$

$$T_{\mathbf{h}}(H^p) = \mathfrak{h}h_1 \oplus \ldots \mathfrak{h}h_p,$$

and, for any $\xi \in \mathfrak{g}$,

$$\xi(\mathbf{h}) = (\xi h_1 - h_1\xi, \ldots, \xi h_p - h_p\xi) = ((\xi - \mathrm{Ad}(h_1)\xi)h_1, \ldots, (\xi - \mathrm{Ad}(h_p)\xi)h_p).$$

So $\xi(\mathbf{h}) \in T_{\mathbf{h}}(H^p)$ iff

$$\xi - \mathrm{Ad}(h_i)\xi \in \mathfrak{h}, \ i = 1, \ldots, p. \tag{4}$$

Let $\mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant complementary subspace of $\mathfrak{h}$ in $\mathfrak{g}$. If

$$\xi = \eta + \zeta \quad (\eta \in \mathfrak{h}, \zeta \in \mathfrak{m}),$$

then (4) implies that

$$\zeta - \mathrm{Ad}(h_i)\zeta = 0, \ i = 1, \ldots, p,$$

whence

$$\xi(\mathbf{h}) = \eta(\mathbf{h}) \in \mathfrak{h}(\mathbf{h}).$$

So the condition (3) is fulfilled in our situation and Lemma 2 gives the desired result. ∎

Corollary 3a. *The morphism ψ is quasi-finite.* ∎

Corollary 3b. *The group of automorphisms of H defined by $N(H)$ is a finite extension of the group of inner automorphisms of H.* ∎

Proof. Take any p-tuple $\mathbf{h} = (h_1, \ldots, h_p)$ generating H. It follows from the proposition that the $N(H)$-orbit of $\mathbf{h}$ decomposes into finitely many H-orbits. Since any automorphism a of H is uniquely defined by $a(\mathbf{h})$, this implies the assertion. ∎

4. Some counterexamples

The following example shows that for $p = 1$ (and H connected) the morphism φ need not be birational.

Let

$$G = \mathrm{SL}_5(k), \quad H = \mathrm{SL}_2(k) \oplus \mathrm{SO}_3(k).$$

Since H has no outer automorphisms, the action of $N(H)$ on H coincides with that of H. It $h_1 \in \mathrm{SL}_2(k)$ has eigenvalues λ, λ^{-1} and $h_2 \in \mathrm{SO}_3(k)$ has eigenvalues $\mu, \mu^{-1}, 1$, then $h = h_1 \oplus h_2$ has eigenvalues $\lambda, \lambda^{-1}, \mu, \mu^{-1}, 1$. The same eigenvalues has $h' = h_1' \oplus h_2'$, where $h_1' \in \mathrm{SL}_2(k)$ has eigenvalues μ, μ^{-1} and $h_2' \in \mathrm{SO}_3(k)$ has eigenvalues $\lambda, \lambda^{-1}, 1$. If $\lambda, \mu \neq \pm 1$ and $\mu \neq \lambda, \lambda^{-1}$, then the H-orbits of h and h' are closed and distinct from each other, while their G-orbits coincide.

The next example shows that even if $p \geqslant 2$ (and H connected) the morphism φ need not be injective.

Let

$$G = \mathrm{SL}_5(k), \quad H = R(\mathrm{SL}_2(k)),$$

where R is the 5-dimensional irreducible representation of $\mathrm{SL}_2(k)$. As in the preceding example, H has no outer automorphisms, so the action of $N(H)$ on H^p coincides with that of H. If $u \in \mathrm{SL}_2(k)$ has eigenvalues λ, λ^{-1}, then $R(u)$ has eigenvalues $\lambda^2, \lambda^{-2}, \lambda, \lambda^{-1}, 1$. Let $\lambda = \exp \frac{2\pi i}{5}$. Then $R(u)$ and $R(u^2)$ have the same eigenvalues, so their G-orbits coincide. At the same time their H-orbits are closed and distinct from each other. The same is true for the p-tuples

$$(R(u), \ldots, R(u)),\ (R(u^2), \ldots, R(u^2)) \in G^p$$

for any p.

5. Toric subvarieties

In order to handle nonconnected groups, we need some generalization of maximal tori. We follow ideas of Gantmacher [2]. For more details see [3].

Let G be a reductive group, G_0 its connected component containing e, and G_1 some connected component of G.

Any coset S of a torus in G, lying in the centralizer of this torus, will be called *a toric subvariety.* Its normalizer (respectively, centralizer) in G_0 will be denoted by $N_0(S)$ (respectively, $Z_0(S)$). The group $W_0(S) = N_0(S)/Z_0(S)$ will be called *the Weyl group* of S. Note that the group $W_0(S)$ is finite, because it is an algebraic group of automorphisms of the subgroup generated by S, which is a direct product of a torus and a finite cyclic group.

Proposition 4. 1) *Any two maximal toric subvarieties of* G_1 *are* G_0*-conjugate.*

2) *Any semisimple element of* G *is contained in a maximal toric subvariety.*

3) *Two elements of a maximal toric subvariety* S *are* G_0*-conjugate iff they are* $W_0(S)$*- conjugate.*

One can easily show (and it follows from the first assertion of the proposition) that any maximal toric subvariety of a connected reductive group is a maximal torus. In this partial case the proof of the proposition is well-known. In the general case only small modifications are needed.

Sketch of the proof. Let S be a maximal toric subvariety contained in G_1, and $g \in S$. We have $S = gT$, where T is a maximal torus in the centralizer $Z_0(g)$ of g in G_0. An element $g \in S$ is called *regular*, if $Z_0(g)$ is a finite extension of T. Obviously, the subvariety S is uniquely determined by any regular element of it. Considering the weight decomposition of the tangent algebra $\mathfrak{g}$ of G with respect to the subgroup generated by S, we see that the regular elements constitute a non-empty open subset in S. Denote it by S^{reg}. A standard computation shows that the map

$$G_0 \times S^{\mathrm{reg}} \to G_1, \quad (g,s) \mapsto gsg^{-1},$$

is smooth and hence its image is open in G_1. As usually, this implies the first assertion of the proposition. The second one is trivial. The proof of the third one does not differ from the usual proof for the case when S is a maximal torus.■

Corollary 4a. *Let* $S \subset G_1$ *be a maximal toric subvariety. Then the natural morphism*

$$S/\!\!/W_0(S) \longrightarrow G_1/\!\!/G_0$$

is an isomorphism. ■

The following consequence of this theory will be used in the proof of Theorem 1.

Proposition 5. *The restriction of the morphism* $\pi_G\colon G \to G/\!\!/G$ *to any toric subvariety* $S \subset G$ *is a finite morphism.*

Proof. Let S_1 be a maximal toric subvariety containing S. If S (and S_1) is contained in a connected component G_1 of G, then the restriction of π_G on S is the composition of the following finite morphisms:

$$S \hookrightarrow S_1 \to S_1/W_0(S_1) \xrightarrow{\sim} G_1/\!\!/G_0 \to G/\!\!/G_0 \to G/\!\!/G. \qquad ■$$

Let now S be a maximal toric subvariety of G, which is a coset of a torus T. It is known [3] that the centralizer of T in G_0 is a maximal torus of G.

Consider the weight decomposition $\mathfrak{g} = \sum_\alpha \mathfrak{g}_\alpha$ of the Lie algebra $\mathfrak{g}$ with respect to the subgroup generated by S. We have $\mathfrak{g}_0 = \mathfrak{t}$ (the tangent algebra of T). The non-zero weights vanishing on T are called *imaginary roots.* Denote the set of imaginary roots by Δ_{im}. The sum $\mathfrak{t} + \sum\limits_{\alpha\in\Delta_{\mathrm{im}}} \mathfrak{g}_\alpha$ is a Cartan subalgebra of $\mathfrak{g}$. The weights not vanishing on T are called *real roots.* Denote the set of real roots by Δ_{re}. It is known [3] that $\dim \mathfrak{g}_\alpha = 1$ for any $\alpha \in \Delta_{\mathrm{re}}$.

Proposition 6. *If G is semisimple, there are only finitely many maximal algebraic subgroups of G containing S.*

Proof. Let $H \subset G$ be such a subgroup. Since $H \subset N(H_0)$, we have $N(H_0) = G$ or H.

In the first case H_0 is a normal subgroup of G_0. Since $H_0 \supset T$, the centralizer of H_0 in G_0 is abelian. Hence $H_0 = G_0$, and there are only finitely many possibilities for H.

In the second case H is completely defined by its tangent algebra $\mathfrak{h}$, which is invariant under S, contains T, and coincides with its normalizer in $\mathfrak{g}$. Let us prove that there are only finitely many such subalgebras in $\mathfrak{g}$.

Let $\mathfrak{h}$ be such a subalgebra, and

$$\Delta_{\mathrm{re}}(\mathfrak{h}) = \{\alpha \in \Delta_{\mathrm{re}} \colon \mathfrak{g}_\alpha \subset \mathfrak{h}\}.$$

We have

$$\mathfrak{h} = \mathfrak{t} + \sum_{\alpha \in \Delta_{\mathrm{im}}} (\mathfrak{h} \cap \mathfrak{g}_\alpha) + \sum_{\alpha \in \Delta_{\mathrm{re}}(\mathfrak{h})} \mathfrak{g}_\alpha.$$

Since $\mathfrak{h}$ coincides with its normalizer in $\mathfrak{g}$,

$$\mathfrak{h} \cap \mathfrak{g}_\alpha = \{\xi \in \mathfrak{g}_\alpha \colon [\xi, \mathfrak{g}_\beta] = 0 \quad \text{for all} \quad \beta \in \Delta_{\mathrm{re}}(\mathfrak{h}) \quad \text{such that} \quad \beta + \alpha \notin \Delta_{\mathrm{re}}(\mathfrak{h})\}$$

for $\alpha \in \Delta_{\mathrm{im}}$. It follows that $\mathfrak{h}$ is completely defined by $\Delta_{\mathrm{re}}(\mathfrak{h})$. ∎

6. Splitting of nonconnected reductive groups

Let G be any reductive group.

Proposition 7. *There exists a finite subgroup $F \subset G$ such that $G = G_0 F$.*

Proof. For an algebraic group H, let $\operatorname{Aut} H$ denote the group of all automorphisms and $\operatorname{Int} H$ the group of inner automorphisms of H. It is well-known that if H is a connected semisimple group, then $\operatorname{Aut} H$ is a semidirect product $\operatorname{Int} H \rtimes A(H)$, where $A(H)$ is some finite group.

Put $(G_0, G_0) = H$ and consider the subgroup of G, consisting of the elements g such that the automorphism $h \mapsto ghg^{-1}$ of H belongs to $A(H)$. It contains representatives of all cosets of H (and the more of all cosets of G_0) and its connected component is contained in the centralizer of H and hence is a torus. Thereby the proof reduces to the case when G_0 is a torus.

Let $G_0 = T$ be a torus, and $G/G_0 = \Gamma$. The extention $T \subset G$ is described by a cocycle $c \in Z^2(\Gamma, T)$. Let T_{fin} denote the subgroup of elements of finite order of T. Since in the group T/T_{fin} each equation $x^n = a$ has a unique solution, $H^2(\Gamma, T/T_{\mathrm{fin}}) = 0$. Therefore the cocycle c is equivalent to a cocycle $c' \in Z^2(\Gamma, T_{\mathrm{fin}})$. In fact all the values of c' belong to a (finitely generated and hence) finite subgroup $\Delta \subset T$. This means that there exists a finite subgroup $F \subset G$ containing representatives of all cosets of T and intersecting T in Δ. ∎

In the notation of the proposition, G is isomorphic to a quotient group of the semidirect product $G_0 \rtimes F$ with respect to some finite normal subgroup.

7. Generators of non-connected reductive groups

To prove Theorem $2'$, we need the following generalization of Proposition 2.

Proposition 8. *Let G_1 and G_2 be (possibly coinciding) connected components of a reductive group G. Then there are elements $g_1 \in G_1$ and $g_2 \in G_2$ generating an algebraic subgroup of finite index in G. Moreover, the set of all such pairs is dense in $G_1 \times G_2$.*

Proof. We may assume that the group G/G_0 is generated by G_1 and G_2. Under this condition we are to prove that there are elements $g_1 \in G_1$ and $g_2 \in G_2$ generating G, and the set of all such pairs is dense in $G_1 \times G_2$.

Denote by Z the center of G_0. It is easy to see that an algebraic subgroup $H \subset G$ coincides with G iff it projects both onto G/Z and $G/(G_0, G_0)$. In such a way the proof of the theorem reduces to two partial cases when G is semisimple or G_0 is a torus.

Let first G be semisimple. Take any maximal toric subvariety $S \subset G_1$. The subgroup generated by S is a direct product of a torus T and a finite cyclic group C. Let c be the generator of C lying in S. Then for any element $t \in T$ generating T, the element $s = tc$ generates $T \times C$. The set of all such elements s is dense in S.

Let s be chosen as above. According to Proposition 6 there are only finitely many maximal algebraic subgroups of G containing s. Note that none of them can contain G_2. If $h \in G_2$ does not belong to any of these subgroups, the pair (s, h) generates G. Obviously, the set of all pairs (s, h) of such type is dense in $G_1 \times G_2$.

Let now $G_0 = T$ be a torus. In view of Proposition 7 it suffices to consider the case $G = T \leftthreetimes \Gamma$, where Γ is some finite group. In this case, we have $G_1 = T\gamma_1,\ G_2 = T\gamma_2$, where $\gamma_1, \gamma_2 \in \Gamma$.

Consider the homomorphism of the free group on two generators to G/T, taking the i-th generator to G_i. Let $w_1, \dots, w_n$ be generators of its kernel. Then for $g_1 \in G_1,\ \ g_2 \in G_2$ the intersection of T with the algebraic subgroup generated by g_1 and g_2 is the algebraic subgroup generated by $w_1(g_1, g_2), \dots, w_n(g_1, g_2)$.

For $i = 1, \dots, n$, the map

$$\varphi\colon T^2 \to T, \quad (t_1, t_2) \mapsto w_i(t_1\gamma_1, t_2\gamma_2),$$

is a homomorphism of algebraic tori. The elements $t_1\gamma_1$ and $t_2\gamma_2$ generate G iff $\varphi_1(t_1, t_2)$, $\dots$, $\varphi_n(t_1, t_2)$ generate T. Note that if (t_1, t_2) generates T^2, then $\varphi_i(t_1, t_2)$ generates $\varphi_i(T^2)$ for each i. So it suffices to prove that the subgroup T' generated by $\varphi_1(T^2), \dots, \varphi_n(T^2)$ coincides with T.

It is easy to see that T' is a normal subgroup of G. Passing to the quotient group, we may assume that $T' = \{e\}$. Under this condition we are to prove that $T = \{e\}$.

The condition $T' = \{e\}$ means that for any t_1, $t_2 \in T$ the intersection of T with the subgroup generated by $t_1\gamma_1$ and $t_2\gamma_2$ reduces to $\{e\}$. For any $\gamma \in \Gamma$ denote by $c(\gamma)$ the (unique) element of T such that $c(\gamma)\gamma$ belongs to this subgroup. Then c is a 1-cocycle on Γ with values in T. By definition $c(\gamma_1) = t_1$, $c(\gamma_2) = t_2$. If follows that $\dim Z^1(\Gamma, T) = 2\dim T$, whence

$$\dim H^1(\Gamma, T) \geqslant \dim Z^1(\Gamma, T) - \dim C^0(\Gamma, T) = \dim T.$$

But the group $H^1(\Gamma, T)$ is finite. Hence $\dim T = 0$. ■

Corollary 8a. *Let $p \geqslant 2$ and some connected components $G_1, \dots, G_p$ of G generate the group G/G_0. Then $G_1 \times \cdots \times G_p$ contains a generating p-tuple of G; moreover, the set of all such p-tuples is dense in $G_1 \times \cdots \times G_p$.* ■

Corollary 8b. *The set of generating p-tuples is dense in $\widetilde{G}^p$ for any p.*
(The definition of $\widetilde{G}^p$ is given in Section 1.)

Proof. For $p \geqslant 2$ this follows immediately from Corollary 1. For $p = 1$ the only case is to be considered when G is a direct product of a torus and a finite cyclic group. This case was handled in the proof of Proposition 8. ■

Remark 1. One can prove that, if G is semisimple, the set of generating p-tuples is open in G^p. But we do not need this fact. ■

Now we can prove Theorem $2'$ by repeating the proof of Theorem 2 given in Section 2, replacing φ by $\tilde{\varphi}$ and H^p by $\widetilde{H}^p$.

8. Stable actions

In order to prove Theorem 1 we need some preliminary results. In this section we recall the notion of a stable action.

Let a reductive group G act on an affine variety X. We admit X to be reducible but require that G act transitively on the set of irreducible components of X (so $X/\!\!/G$ is irreducible). Such a G-variety will be called *G-irreducible.*

Let m_G be the maximal (=typical) dimension of orbits of the action $G : X$, and $X^{\deg}$ the union of orbits of lesser dimension. Obviously, $X^{\deg}$ is a proper G-invariant closed subvariety of X.

We put $X/\!\!/G = Y$.

Proposition 9. *The following properties of the action $G : X$ are equivalent:*

(a) *there exists a nonempty open subset $\widetilde{Y} \subset Y$ such that for any $y \in \widetilde{Y}$ the fiber $\pi_G^{-1}(y)$ consists of a single orbit;*

(b) *there exists an invariant non-empty open subset $\widetilde{X} \subset X$ consisting of closed orbits;*

(c) *there is a closed orbit of dimension m_G;*

(d) $\pi_G(X^{\deg}) \neq Y$.

These equivalences are well-known: see [1] and references there. Nevertheless we give their proof for convenience of the reader.

Proof. (a) $\Rightarrow$ (b). Take $\widetilde{X} = \pi_G^{-1}(\widetilde{Y})$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (d). If O is a closed orbit of dimension m_G, then $\pi_G(O) \notin \pi_G(X^{\deg})$.

(d) $\Rightarrow$ (a). Note that $\pi_G(X^{\deg})$ is a closed subvariety of Y. Let $\widetilde{Y}$ be the complement of this subvariety. Then for any $y \in \widetilde{Y}$ the fiber $\pi_G^{-1}(y)$ consists of orbits of the maximal dimension, hence of a single orbit. ∎

An action, satisfying the equivalent conditions of Proposition 9, is called *stable.* In the case $k[X]^G = k$ the stability of the action means its transitivity.

For a stable action, orbits in general position are separated by invariants, which implies that

$$\dim X/\!\!/G = \dim X - m_G. \tag{6}$$

9. The dimension of $H^p/\!\!/G$

For the action $G\colon G^p$ considered above, all the stabilizers contain the center $Z(G)$ of G. Moreover, if a p-tuple generates G, then its stabilizer coincides with $Z(G)$ and its orbit is closed. So the action of G on any G-irreducible component of G^p lying in $\widetilde{G}^p$ is stable and applying formula (6) yields that all irreducible components of $\widetilde{G}^p/\!\!/G$ have the dimension equal to

$$\dim \widetilde{G}^p/\!\!/G = (p-1)\dim G + \dim Z(G). \tag{7}$$

Let now H be a connected reductive subgroup of G. It was proved in Section 3, that the natural dominant morphism $H^p/\!\!/H \to H^p/\!\!/G$ is quasi-finite. Hence

$$\dim \widetilde{H}^p/\!\!/G = \dim \widetilde{H}^p/\!\!/H = (p-1)\dim H + \dim Z(H), \tag{8}$$

where $Z(H)$ denotes the center of H.

10. Fields of definition of reductive groups

The results presented in this section are, of course, known to specialists, but I have not found an appropriate reference.

Let as above k be an algebraically closed field of characteristic zero, and K an algebraically closed extension of k.

Any algebraic k-variety X (respectively, algebraic k-group G) defines in a natural way an algebraic K-variety $X(K)$ (respectively, an algebraic K-group $G(K)$). Such an algebraic K-variety (respectively, an algebraic K-group) is called *defined over* k.

Any morphism $X \to Y$ of algebraic k-varieties (respectively, a homomorphism $G \to H$ of algebraic k-groups, an action of an algebraic k-group G on an algebraic k-variety X) defines in a natural way a morphism $X(K) \to Y(K)$ (respectively, a homomorphism $G(K) \to H(K)$, an action of $G(K)$ on $X(K)$). Such a morphism (respectively, a homomorphism, an action) is called *defined over* k.

Two homomorphism $\varphi, \psi : G \to H$ of algebraic groups will be called *equivalent,* if there exists $h \in H$ such that

$$\psi(g) = h\varphi(g)h^{-1}$$

for any $g \in G$.

Proposition 10. 1) *Any reductive K-group is isomorphic to a K-group defined over k.*

2) *Any homomorphism of reductive K-groups defined over k is equivalent to a homomorphism defined over k.*

3) *Any reductive subgroup of an algebraic K-group defined over k is conjugate to a subgroup defined over k.*

Before proving the proposition we prove some lemmas.

Lemma 3. *For a reductive k-group G, any linear representation of $G(K)$ is equivalent to a representation defined over k.*

Proof. It suffices to prove the assertion for irreducible representations. Decompose the regular representation of G into a sum of irreducible representations. Tensoring this decomposition by K, we obtain a decomposition of the regular representation of $G(K)$ into a sum of irreducible representations, each of them is defined over k. Since the regular representation of a reductive group contains all its irreducible representations, we obtain the desirable result. ∎

Lemma 4. *For a reductive k-group G, any normal algebraic subgroup of $G(K)$ is defined over k.*

Proof. Since any normal algebraic subgroup of an algebraic group is the kernel of some linear representation of the group, the assertion follows from Lemma 4.∎

Lemma 5. *Let an algebraic k-group G act on an algebraic k-variety X. If the group $G(K)$ acts on $X(K)$ with finitely many orbits, then all these orbits are defined over k.*

Proof. We may assume that G acts transitively on the set of irreducible components of X (and hence $G(K)$ acts transitively on the set of irreducible components of $X(K)$). Let $\dim X = m$. Then the (only) open orbit of $G(K)$ in $X(K)$ is the complement of the subvariety

$$\begin{aligned} X(K)^{\deg} &= \{x \in X(K) \colon \dim G(K)x < m\} = \\ &= \{x \in X(K) \colon \dim \mathfrak{g}(K)(x) < m\} \end{aligned}$$

which is defined over k. Hence the open orbit is also defined over k. Proceeding by induction on $\dim X$, we obtain the desirable result. ∎

Proof of Proposition 10. We start with the second assertion. Let G and H be reductive k-groups and $\varphi: G(K) \to H(K)$ a homomorphism. We may assume that $H \subset \mathrm{GL}_n(k)$, so $H(K) \subset \mathrm{GL}_n(K)$. Then the composition of φ and the latter embedding yields a linear representation

$$\rho: G(K) \to \mathrm{GL}_n(K),$$

which is by Lemma 3 equivalent to a representation defined over k.

Let the group G be generated by a p-tuple $\mathbf{g} = (g_1, \dots, g_p) \in G^p$. The variety O of all representations equivalent to ρ, can be identified with the $\mathrm{GL}_n(K)$-orbit of the p-tuple

$$\rho(\mathbf{g}) = \varphi(\mathbf{g}) \in \mathrm{GL}_n(K)^p,$$

a representation $\sigma \in O$ being identified with the p-tuple $\sigma(\mathbf{g})$. Since the orbit O contains a k-point, it is defined over k.

The intersection $O \cap H(K)^p$ is, of course, also defined over k. According to Proposition 3, it decomposes into finitely many $H(K)$-orbits. Each of them is by Lemma 5 defined over k. In particular, the $H(K)$-orbit of $\varphi(\mathbf{g})$ is defined over k and, consequently, contains a k-point. This means that the homomorphism φ is equivalent to a homomorphism defined over k.

Now we prove the first assertion of the proposition. Lemma 4 implies that if a reductive K-group is defined over k, then any quotient group of it is also defined over k. Any reductive K-group is isomorphic to a quotient group of a group of the form

$$G = (H \times T) \leftthreetimes F, \tag{9}$$

where H is a simply connected semisimple group, T is a torus, F is a finite group and $\leftthreetimes$ denotes a semidirect product (see Section 6). So it suffices to prove the assertion for groups of the form (9).

It follows from the classification of semisimple groups that any simply connected semisimple K-group is isomorphic to a group defined over k. The same is, of course, true for tori. So we may assume that the group $H \times T = G_0$ is defined over k.

Finally, the homomorphism $\varphi: F \to \operatorname{Aut} G_0$, defining the semidirect product in (9), by the above is equivalent to a homomorphism defined over k. Hence, the group G is isomorphic to a group defined over k.

The third assertion of the proposition follows from the first two ones. ∎

11. The embeddings $G^p /\!\!/ G \subset G^q /\!\!/ G$

For any group G, let us identify a p-tuple $(g_1, \dots, g_p) \in G^p$ with the $(p+1)$-tuple $(g_1, \dots, g_p, e) \in G^{p+1}$. In such a way we get G-equivariant embeddings

$$G^1 \subset G^2 \subset \dots \subset G^p \subset G^{p+1} \subset \dots \tag{10}$$

Let now G be a reductive algebraic group. Then the embeddings (10) are closed and give rise to closed embeddings

$$G^1/\!\!/G \subset G^2/\!\!/G \subset \cdots \subset G^p/\!\!/G \subset G^{p+1}/\!\!/G \subset \ldots$$

For a reductive subgroup $H \subset G$ we have

$$H^1/\!\!/G \subset H^2/\!\!/G \subset \cdots \subset H^p/\!\!/G \subset H^{p+1}/\!\!/G \subset \ldots$$

Proposition 11. *$(H^q/\!\!/G) \cap (G^p/\!\!/G) = H^p/\!\!/G$ for $q \geqslant p$.*

Proof. It suffices to prove that

$$(H^{p+1}/\!\!/G) \cap (G^p/\!\!/G) = H^p/\!\!/G.$$

Consider the G-equivariant projection

$$\rho\colon G^{p+1} \to G^p, \quad (g_1, \ldots, g_p, g_{p+1}) \mapsto (g_1, \ldots, g_p).$$

It defines a projection

$$\rho_G\colon G^{p+1}/\!\!/G \to G^p/\!\!/G.$$

Since $\rho_G(H^{p+1}/\!\!/G) \subset H^p/\!\!/G$ and ρ_G is the identity map on $G^p/\!\!/G$, we get

$$(H^{p+1}/\!\!/G) \cap (G^p/\!\!/G) \subset H^p/\!\!/G.$$

The opposite inclusion is evident. ∎

Let us now prove the analogous property for $\widetilde{H}^p$'s (see the notation in Section 1). Obviously, we have

$$\widetilde{H}^1/\!\!/G \subset \widetilde{H}^2/\!\!/G \subset \cdots \subset \widetilde{H}^p/\!\!/G \subset \widetilde{H}^{p+1}/\!\!/G \subset \ldots$$

Proposition 12. *$(\widetilde{H}^q/\!\!/G) \cap (G^p/\!\!/G) = \widetilde{H}^p/\!\!/G$ for $q \geqslant p \geqslant 2$.*

Let H_0 be the connected component of H, containing e, and H_1 some other connected component. We put

$$H_1/\!\!/G = \overline{\pi_G(H_1)}.$$

Lemma 6. *$H_1/\!\!/G \not\ni \pi_G(e)$.*

Proof. Let S be a maximal toric subvariety of H_1. Then $\pi_G(H_1) = \pi_G(S)$. According to Proposition 5, π_G is finite on S. Hence $\pi_G(S)$ is closed and $H_1/\!\!/G = \pi_G(S)$. Since S consists of semisimple elements distinct from e, $\pi_G(S) \not\ni e$. ∎

Proof of Proposition 12. It suffices to prove that

$$(\widetilde{H}^{p+1}/\!\!/G) \cap (G^p/\!\!/G) = \widetilde{H}^p/\!\!/G.$$

Consider the G-equivariant projection

$$\sigma\colon G^{p+1} \to G, \quad (g_1, \ldots, g_p, g_{p+1}) \mapsto g_{p+1}.$$

It defines a projection

$$\sigma_G\colon G^{p+1}/\!\!/G \to G/\!\!/G.$$

Let $H' = H \setminus H_0$. In view of Corrollary 1 to Proposition 8

$$\widetilde{H}^{p+1} \subset (\widetilde{H}^p \times H_0) \cup (H^p \times H')$$

and hence

$$\widetilde{H}^{p+1}/\!\!/G = \overline{\pi_G(\widetilde{H}^{p+1})} \subset \overline{\pi_G(\widetilde{H}^p \times H_0)} \cup \overline{\pi_G(H^p \times H')}.$$

By Lemma 6 we have

$$\sigma_G\overline{(\pi_G(H^p \times H'))} \subset \overline{\sigma_G\pi_G(H^p \times H')} = \overline{\pi_G(H')} \not\ni \pi_G(e),$$

which implies that

$$\overline{\pi_G(H^p \times H')} \cap (G^p/\!\!/G) = \varnothing$$

Thus

$$(\widetilde{H}^{p+1}/\!\!/G) \cap (G^p/\!\!/G) = \overline{\pi_G(\widetilde{H}^p \times H_0)} \cap (G^p/\!\!/G).$$

Now we can do as in the proof of Proposition 11. Namely, since

$$\rho_G(\overline{\pi_G(\widetilde{H}^p \times H_0)}) \subset \widetilde{H}^p/\!\!/G$$

and ρ_G is the identity map on $G^p/\!\!/G$, we get

$$(\widetilde{H}^{p+1}/\!\!/G) \cap (G^p/\!\!/G) \subset \widetilde{H}^p/\!\!/G.$$

The opposite inclusion is evident. ∎

12. The action $\operatorname{Aut} F_p\colon G^p$

Let F_p be the free group on generators $u_1, \dots, u_p$, and $\Gamma_p = \operatorname{Aut} F_p$ the group of its automorphisms. For any group G, one can define a right action of Γ_p on G^p as follows. Let $\gamma \in \Gamma_p$. Then

$$\gamma(u_i) = w_i(u_1, \dots, u_p),$$

where $w_1, \dots, w_p$ are some words in p letters. We put

$$(g_1, \dots, g_p)^\gamma = (w_1(g_1, \dots, g_p), \dots, w_p(g_1, \dots, g_p)).$$

Obviously, this action commutes with the action of G by simultaneous conjugations. If $H \subset G$ is a subgroup, then the subset $H^p \subset G^p$ is invariant under Γ_p.

We have natural embeddings

$$F_1 \subset F_2 \subset \cdots \subset F_p \subset F_{p+1} \subset \dots,$$
$$\Gamma_1 \subset \Gamma_2 \subset \cdots \subset \Gamma_p \subset \Gamma_{p+1} \subset \dots,$$

where $\gamma \in \Gamma_p$ is extended to an automorphism of F_{p+1} by $\gamma(u_{p+1}) = u_{p+1}$. Obviously, the embedding $G^p \subset G^{p+1}$ is Γ_p-equivariant.

Let now G be a reductive algebraic group. Then any element of Γ_p acts as an automorphism of the algebraic variety G^p, so the action of the group Γ_p on G^p induces its action on $G^p/\!\!/G$. Note that $\widetilde{G}^p$, and hence $\widetilde{G}^p/\!\!/G$, is invariant under Γ_p. The embedding $G^p/\!\!/G \subset G^{p+1}/\!\!/G$ is Γ_p-equivariant.

If $H \subset G$ is a reductive subgroup, then $H^p/\!\!/G$ and $\widetilde{H}^p/\!\!/G$ are invariant under Γ_p.

Proposition 13. *Assume that $\mathbf{g} = (g_1, \dots, g_p) \in G^p$ (topologically) generates G. Then for sufficiently large $q \geqslant p$ the closure of the Γ_q-orbit of $\mathbf{g}$ coincides with $\widetilde{G}^q$.*

Proof. Denote the Γ_q-orbit of $\mathbf{g}$ by O and its closure by $\bar{O}$. Obviously, any element of O generates G and hence belongs to $\widetilde{G}^q$. Thus, $\bar{O} \subset \widetilde{G}^q$. We are to prove that $\bar{O} \supset \widetilde{G}^q$ for sufficiently large q.

Note that the group Γ_q contains (and is generated by) the following two types of transformations of G^q:

(T1) multiplication of some component by some word in other components;

(T2) permutation of the components.

Let $q = p + r$. Applying to $\mathbf{g}$ transformations of type (T1), we can get any q-tuple of the form

$$(g_1, \dots, g_p, h_1, \dots, h_r),$$

where $h_1, \dots, h_r$ belong to the subgroup Γ algebraically generated by $\mathbf{g}$. Since Γ is dense in G, this implies that

$$(g_1, \dots, g_p, h_1, \dots, h_r) \in \bar{O}$$

for any $h_1, \dots, h_r \in G$.

If $h_1, \dots, h_r$ generate G, we can prove in the same way that

$$(f_1, \dots, f_p, h_1, \dots, h_r) \in \bar{O}$$

for any $f_1, \dots, f_p \in G$. Since the set of generating r-tuples is dense in $\widetilde{G}^r$ (Corollary 8b of Proposition 8), we obtain that

$$G^p \times \widetilde{G}^r \subset \bar{O}. \tag{11}$$

Now take $r \geqslant 2$ large enough for the following condition to be satisfied: *Any generating set of $\geqslant r$ elements of the group G/G_0 contains a generating subset of r elements.* (For example, we can take $r = \max\{2, |G/G_0|\}$.) Then (11) and Corollary 8a of Proposition 8 imply that $\widetilde{G}^q \subset \bar{O}$. ■

Corollary 13a. *If $\mathbf{h} = (h_1, \dots, h_p) \in G^p$ generates a reductive subgroup H, then for sufficiently large $q \geqslant p$ the Γ_q-orbit of $\pi_G(\mathbf{h})$ is dense in $\widetilde{H}^q /\!\!/ G$.* ■

13. Approximation of subgroups

Let H and F be reductive subgroups of a reductive group G.

Lemma 7. *The following conditions are equivalent:*

(a) *$\pi_G(\mathbf{f}) \in \widetilde{H}^p /\!\!/ G$ for some generating set $\mathbf{f} = (f_1, \dots, f_p)$ of F;*

(b) *$\widetilde{F}^p /\!\!/ G \subset \widetilde{H}^p /\!\!/ G$ for all $p \geqslant 2$.*

(c) *$\widetilde{F}^p /\!\!/ G \subset \widetilde{H}^p /\!\!/ G$ for some p such that $\widetilde{F}^p \neq \varnothing$.*

Proof. (a) $\Rightarrow$ (b). According to Corollary 13a of Proposition 13 the Γ_q-orbit of $\pi_G(\mathbf{f})$ is dense in $\widetilde{F}^q /\!\!/ G$ for sufficiently large q. Since $\widetilde{H}^q /\!\!/ G$ is Γ_q-invariant and closed in $G^q /\!\!/ G$, this implies that $\widetilde{F}^q /\!\!/ G \subset \widetilde{H}^q /\!\!/ G$. But then (b) follows in view of Proposition 12.

(b) $\Rightarrow$ (c) is evident.

(c) $\Rightarrow$ (a) is also evident. ∎

If the equivalent properties of Lemma 7 are satisfied, we shall say that the subgroup F *is approximated by* H and write $F \prec H$. Obviously, this relation is transitive. It is also clear that if $F \subset H$ and $FH_0 = H$ (i.e., F intersects each connected component of H), then $F \prec H$. (We shall see later that the converse is also true up to conjugacy.)

If follows from formula (8) applied for sufficiently large p that if $F \prec H$ then

$$\dim F \leqslant \dim H.$$

Lemma 8. *If $F \prec H$, then $(F,F) \prec (H,H)$.*

Proof. Let $\mathbf{f} = (f_1, \dots, f_p)$ be a generating set of F. Then $\pi_G(\mathbf{f}) \in \overline{\pi_G(H_1 \times \cdots \times H_p)}$, where $H_1, \dots, H_p$ are connected components of H such that $H_1 \times \cdots \times H_p \subset \widetilde{H}^p$.

Let $v_1, \dots, v_r$ $(r \geqslant 2)$ be some words in the commutator subgroup of the free group on p generators. Define a G-equivariant morphism $\theta\colon G^p \to (G,G)^r$ by

$$\theta(g_1, \dots, g_p) = (v_1(g_1, \dots, g_p), \dots, v_r(g_1, \dots, g_p)).$$

It induces a morphism $\theta_G\colon G^p /\!\!/ G \to (G,G)^r /\!\!/ G$. The words $v_1, \dots, v_r$ can be choosen in such a way that $\theta(\mathbf{f})$ generate (F,F) and $\theta(\mathbf{h})$ generate

$$(H,H)/(H,H)_0 \quad \text{for} \quad \mathbf{h} \in H_1 \times \cdots \times H_p.$$

Then $\theta(H_1 \times \cdots \times H_p) \subset \widetilde{(H,H)}^r$ and

$$\begin{aligned}\pi_G(\theta(\mathbf{f})) = \theta_G(\pi_G(\mathbf{f})) \in \theta_G(\overline{\pi_G(H_1 \times \cdots \times H_p)}) &\subset \overline{\pi_G(\theta(H_1 \times \cdots \times H_p))} \\ &\subset \widetilde{(H,H)}^r /\!\!/ G,\end{aligned}$$

so (F,F) is approximated by (H,H). ∎

Lemma 9. *If $F \prec H$ and $\dim F = \dim H$, then $\dim Z(F) \geqslant \dim Z(H)$.* (We shall see below that in fact such a situation is impossible, unless $F = H$.)

Proof. This is an immediate consequence of Lemma 8 and the following lemma. ∎

Lemma 10. *For any reductive group G,*

$$\dim(G,G) + \dim Z(G) = \dim G. \tag{12}$$

Proof. Consider the group $G/(G,G_0)$. Its center contains $G_0/(G,G_0)$ and hence has finite index. By a theorem of Schur the commutator subgroup of $G/(G,G_0)$ is finite. This means that (G,G) is a finite extension of (G,G_0). (In fact $(G,G_0)=(G,G)_0$.)

Consider now the decomposition of the tangent algebra $\mathfrak{g}$ of G into the direct sum of the subspace $\mathfrak{g}^G$ of $\mathrm{Ad}(G)$-invariant elements and the subspace $\mathfrak{g}_G$ spanned by the elements $\mathrm{Ad}(g)\xi-\xi$, $(g\in G,\quad \xi\in\mathfrak{g})$. Clearly, $\mathfrak{g}^G$ is the tangent algebra of $Z(G)$. The subspace $\mathfrak{g}_G$ contains $[\mathfrak{g},\mathfrak{g}]$ and can be characterized as the least ideal of $\mathfrak{g}$ such that $\mathrm{Ad}(G)$ acts trivially on the corresponding quotient algebra. On the other hand, (G,G_0) is the least G-invariant algebraic subgroup of G_0 such that G acts trivially on the corresponding quotient group. Hence $\mathfrak{g}_G$ is the tangent algebra of (G,G_0) or, which is the same, of (G,G). The equality (12) follows, since

$$\dim\mathfrak{g}^G+\dim\mathfrak{g}_G=\dim\mathfrak{g}. \qquad \blacksquare$$

14. Structure of quasi-finite morphisms

We recall the structure of quasi-finite morphisms given by the principal Zariski theorem.

Any quasi-finite morphism $\psi\colon X\to Y$ of irreducible algebraic varieties admits a decomposition

$$\psi=\psi_1\psi_0, \tag{13}$$

where $\psi_0\colon X\to Z$ is an open embedding and $\psi_1\colon Z\to Y$ a finite morphism. If X is normal, one may assume Z to be normal as well, and under this condition the decomposition (13) is unique. Moreover, for any other decomposition $\psi=\tilde\psi_1\tilde\psi_0$ of type (13) we have

$$\tilde\psi_0=\nu\psi_0,\quad \psi_1=\tilde\psi_1\nu,$$

where $\nu\colon Z\to\tilde Z$ is a normalization, whose restriction to $\psi_0(X)$ is an open embedding.

Suppose now that X is an affine variety. Then $S_0=Z\setminus\psi_0(X)$ is a divisor in Z. Moreover, if ψ (and hence ψ_1) is dominant, $S=\psi_1(S_0)$ is a divisor in Y. It follows from the above description of all decompositions of type (13) that the divisor S does not depend on the choice of such a decomposition (that is, we need not suppose Z to be normal). We shall call S *the singular divisor* of ψ.

For our purposes it is convenient to extend this definition to reducible varieties. Let X be an unmixed normal affine variety (the normality means, in particular, that the irreducible components of X do not intersect), and $\psi\colon X\to Y$ a quasi-finite dominant morphism. We define the singular divisor of ψ as the union of the singular divisors of the induced morphisms

$$\psi_i\colon X_i\to\overline{\psi(X_i)},$$

where X_i runs over the irreducible components of X.

Let X, Y, ψ be as above, and X' an unmixed normal closed subvariety of X. Then the induced morphism

$$\psi' \colon X' \to Y' = \overline{\psi(X')}$$

is also quasi-finite and dominant. Denote by S' its singular divisor.

Lemma 11. $S' \subset S$.

Proof. If suffices to consider the case when X and X' are irreducible. In this case, any decomposition $\psi = \psi_1\psi_0$ of type (13) gives rise to a decomposition $\psi' = \psi_1^{'}\psi_0^{'}$,where $\psi_0^{'} \colon X' \to Z' = \overline{\psi_0(X')}$ and $\psi_1^{'} \colon Z' \to Y'$ are the restrictions of ψ_0 and ψ_1, respectively. Obviously, $\psi_0^{'}$ is an open embedding and $\psi_1^{'}$ a finite morphism. It follows that

$$S' = \psi_1^{'}(Z' \setminus \psi_0^{'}(X')) \subset \psi_1(Z \setminus \psi_0(X)) = S. \qquad \blacksquare$$

15. Proof of Theorem 1

Since the restriction of a finite morphism to any closed subvariety is also finite, it suffices to prove the theorem for sufficiently large p. In particular, we may (and shall) assume that $p \geqslant 2$.

In view of Corollary 8a to Proposition 8, H^p is the disjoint union of (closed) subsets $\widetilde{F}^p$, where F runs over the subgroups of finite index in H. Consequently, it suffices to prove that for any reductive sugroup $H \subset G$ the natural morphism

$$\tilde{\psi} \colon \widetilde{H}^p /\!\!/ H \to \widetilde{H}^p /\!\!/ G$$

is finite. We already proved in Section 3 that it is quasi-finite. Let $S_p \subset \widetilde{H}^p /\!\!/ G$ denotes its singular divisor (see the preceding section). We are to prove that $S_p = \varnothing$.

Since the group Γ_p naturally acts on all the involved varieties, the divisor S_p is Γ_p-invariant.

Suppose that $S_p \neq \varnothing$, and take any irreducible component T of S_p. Let $K = k(T)$ and $\bar{K}$ be the algebraic closure of K. We shall think of elements of $\bar{K}$ as of algebraic functions on T.

Let t be the K-point of $\widetilde{H}^p /\!\!/ G$ defined by the restriction of functions to T. Take any semisimple point

$$\mathbf{f}(t) = (f_1(t), \ldots, f_p(t)) \in G^p(\bar{K}),$$

projecting to t. Let $F(t)$ be the (reductive) subgroup of $G(\bar{K})$ generated by $\mathbf{f}(t)$. According to Proposition 8 there exists a reductive subgroup $F \subset G$ such that $F(t)$ is conjugate to $F(\bar{K})$ in $G(\bar{K})$. It follows that

$$t \in \widetilde{F}^p(\bar{K}) /\!\!/ G(\bar{K}) = (\widetilde{F}^p /\!\!/ G)(\bar{K}),$$

whence

$$T \subset \widetilde{F}^p /\!\!/ G.$$

According to Corollary 13a of Proposition 13, Γ_q-orbit of $\pi_{G(\bar{K})}(\mathbf{f}(t))$ is dense in $(\widetilde{F}^q /\!\!/ G)(\bar{K})$ for sufficiently large q. Since $\pi_{G(\bar{K})}(\mathbf{f}(t)) \in S_p(\bar{K}) \subset S_q(\bar{K})$ and $S_q(\bar{K})$ is Γ_q-invariant, we get

$$(\widetilde{F}^q /\!\!/ G)(\bar{K}) \subset S_q(\bar{K}),$$

which implies that

$$\widetilde{F}^q /\!\!/ G \subset S_q \subset \widetilde{H}^q /\!\!/ G. \tag{14}$$

In particular, F is approximated by H, and

$$\widetilde{F}^p /\!\!/ G \subset \widetilde{H}^p /\!\!/ G.$$

If $\widetilde{F}^p /\!\!/ G$ contains an irreducible component of $\widetilde{H}^p /\!\!/ G$, then it contains a point of the form $\pi_G(\mathbf{h})$, where $\mathbf{h} \in \widetilde{H}^p$ generates H. Since the Γ_q-orbit of such point is dense in $\widetilde{H}^q /\!\!/ G$ for sufficiently large q, we get $\widetilde{F}^q /\!\!/ G = \widetilde{H}^q /\!\!/ G$, which contradicts (14). Hence, $\widetilde{F}^p /\!\!/ G$ is a divisor in $\widetilde{H}^p /\!\!/ G$, that is

$$\dim \widetilde{F}^p /\!\!/ G = \dim \widetilde{H}^p /\!\!/ G - 1. \tag{15}$$

By formula (8) we have

$$\dim \widetilde{H}^p /\!\!/ G = (p-1) \dim H + \dim Z(H),$$
$$\dim \widetilde{F}^p /\!\!/ G = (p-1) \dim F + \dim Z(F).$$

Note that $\dim Z(F) \leqslant \operatorname{rk} G$. Therefore, if $p \geqslant \operatorname{rk} G + 3$, the equality (15) is impossible, unless $\dim F = \dim H$. But Lemma 10 shows that (15) is also impossible if $\dim F = \dim H$. We came to a contradiction, which proves that $S_p = \varnothing$.

Corollary 1 of Theorem 1. *Let H and F be reductive subgroups of a reductive group G. The subgroup F is approximated by H iff it is conjugate to a subgroup of H intersecting all connected components of H.*

Proof. The "if" part is clear due to Corollary 1 to Proposition 8.

Assume now that $F \prec H$, and let $\mathbf{f} = (f_1, \ldots, f_p)$, $p \geqslant 2$, be a generating set of F. Then Theorem 1 shows that $\pi_G(\mathbf{f}) \in \pi_G(\widetilde{H}^p)$. This means that $\mathbf{f}$ is conjugate to a p-tuple of $\widetilde{H}^p$, and hence F is conjugate to a subgroup of H, projecting onto H/H_0. ■

16. Varieties of characters of finitely generated groups

One can look at Theorem 1 from another point of view. A p-tuple $\mathbf{g} = (g_1, \ldots, g_p) \in G^p$ can be identified with the representation $F_p \to G$ taking the

i-th generator of F_p to g_p. In such a way the variety $R(F_p, G)$ of all representations of F_p into G is identified with G^p. If Γ is any group with p generators, one can consider Γ as a quotient group of F_p, and thereby the variety $R(\Gamma, G)$ of representations of Γ into G is identified with a closed G-invariant subvariety of $R(F_p, G)$.

The quotient $X(\Gamma, G) = R(\Gamma, G)/\!\!/G$ is called the variety of G-characters of Γ. It is naturally identified with a closed subvariety of $X(F_p, G)$. In view of the description of invariants for the action $SL_n(k) : \mathrm{SL}_n(k)^p$ (see the introduction) the elements of $X(\Gamma, SL_n(k))$ are interpreted as the usual characters of (unimodular) n-dimensional linear representations of Γ. (In the case of $GL_n(k)$, one should consider the inverse determinant of the representation together with its character.)

It is just a reformulation of Theorem 1 that, for a reductive subgroup $H \subset G$, the natural morphism $X(F_p, H) \to X(F_p, G)$ is finite. Moreover, since the restriction of a finite morphism to a closed subvariety is finite as well, we obtain

Corollary 2 of Theorem 1. *Let Γ be a finitely generated group, and H a reductive subgroup of a reductive group G. Then the natural morphism $X(\Gamma, H) \to X(\Gamma, G)$ is finite. In particular, for any reductive subgroup $H \subset SL_n(k)$ the set of characters of representations of Γ into H is closed in $X(\Gamma, SL_n(k))$.* ■

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Chair of Algebra
Moscow University
119899 Moscow
Russia
vinberg@ebv.pvt.msu.su

Received October 23, 1995
and in final form October 4, 1996