Journal of Lie Theory
Volume **7** (1997) 287–292

Lie bialgebras real Cohomology

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Communicated by K.-H. Neeb

Abstract. Let $\mathfrak{g}$ be a finite dimensional real Lie bialgebra. We introduce an $\mathbb{R}$-valued cohomology of $\mathfrak{g}$ for which the space of all inequivalent Lie bialgebra central extensions of $\mathfrak{g}$ by $\mathbb{R}$ is isomorphic to that second cohomology group of $\mathfrak{g}$. Furthermore, we study the natural projection of this group on the $\mathbb{R}$-valued Lie algebra second cohomology group of $\mathfrak{g}$.

Introduction

Lie bialgebras ([1], [2], [4]) are modern objects which generalize Lie algebras. In ([3]), we have introduced the notion of central extensions of those new objects that we have classified explicitly up to equivalence. The aim of this paper is to give an intrinsic treatment of the result we have developped in the previous work. More precisely, it is well known that the space of all inequivalent central extensions of a Lie algebra by $\mathbb{R}$ is isomorphic to the real second cohomology group of the given Lie algebra. We have asked ourselves: is it possible to do the same for Lie bialgebras? The answer is yes, as we shall prove, and is given as follows. Let $\mathfrak{g}$ be a finite dimensional real Lie bialgebra and denote by $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ the space of all inequivalent central extensions of $\mathfrak{g}$ by $\mathbb{R}$. For any finite dimensional real Lie algebra $\mathfrak{a}$ let $\wedge\mathfrak{a}^*$ denote the standard differential complex, $\mathbb{R} \to \mathfrak{a}^* \to \wedge^2\mathfrak{a}^* \to \wedge^3\mathfrak{a}^* \to \cdots \to 0$, defining the Lie algebra cohomology of $\mathfrak{a}$ with values in the trivial $\mathfrak{a}$-module $\mathbb{R}$. Let $\mathfrak{D} = \mathfrak{g} \bowtie \mathfrak{g}^*$ be the double of the Lie bialgebra $\mathfrak{g}$ ($\mathfrak{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ as vector spaces) and consider the natural projection $\wedge\mathfrak{D} \to \wedge\mathfrak{g}$. This projection intertwines the differentials of those complexes so the kernel $\ker(\wedge\mathfrak{D} \to \wedge\mathfrak{g})$ of this projection is a differential subcomplex of $\wedge\mathfrak{D}$. We define the k-th cohomology group $\mathcal{H}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ of the Lie bialgebra $\mathfrak{g}$ with values in $\mathbb{R}$ as the k-th cohomology group of the complex $\ker(\wedge\mathfrak{D} \to \wedge\mathfrak{g})$. We prove that $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ is isomorphic to $\mathcal{H}^2_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$. If $\mathcal{H}^2_{\mathrm{alg}}(\mathfrak{g}, \mathbb{R})$ denotes the real second cohomology group of the Lie algebra $\mathfrak{g}$ then we have a natural projection $\mathcal{H}^2_{\mathrm{big}}(\mathfrak{g}, \mathbb{R}) \to \mathcal{H}^2_{\mathrm{alg}}(\mathfrak{g}, \mathbb{R})$ which, in general, is neither injective nor surjective. All vector spaces considered here are real finite dimensional.

ISSN 0949–5932 / \$2.50

Preliminaries

A Lie bialgebra is a Lie algebra $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ for which the dual vector space $\mathfrak{g}^*$ is a Lie algebra $(\mathfrak{g}^*, [\ ,\ ]_{\mathfrak{g}^*})$, and those Lie brackets satisfy the Drinfeld compatibility:

$$\begin{aligned}\langle [\xi,\eta]_{\mathfrak{g}^*}, [x,y]_{\mathfrak{g}}\rangle = &- \langle [\operatorname{coad}_x \xi, \eta]_{\mathfrak{g}^*}, y\rangle \\ &- \langle [\xi, \operatorname{coad}_x \eta]_{\mathfrak{g}^*}, y\rangle + \langle [\operatorname{coad}_y \xi, \eta]_{\mathfrak{g}^*}, x\rangle + \langle [\xi, \operatorname{coad}_y \eta]_{\mathfrak{g}^*}, x\rangle;\end{aligned}$$

where coad denotes the coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$. For other equivalent formulations of this compatibility we refer the reader to ([1]).

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie bialgebras. A linear map $u : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie bialgebra morphism if $u : (\mathfrak{g}_1, [\ ,\ ]_{\mathfrak{g}_1}) \to (\mathfrak{g}_2, [\ ,\ ]_{\mathfrak{g}_2})$ is a Lie algebra morphism, and its transpose $u^* : (\mathfrak{g}_2^*, [\ ,\ ]_{\mathfrak{g}_2^*}) \to (\mathfrak{g}_1^*, [\ ,\ ]_{\mathfrak{g}_1^*})$ is also a Lie algebra morphism. A Lie bialgebra isomorphism is a bijective morphism of Lie bialgebras.

The double $\mathfrak{D} = \mathfrak{g} \bowtie \mathfrak{g}^*$ of a Lie bialgebra $\mathfrak{g}$ is the vector space $\mathfrak{D} = \mathfrak{g} \oplus \mathfrak{g}^*$ endowed with the Lie bracket: $[(x,\xi),(y,\eta)]_{\mathfrak{D}} = ([x,y]_{\mathfrak{g}} + \operatorname{coad}_\xi y - \operatorname{coad}_\eta x, [\xi,\eta]_{\mathfrak{g}^*} + \operatorname{coad}_x \eta - \operatorname{coad}_y \xi)$. It is to be understood that $\operatorname{coad}_\eta x$ denotes the coadjoint action of $\eta \in \mathfrak{g}^*$ on $x \in \mathfrak{g} \cong (\mathfrak{g}^*)^*$ (because $\mathfrak{g}$ is finite dimensional) and that $\operatorname{coad}_x \eta$ denotes the coadjoint action of $x \in \mathfrak{g}$ on $\eta \in \mathfrak{g}^*$.

Throughout the rest of this paper $\mathfrak{g}$ shall denote a fixed but arbitrary finite dimensional real Lie bialgebra.

Definition 1.1. A Lie bialgebra $\widehat{\mathfrak{g}}$ is called a central extension of $\mathfrak{g}$ by $\mathbb{R}$ if there exists an exact sequence $0 \to \mathbb{R} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$ in which i and π are morphisms of Lie bialgebras such that $i(\mathbb{R})$ is contained in the center of the Lie algebra $\widehat{\mathfrak{g}}$. Two central extensions $\widehat{\mathfrak{g}_1}$ and $\widehat{\mathfrak{g}_2}$ of $\mathfrak{g}$ by $\mathbb{R}$ will be called equivalent if there exists an isomorphism of Lie bialgebras $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ for which the following diagram commutes:

$$\begin{array}{ccccccccc} & & & & \widehat{\mathfrak{g}}_1 & & & & \\ & & & {}^{i_1}\nearrow & & \searrow^{\pi_1} & & & \\ 0 & \to & \mathbb{R} & & \downarrow \rho & & \mathfrak{g} & \to & 0\,. \\ & & & {}_{i_2}\searrow & & \nearrow_{\pi_2} & & & \\ & & & & \widehat{\mathfrak{g}}_2 & & & & \end{array}$$

We denote by $\operatorname{Ext}_{\text{big}}(\mathfrak{g},\mathbb{R})$ the space of all inequivalent Lie bialgebra central extensions of $\mathfrak{g}$ by $\mathbb{R}$. In ([3]) we have described explicitely this space as follows. Let $\mathcal{Z}^2_{\text{alg}}(\mathfrak{g},\mathbb{R})$ denote the space of $\mathbb{R}$-valued 2-cocycles of the Lie algebra $\mathfrak{g}$ and let $\operatorname{Der}(\mathfrak{g}^*)$ denote the space of all derivations of the Lie algebra $\mathfrak{g}^*$. We will say that $\gamma \in \mathcal{Z}^2_{\text{alg}}(\mathfrak{g},\mathbb{R})$ and $f \in \operatorname{Der}(\mathfrak{g}^*)$ are Drinfeld compatible if the transpose of f, denoted by $f^* : \mathfrak{g} \to \mathfrak{g}$ (N.B $(\mathfrak{g}^*)^* \cong \mathfrak{g}$ because $\mathfrak{g}$ is finite dimensional), satisfies the following condition:

$$\forall x, y \in \mathfrak{g} : f^*([x,y]) - [f^*(x), y] - [x, f^*(y)] = \operatorname{coad}_{\widetilde{\gamma(y)}}(x) - \operatorname{coad}_{\widetilde{\gamma(x)}}(y)\ ;$$

where $\widetilde{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ is defined by $\langle \widetilde{\gamma}(x), y \rangle = \gamma(x, y)$, coad denoting the coadjoint action of the Lie algebra $\mathfrak{g}^*$ on its dual Lie algebra $\mathfrak{g}$. This relation is exactly the Drinfeld compatibilty of Lie brackets defined by γ and f on a Lie bialgebra central extension of $\mathfrak{g}$ (see [3]); and that fact justifies our terminology.

Theorem 1.2 ([3]). *There is a 1-1 correspondence between* $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ *and the quotient of* $\{(\gamma, f) \in \mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g}, \mathbb{R}) \times \mathrm{Der}(\mathfrak{g}^*) \mid \gamma, f$ *Drinfeld-compatible*$\}$ *by* $\{(\delta\varphi, \mathrm{ad}_\varphi) \mid \varphi \in \mathfrak{g}^*\}$ *where* δ *denotes the coboundary operator in the* $\mathbb{R}$*-valued Lie algebra cohomology of* $\mathfrak{g}$ *and where* ad *denotes the adjoint action of the Lie algebra* $\mathfrak{g}^*$ *on itself.*

Remark . The Drinfeld compatibility of Lie brackets in the Lie bialgebra $\mathfrak{g}$ is equivalent to the condition that for each $\varphi \in \mathfrak{g}^*$, $\delta\varphi$ and ad_φ are Drinfeld compatible.

It is well known that the space of all inequivalent central extensions of a Lie algebra by $\mathbb{R}$ is isomorphic to the real second cohomology group of the given Lie algebra ([8]). In the next section we shall prove that $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ has the same type of description in terms of a real second cohomology group.

2. Real Cohomology of $\mathfrak{g}$

Let $(\wedge\mathfrak{g}, \Delta)$ be the standard differential complex,

$$\mathbb{R} \longrightarrow \mathfrak{g} \xrightarrow{\Delta_1} \wedge^2\mathfrak{g} \xrightarrow{\Delta_2} \wedge^3\mathfrak{g} \xrightarrow{\Delta_3} \quad \dots \quad \longrightarrow 0,$$

which defines the cohomology of the Lie algebra $\mathfrak{g}^*$ with values in the trivial module $\mathbb{R}$. For each $k \in \mathbb{N}$, $\Delta_k = \Delta$ shall denote the coboundary operator for this cohomology. Let $\mathfrak{D} = \mathfrak{g} \bowtie \mathfrak{g}^*$ be the double of the Lie bialgebra $\mathfrak{g}$ and let $(\wedge\mathfrak{D}, \widetilde{\delta})$ denote the differential complex defining the Lie algebra cohomology of $\mathfrak{D}^*$ ($\cong \mathfrak{D}$). One can easily verify that the natural projection $(\wedge\mathfrak{D}, \widetilde{\delta}) \xrightarrow{\pi} (\wedge\mathfrak{g}, \Delta)$ intertwines the differentials of those complexes, i.e., that the following diagram commutes:

$$\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & \mathfrak{D} & \xrightarrow{\widetilde{\delta}_1} & \wedge^2\mathfrak{D} & \xrightarrow{\widetilde{\delta}_2} & \wedge^3\mathfrak{D} & \xrightarrow{\widetilde{\delta}_3} & \dots \\
\downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_3} & & \\
\mathbb{R} & \longrightarrow & \mathfrak{g} & \xrightarrow{\Delta_1} & \wedge^2\mathfrak{g} & \xrightarrow{\Delta_2} & \wedge^3\mathfrak{g} & \xrightarrow{\Delta_3} & \dots
\end{array}$$

Hence, for each $k \in \mathbb{N}$ the $\widetilde{\delta_k}$ ($= \widetilde{\delta}$)-range of the kernel $N_k = \ker \pi_k$ is a subspace of the kernel $N_{k+1} = \ker \pi_{k+1}$. So the complex $N(\mathfrak{g}, \mathbb{R})$:

$$0 \longrightarrow N_1 = \{0\} \oplus \mathfrak{g}^* \xrightarrow{\widetilde{\delta}_1} N_2 \xrightarrow{\widetilde{\delta}_2} N_3 \xrightarrow{\widetilde{\delta}_3} \quad \dots$$

defined as the kernel of the projection $\pi\colon (\wedge\mathfrak{D}, \widetilde{\delta}) \to (\wedge\mathfrak{g}, \Delta)$ is a differential subcomplex of $(\wedge\mathfrak{D}, \widetilde{\delta})$. Let $\mathcal{Z}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R}) = \ker(\widetilde{\delta}_k | N_k) \subset N_k$ denote the subspace of k-cocycles and let $\mathcal{B}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R}) = \mathrm{im}(\widetilde{\delta}_{k-1} | N_{k-1}) \subset N_k$ denote the subspace of k-coboundaries of this differential complex.

Definition 2.1. The k-th cohomology group $\mathcal{H}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g},\mathbb{R})$ of the Lie bialgebra $\mathfrak{g}$ with values in $\mathbb{R}$ is the k-th cohomology group of the complex $N(\mathfrak{g},\mathbb{R})$; i.e., $\mathcal{H}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) = \mathcal{Z}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g},\mathbb{R})/\mathcal{B}^{\mathrm{k}}_{\mathrm{big}}(\mathfrak{g},\mathbb{R})$.

Lemma 2.2.

1) $\mathcal{Z}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) \cong \{(\gamma, f) \in \mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g},\mathbb{R}) \times \mathrm{Der}(\mathfrak{g}^*) \mid \gamma, f$ *Drinfeld-compatible*$\}$.

2) $\mathcal{B}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) \cong \{(\delta\varphi, \mathrm{ad}_\varphi) \mid \varphi \in \mathfrak{g}^*\}$.

Proof. Let ω be in $\wedge^2\mathfrak{D}$; N.B: $\mathfrak{D} \cong \mathfrak{D}^*$. So ω defines an element γ of $\wedge^2\mathfrak{g}^*$ by $\gamma(x,y) = \omega((x,0),(y,0))$, x, $y \in \mathfrak{g}$, and an endomorphism f of $\mathfrak{g}^*$ by $\langle f(\eta), x\rangle = \omega((x,0),(0,\eta))$, $x \in \mathfrak{g}$, $\eta \in \mathfrak{g}^*$. If $\omega \in N_2 := \ker \pi_2$, we then have $\forall (x,\xi),(y,\eta) \in \mathfrak{D}$

$$\omega((x,\xi),(y,\eta)) = \gamma(x,y) + \langle f(\eta), x\rangle - \langle f(\xi), y\rangle.$$

The map $N_2 \to \wedge^2\mathfrak{g}^* \oplus \mathrm{End}(\mathfrak{g}^*)$ which associates (γ, f) to ω is a vector space isomorphism. It is not difficult to prove that the condition that $\omega \in \mathcal{Z}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) \subset N_2$ is equivalent to the following condition that: $\gamma \in \mathcal{Z}^2_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$, that $f \in \mathrm{Der}(\mathfrak{g}^*)$, and that γ and f are Drinfeld-compatible. This establishes assertion 1).

An element θ of N_1 is of the form $\theta = (0,\varphi)$ with $\varphi \in \mathfrak{g}^*$. An easy computation shows that $\forall (x,\xi),(y,\eta) \in \mathfrak{D}$

$$(\widetilde{\delta}\theta)((x,\xi),(y,\eta)) = (\delta\varphi)(x,y) + \langle [\varphi,\eta], x\rangle - \langle [\varphi,\xi], y\rangle.$$

Hence $\mathcal{B}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) = \mathrm{im}(\widetilde{\delta}|N_1) \subset N_2$ is isomorphic to $\{(\delta\varphi, \mathrm{ad}_\varphi) \mid \varphi \in \mathfrak{g}^*\}$. ■

As a consequence of this lemma and Theorem 1.2 we obtain the following result.

Theorem 2.3. $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g},\mathbb{R})$ *is isomorphic to* $\mathcal{H}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R})$.

Every Lie bialgebra $\widehat{\mathfrak{g}}$ central extension of $\mathfrak{g}$ by $\mathbb{R}$ is in particular a Lie algebra central extension of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}$, and equivalent Lie bialgebra central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are also equivalent Lie algebra central extensions of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}$. Let $\mathrm{Ext}_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$ be the space of all inequivalent Lie algebra central extensions of the Lie algebra $\mathfrak{g}$ by $\mathbb{R}$. Then we have a natural projection $\pi_{\mathrm{big}} : \mathrm{Ext}_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) \to \mathrm{Ext}_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$. It follows from the previous theorem and the well known fact that $\mathrm{Ext}_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$ is isomorphic to the $\mathbb{R}$-valued Lie algebra second cohomology group $\mathcal{H}^2_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$ of $\mathfrak{g}$ (see [8]) this projection is given in cohomological terms by: $\pi_{\mathrm{big}} : \mathcal{H}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R}) \to \mathcal{H}^2_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$; $\pi_{\mathrm{big}}([[(\gamma,f)]]) = [[\gamma]]$; where double brackets denote the equivalence classes in $\mathcal{H}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R})$ and $\mathcal{H}^2_{\mathrm{alg}}(\mathfrak{g},\mathbb{R})$ respectively.

Proposition 2.4. *The kernel of* π_{big} *is the quotient of* $\{f \in \mathrm{Der}(\mathfrak{g}^*) \mid f^* \in \mathrm{Der}(\mathfrak{g})\}$ *by* $\{\mathrm{ad}_\varphi; \varphi \in \mathfrak{g}^* \mid \mathrm{ad}^*_\varphi \in \mathrm{Der}(\mathfrak{g})\}$.

Proof. Let $[[(\gamma,f)]] \in \mathcal{H}^2_{\mathrm{big}}(\mathfrak{g},\mathbb{R})$ be in the kernel of π_{big}, i.e., suppose that: $[[\gamma]] = [[0]]$. Then there exists $\psi \in \mathfrak{g}^*$ such that $\gamma = \delta\psi$, so $[[(\gamma,f)]] = [[(0, f - \mathrm{ad}_\psi)]]$. It is obvious that $\tilde{f} = f - \mathrm{ad}\,\psi$ is a derivation of $\mathfrak{g}^*$. The Drinfeld compatibility of $\tilde{f}$ with the null cocycle is just the condition that $\tilde{f}^* \in \mathrm{Der}(\mathfrak{g})$. We also have $[[(\gamma, f)]] = [[(0, \tilde{f} - \mathrm{ad}_\varphi)]]$ for all $\varphi \in \mathfrak{g}^*$. The Drinfeld compatibility of the null cocycle with $\tilde{f} - \mathrm{ad}_\varphi$ is equivalent to $\mathrm{ad}^*_\varphi \in \mathrm{Der}(\mathfrak{g})$. This establishes the proposition. ■

Remark . If $\delta\varphi = 0$; $\varphi \in \mathfrak{g}^*$, then the Drinfeld compatibility of $\delta\varphi$ with ad_φ implies that ad^*_φ is a derivation of $\mathfrak{g}$.

We now give a few examples of cases where π_{big} is either not injective or is not surjective.

Example 2.5. Let us consider $\mathfrak{g} = sl(2, \mathbb{R})$ with its standard Lie bialgebra structure given in terms of the canonical basis $\{H, X_+, X_-\}$ of $\mathfrak{g}$ and its dual basis $\{H^*, X^*_-, X^*_+\}$. An easy computation shows that $\{\mathrm{ad}_\varphi; \varphi \in \mathfrak{g}^* \mid \mathrm{ad}^*_\varphi \in \mathrm{Der}(\mathfrak{g})\} = \{0\}$ and that the kernel of π_{big} is the subspace of endomorphisms of $\mathfrak{g}^*$ given w.r.t. the ordered basis (H^*, X^*_+, X^*_-) by the matrices of the form: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix}$, with $a \in \mathbb{R}$. We conclude that π_{big} is not injective in the case of $sl(2, \mathbb{R})$.

Example 2.6. Endow $\mathfrak{g} = \mathbb{R}^2$ with a Lie bialgebra structure as follows: Consider the abelian Lie algebra structure on $\mathfrak{g}$ and define a bracket on $\mathfrak{g}^* = \mathbb{R}^2$ by setting $[e_1, e_2] = e_2$ where (e_1, e_2) denotes the canonical ordered basis of $\mathbb{R}^2$. Consider the canonical symplectic 2-form γ of $\mathbb{R}^2$; it is a 2-cocycle because $\mathfrak{g}$ is an abelian Lie algebra. The Drinfeld-compatibility of γ and a derivation f of $\mathfrak{g}^*$ reduces to the requirement that $\forall x, y \in \mathbb{R}^2 : \mathrm{coad}_{\widetilde{\gamma(y)}}(x) - \mathrm{coad}_{\widetilde{\gamma(x)}}(y) = 0$, which, however, is not satified; so $[[\gamma]]$ has no preimage under π_{big} and thus π_{big} is not surjective in this case.

3. Comparison with other works.

For reasons distinct from ours, Drinfeld ([5]) has introduced a cohomological obstruction for Lie bialgebras. It is the cohomology of the complex kernel of the natural projection $\wedge\mathfrak{D} \to \wedge\mathfrak{g} \oplus \wedge\mathfrak{g}^*$. Using arguments similar to those in the proof of Theorem 2.3 one can easily see that the second group of Drinfeld's cohomology is given by $\{f \in \mathrm{Der}(\mathfrak{g}^*) | f^* \in \mathrm{Der}(\mathfrak{g})\}$. This space is clearly not isomorphic to $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$, so that cohomology does not answer our question.

After this work, C. Roger and P. Lecomte ([6]) introduced Lie bialgebra cohomology with values in a module. The module category is the double-module category and the complex defining this cohomology is the kernel of the natural projection $\wedge\mathfrak{D} \to \mathfrak{g}^*$. Its relation to our cohomology is that it gives the cohomology of the bialgebra $\mathfrak{g}^*$ rather than of $\mathfrak{g}$ itself. By analogy to Theorem 2.3, the second group of this cohomology is isomorphic to $\mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}^*, \mathbb{R})$. The notion of central extensions of Lie bialgebras is not self-dual. Hence, their last cohomology does not answer our question. Roger and Lecomte modified their cohomology in ([7]) by change of the Lie bialgebra module category. In the case of trivial $\mathfrak{D}$-modules (our case), they get Drinfeld's cohomology .

Aknowledgements

I would like to thank G. Tuynman and Y. Kosmann-Schwarzbach for their interest in the present work. I also thank the referee for his suggestions regarding the manuscript.

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Received February 27, 1997