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The surjectivity question for the exponential function of real Lie groups: A status report

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Abstract. The present knowledge of the structure of the exponential function of a real Lie group is surveyed. The emphasis is on the image of the exponential function: Which powers of a group element are contained in it, when is it dense, when does it fill the group? Some so far unobserved errors in the literature are pointed out, results are described, and some conjectures are formulated.

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0. The basic definitions and introductory remarks

The Lie algebra of a Lie group G will be denoted by $\mathfrak{L}(G)$ or more succinctly by $\mathfrak{g}$ if the context makes it clear what is meant.

Definition 0.1. Let G be a (real) Lie group, $\mathfrak{g}$ its Lie algebra and $\exp_G\colon \mathfrak{g} \to G$ its exponential map. We set $E_G = \{\exp_G(X) : X \in \mathfrak{g}\} = \exp_G \mathfrak{g}$ and say that G is *exponential* if $E_G = G$, and that G is *weakly exponential* if $\overline{E_G} = G$, i.e., if E_G is dense in G.

For an arbitrary Lie group G and $x \in G$ we define

$$\operatorname{ind}_G(x) = \begin{cases} \min\{k \in \mathbb{N} : x^k \in E_G\} & \text{if this minimum exists,} \\ \infty & \text{otherwise,} \end{cases}$$
$$\operatorname{ind}(G) = \{\operatorname{ind}_G(x) : x \in G\}.$$

We say that $\operatorname{ind}_G(x)$ is the *G-index of x* and $\operatorname{ind}(G)$ the *index set of G*. ∎

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Thus G *is exponential if and only if* $\operatorname{ind}(G) = \{1\}$. We warn the reader that this definition of $\operatorname{ind}(G)$ is different from the one which LAI adopted [46, 47] and also used in [22]. A forerunner of the index appears to have been introduced by POLISHCHUK in [72] who showed that for a semisimple complex Lie groups there is a natural number n such that $x^n \in E_G$ for all $x \in G$, and he exhibits such numbers for complex simple Lie groups and their universal covering groups. This paper appears to have either passed unnoticed or to have been forgotten before the index was introduced by GOTO and LAI as the least common multiple of the numbers in $\operatorname{ind}(G)$ if it exists, or infinity, otherwise. Note that weakly exponential groups are necessarily connected. We also alert the reader to the fact that some authors [6, 43, 56] use the term "exponential group" in a different sense, namely to denote the Lie groups G for which the exponential map $\exp_G\colon \mathfrak{g} \to G$ is a diffeomorphism.

In a connected compact Lie group every point is contained in a maximal torus; hence compact connected groups are exponential: a classical result. All connected complex and all connected solvable Lie groups are weakly exponential [39, 31, 71]. It is an elementary and well known fact that, for instance, if $G = \mathrm{SL}(n, \mathbb{R})$, $n \geq 2$, then $E_G \neq G$, while $x^2 \in E_G$ for all $x \in G$.

As is not uncommon in the structure theory of Lie algebras and Lie groups, the substantial results on the structure of the exponential function roughly fall into two classes: (A) Characterisation Theorems, and (B) Classification (or Catalog) Theorems. Characterisation Theorems specify necessary and sufficient conditions for certain properties such as being weakly exponential or for being exponential, while Catalog Theorems provide complete lists (usually within preassigned classes) of types of Lie algebras or connected Lie groups whose exponential function has such properties. Catalog Theorems are sensible in the context of special classes such as simple connected Lie groups where a full classification of all types exists. Results pertaining to *all* Lie algebras and Lie groups are likely to be Characterisation Theorems simply because a classification of nilpotent, let alone solvable, connected Lie groups is out of the question in general and because, the LEVI-MALCEV-IWASAWA Theorems on the semidirect (near) splitting of the radical notwithstanding, (nearly) semidirect extensions are not a trivial matter as regards the exponential function.

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1. The index set of classical groups

We recall an old result of J. SIBUYA [74].

Proposition 1.1. *Let $K_1, \ldots, K_m$ be real n by n matrices and let G be the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of all invertible matrices X such that*

$$X' K_i X = K_i, \quad i = 1, \ldots, m;$$

where X' denotes the transpose of X. Then $X^2 \in E_G$ for all $X \in G$, and $X \in E_G$ for all those $X \in G$ that have no real negative eigenvalues. ∎

For instance, G could be $\mathrm{O}(p,q)$, $p+q=n$, or $\mathrm{Sp}(n,\mathbb{R})$ if n is even.

This result motivated L. MARKUS [52] to ask whether it applied to certain other classes of groups, and he considered related questions. His results include the following.

If G is an algebraic subgroup of $\mathrm{GL}(n,\mathbb{R})$ *and $x \in G$, then there exists a $k \in \mathbb{N}$ such that $x^k \in E_G$. If, in addition, G is abelian, k can be chosen independently of x.* He also shows that *there is a connected and simply connected solvable Lie subgroup G of* $\mathrm{GL}(4,\mathbb{R})$ *containing an element x such that $x^k \notin E_G$ for all nonzero integers k.*

M. GOTO [23] strengthened MARKUS' result by showing

Proposition 1.2. *For every algebraic subgroup G of* $\mathrm{GL}(n,\mathbb{R})$ *there exists a $k \in \mathbb{N}$ such that $x^k \in E_G$ for all $x \in G$.* ■

The conclusion is also correct for each semisimple linear real or complex Lie group as GORBATSEVICH, ONISHCHIK and VINBERG develop in their Encyclopedia article [22], p. 164. The pioneer of the idea of the G-index of a group element and the index set of a Lie group was LAI beginning with his dissertation written under the direction of M. GOTO. However, the proof of the main result in [46, 47] contained gaps. In a subsequent note [48] LAI tried to fill these gaps. The key lemma of his note is

Lemma C. (Lai 1980) *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{g}_1$ be a semisimple subalgebra of maximal rank. Let $A \in \mathfrak{g}_1$ be a nilpotent element which is regular in $\mathfrak{g}_1$. Then each element of the centralizer $\mathfrak{z}(A,\mathfrak{g}) \stackrel{\mathrm{def}}{=} \{x \in \mathfrak{g} : [x,A]=0\}$ of A in $\mathfrak{g}$ is nilpotent.* ■

Although LAI's note was published more than 15 years ago, it remained unnoticed so far that Lemma C is false. Lemma C holds if $\mathfrak{g}$ is of type G_2, but it is false if $\mathfrak{g}$ is of type F_4, E_6, E_7, or E_8. A nilpotent element $A \in \mathfrak{g}$ is called *distinguished* if $\mathfrak{z}(A,\mathfrak{g})$ contains no nonzero semisimple elements (see [5, 8]) or if, equivalently, all elements of $\mathfrak{z}(A,\mathfrak{g})$ are nilpotent. By using the tables in [20] and [5], Corollary 5.7.5, it is easy to verify that, for instance, if

$$\begin{array}{lllllll} \mathfrak{g} & = & F_4, & E_6, & E_7, & E_8, & \text{and} \\ \mathfrak{g}_1 & = & 4A_1, & 3A_2, & 2A_3+A_1, & A_7+A_1, & \text{respectively,} \end{array}$$

then the principal nilpotent element of $\mathfrak{g}_1$ is not distinguished in $\mathfrak{g}$. Hence the claims made by LAI [46, 47] concerning the sets $\mathrm{ind}(G)$ for almost simple complex Lie groups G have to be re-examined (see Theorems 1.4 and 1.5 below).

For any Lie group G, we denote by G° its identity component. If G is also an affine algebraic group over $\mathbb{C}$, then G° coincides with the identity component of G in the Zariski topology. We write $Z(x,G)=\{g \in G : gx=xg\}$ for the centralizer of x in G. The Jordan decomposition of an element x in an affine algebraic group will be written $x = x_s x_u$ where x_s is the unique semisimple factor and x_u the unique unipotent one; recall $x_s x_u = x_u x_s$. We now have the following result:

Theorem 1.3. *Let $x = x_s x_u$ be the Jordan decomposition of an element in an affine algebraic group G over $\mathbb{C}$. If k is the order of $\overline{x} \stackrel{\text{def}}{=} xZ(x_u, G)^\circ$ in the quotient group $Z(x_u, G)/Z(x_u, G)^\circ$, then $\operatorname{ind}_G(x) = k$.*

Proof. For any $m \in \mathbb{N}$, the power x^m has the Jordan decomposition $x_s^m x_u^m$. Since the Zariski closures of $\langle x_u \rangle$ and $\langle x_u^m \rangle$ coincide, we have $Z(x_u^m, G) = Z(x_u, G)$. Theorem 3.2 of [15] is valid for arbitrary complex affine algebraic groups because the hypothesis on the group to be reductive which was made there was not used in the proof. Therefore we conclude that $x^m \in E_G$ if and only if x_s^m (or, equivalently, x^m) belongs to $Z(x_u^m, G)^\circ = Z(x_u, G)^\circ$. Consequently, $\operatorname{ind}_G(x) = k$. ∎

This is a Characterisation Theorem, but it leads expeditiously to Catalog Theorems. Indeed, when G is an almost simple complex Lie group of exceptional type, then the groups $Z(x_u, G)/Z(x_u, G)^\circ$ have been computed by Alekseevskiĭ [1]. The cases where $Z(x_u, G)$ is not connected are also listed in [15]. By consulting these tables we obtain the results tabulated below, where the superscript ad denotes the adjoint group and sc the simply connected group.

Theorem 1.4. *The indices of exceptional complex Lie groups are listed in the following*

Table of the index set $\operatorname{ind}(G)$.

Type of G						
G_2	1	2	3			
F_4	1	2	3	4		
E_6^{ad}	1	2	3			
E_6^{sc}	1	2	3	6		
E_7^{ad}	1	2	3			
E_7^{sc}	1	2	3	6		
E_8	1	2	3	4	5	6

∎

In the last three cases (E_7^{ad}, E_7^{sc}, E_8) these results do not agree with those of Lai [46, 47] (see also [22], pp. 165–166). The following theorem is due to Lai [44].

Theorem 1.5. *For complex adjoint groups G of classical type the set* $\operatorname{ind}(G)$ *is*

1. $\{1\}$ *if G is of type A_n, $n \geq 1$,*
2. $\{1, 2\}$ *for types B_n, $n \geq 2$, C_n, $n \geq 3$, and D_n, $n \geq 4$.*

Sketch of the Proof. In the case of A_n this follows from the commutativity of the diagram

$$\begin{tikzcd} \mathrm{GL}(n,\mathbb{C}) \arrow[r, "\mathrm{quot}"] & \mathrm{PSL}(n,\mathbb{C}) \\ \mathfrak{gl}(n,\mathbb{C}) \arrow[u, "\exp_{\mathrm{GL}(n,\mathbb{C})}"] \arrow[r, "\mathrm{quot}"'] & \mathfrak{sl}(n,\mathbb{C}) \arrow[u, "\exp_{\mathrm{PSL}(n,\mathbb{C})}"'] \end{tikzcd}$$

and the fact that $\exp_G$ is surjective for $G = \mathrm{GL}(n, \mathbb{C})$. For types B_n, C_n, and D_n one can consult the article [13]. Indeed, since $\mathrm{SO}(2n+1, \mathbb{C})$ is the adjoint group of type B_n, in that case the claim follows from [13], Corollary 5.2. The claim in the cases C_n and D_n can be easily derived from loc. cit., Theorems 8.1 and 6.3, respectively. ∎

In a recent note [76], Wüstner observes that the simple complex centerfree Lie group $\mathrm{Sp}(4, \mathbb{C})/Z_2 \simeq \mathrm{SO}(5, \mathbb{C})$ is not exponential, thus providing a counterexample to the statement made by Moskowitz in [57] that all connected semisimple complex centerfree Lie groups are exponential. Apparently neither Wüstner nor Moskowitz were aware of the fact that Lai [44] and Đoković (in [13], Corollary 5.2) had already stated that the groups $\mathrm{SO}(n, \mathbb{C})$, $n \geq 4$, fail to be exponential.

In [13] Đoković explicitly described the image E_G of the exponential function of complex and real classical groups G. With the aid of this description, it is not hard to see which of these groups are exponential. For instance, by this method, Nishikawa [68] has shown

Proposition 1.6. *For the groups $G = \mathrm{SO}(p, q)^\circ$, $p \geq q \geq 1$, the index set is* $\mathrm{ind}(G) = \{1\}$ *if* $q = 1$, *and* $\mathrm{ind}(G) = \{1, 2\}$ *otherwise.* ∎

Hence these groups are exponential if and only if $q = 1$. Note also that Sibuya's theorem implies that $\mathrm{ind}(G) \subseteq \{1, 2\}$ for each of the groups $\mathrm{O}(p, q)$ and $\mathrm{Sp}(2n, \mathbb{R})$.

In his papers [66, 67, 68], Nishikawa studied the structure of the interior and the closure of the image E_G of the exponential function in G for the groups $\mathrm{GL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$, and also for groups $\mathrm{O}(p, q)$ when $p, q \leq 3$. Subsequently, Đoković completely determined in [14] the interior and the closure of E_G in G for all complex and real classical Lie groups G.

2. Special subgroups of a real Lie group

In the characterisation theorems for the properties of being weakly exponential and of being exponential we need the concepts of certain canonically defined subgroups of a real Lie group which are generally known and accepted for semisimple connected Lie groups and for algebraic groups but which appear not to have been commonly known in the general case—or at best in the form of folklore.

∘ *Cartan Subgroups.* (Cartan subgroups will be used in Theorem 3.1 on the characterisation of weakly exponential Lie groups below.) A first and purely group theoretical definition of a Cartan subgroup of an arbitrary group was given by Chevalley [7]. We denote the normalizer of a subgroup S of a group G by $N(S, G)$.

Definition 2.1. According to Chevalley, a subgroup H of a group G is called a *Cartan subgroup of G* if it is maximal among the nilpotent subgroups of G and every normal subgroup S of H with $|H/S| < \infty$ satisfies $|N(S, G)/S| < \infty$. ∎

Since $H \subseteq N(S,G)$ we note $|N(S,G)/S| < \infty$ iff $|N(S,G)/H| < \infty$.

$$\begin{array}{cl} G & \\ | & \\ N(S,G) & \\ | & \big\}\ \text{finite} \\ H & \\ | & \big\}\ \text{finite} \\ S & \\ | & \\ \{1\} & \end{array}$$

Another definition has been given more recently by HOFMANN [32], Definition 1.1.a, and NEEB [63], p.153 and [65], Definition I.1(c). (See also [37], Definition 5.1). The proof that for connected Lie groups this definition is equivalent to CHEVALLEY's is nontrivial and was accomplished by NEEB [65]. In order to record his result, let us recall that for any Cartan subalgebra $\mathfrak{h}$ of a real Lie algebra $\mathfrak{g}$ we have a set $\Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of roots $\lambda\colon \mathfrak{h}_{\mathbb{C}} \to \mathbb{C}$ of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. If G is a Lie group with Lie algebra $\mathfrak{g}$ then we define a subgroup of G by

$$C(\mathfrak{h}) \stackrel{\text{def}}{=} \{g \in N(\exp \mathfrak{h}, G) : \big(\forall \lambda \in \Lambda(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\big)\, \lambda \circ \mathrm{Ad}(g)_{\mathbb{C}}|\mathfrak{h}_{\mathbb{C}} = \lambda\}.$$

Theorem 2.2. *For an arbitrary connected Lie group G and a closed subgroup H, the following conditions are equivalent:*

(1) *H is a Cartan subgroup of G.*

(2) *There is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $H = C(\mathfrak{h})$.* ■

Condition (2) characterizes the elements of a Cartan subgroup by a centralizer-like condition; it was this condition that served to define the concept of a Cartan subgroup of an arbitrary Lie group in [32, 63, 65, 37]. Therefore, for reductive groups this reduces to the more traditional concept of a Cartan subgroup as the centralizer of $\exp \mathfrak{h}$ for some Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{L}(G)$. NEEB's Theorem shows, in particular, that *in a connected Lie group G, the subgroups $C(\mathfrak{h})$ are maximal nilpotent.* Every finite group, in particular every finite nilpotent group is isomorphic to a subgroup of some $\mathrm{U}(n)$; the subgroup H of all diagonal $\mathrm{SO}(3)$ matrices is a maximal abelian and maximal nilpotent subgroup of $\mathrm{SO}(3)$ isomorphic to KLEIN's four group which is not contained in any maximal torus. Every proper subgroup S of H is cyclic and therefore is contained in a maximal torus T. Thus $S \subseteq \langle H \cup T \rangle \subseteq N(S,G)$, and therefore H fails to satisfy CHEVALLEY's condition.

∘ *Near-Cartan Subalgebras and near-Cartan Subgroups.* (Near-Cartan subgroups will be used in Theorem 4.2 on the characterisation of solvable exponential Lie groups below, and in the formulation of certain conjecture concerning exponential Lie groups.) Some generalisations are in order which are relevant for a possible general characterisation of the property of a real Lie group of being exponential. We observe that the vector subspaces of dimension $\operatorname{rank} \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ form a compact Grassmann manifold $\mathcal{M}$ containing the set $\mathcal{H}(\mathfrak{g})$ of Cartan subalgebras. Much more generally, the set of all closed subgroups of a locally compact group G is a compact Hausdorff space $\Sigma(G)$ with respect to a suitably chosen topology about which one may find relevant information in Chapter IX of [40]. For a connected Lie group G, the set $\mathcal{H}(G)$ of all Cartan subgroups of G is a subspace of the space $\Sigma(G)$. HOFMANN proposed the following definition in [31, 32].

Definition 2.3. (i) A member of $\overline{\mathcal{H}(\mathfrak{g})} \subseteq \mathcal{M} \subseteq \Sigma(\mathfrak{g})$ is called *near-Cartan subalgebra.*

(ii) A member of $\overline{\mathcal{H}(G)} \subseteq \Sigma(G)$ is called *near-Cartan subgroup.* ∎

Every near-Cartan subalgebra of $\mathfrak{g}$ is nilpotent and has $\operatorname{rank} \mathfrak{g}$ as dimension, and $\mathfrak{g} = \bigcup \overline{\mathcal{H}(\mathfrak{g})}$. Every near-Cartan subgroup has a near-Cartan subalgebra as Lie algebra and $G = \bigcup \overline{\mathcal{H}(G)}$. An algebraic version of the concept of a near-Cartan subalgebra is discussed in [4]; the relation between the two concepts is not entirely clear yet but is under investigation [24].

∘ *Borel subalgebras and Borel subgroups.* (Borel subgroups occur below in Theorem 4.5 on the characterisation of connected reductive complex linear groups which are exponential.) A *Borel subalgebra of a complex Lie algebra* is a maximal solvable subalgebra. A *Borel subalgebra* $\mathfrak{b}$ *of a real Lie algebra* $\mathfrak{g}$ is a subalgebra whose complexification $\mathfrak{b}_{\mathbb{C}}$ in the complexification $\mathfrak{g}_{\mathbb{C}}$ is a Borel subalgebra. Every Borel subalgebra is a maximal solvable subalgebra. But a maximal abelian subalgebra $\mathfrak{t}$ in a compact Lie algebra $\mathfrak{g}$ is a maximal solvable subalgebra, but is not a Borel subalgebra; hence $\mathfrak{g}$ has no Borel subalgebras at all. Since the sum of a solvable subalgebra and a solvable ideal is a solvable subalgebra, any maximal solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ contains the radical $\mathfrak{r}$, and $\mathfrak{b}/\mathfrak{r}$ is a maximal solvable subalgebra of the semisimple algebra $\mathfrak{g}/\mathfrak{b}$. Thus the maximal solvable subalgebras of $\mathfrak{g}$ are the full inverse images of the maximal solvable subalgebras of the semisimple $\mathfrak{g}/\mathfrak{r}$. The maximal solvable subalgebras of the real forms of complex simple Lie algebras were classified by MATSUMOTO [53]. Every nilpotent subalgebra, hence every near-Cartan subalgebra is contained in some maximal solvable subalgebra. Since the closure of a solvable subgroup in a topological group is solvable, for each maximal solvable subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ the analytic subgroup $\langle \exp \mathfrak{b} \rangle$ is closed and is a maximal solvable connected subgroup. Conversely, if B is a maximal *connected* solvable Lie subgroup of G, then B is closed and $\mathfrak{L}(B)$ is a maximal solvable subalgebra. The concept of a Borel subgroup, common as it is in the area of algebraic groups is not commonly defined for real Lie groups. The following definition therefore is tentative. A subgroup B of a real Lie group G is called a *Borel subgroup* if it is of the form $B = \langle \exp \mathfrak{b} \rangle$ for a Borel subalgebra $\mathfrak{b}$. It is not clear at this stage of our knowledge exactly which role Borel subgroups or maximal connected solvable subgroups will eventually play in a more accomplished theory of exponential real Lie groups.

∘ *Parabolic subgroups.* (Minimal parabolic subgroups will emerge in Theorem 3.1 on the characterisation of weakly exponential Lie groups.)

If G is a semisimple real connected Lie group, consider an Iwasawa decomposition $G = KAN$ and let $M = Z(A, K)$ denote the centralizer of A in K. Then $P \stackrel{\text{def}}{=} MAN$ is called a *minimal parabolic subgroup of* G. A subgroup P of a connected Lie group G is called a *minimal parabolic subgroup of* G if it contains the radical R of G and P/R is a minimal parabolic subgroup of the semisimple group G/R.

In the context of the exponential function it is useful to record a fact on

∘ *The Center.* While the center $Z(G)$ of a connected Lie group is not connected in general, by a result of HOCHSCHILD's [28] (p. 189, Theorem 1.2), it is nevertheless always contained in an analytic abelian subgroup. Therefore it is contained in the image E_G of the exponential function. For later reference we record:

Proposition 2.4. *If G is a connected real Lie group, then $Z(G) \subseteq E_G$.* ∎

3. Weakly exponential groups

The definition of "exponential" and "weakly exponential Lie groups" adopted in Definition 0.1 was probably first proposed in [39] by HOFMANN and MUKHERJEA where a systematic study of these classes of groups was initiated. In particular these authors showed that if N is a closed subgroup of a connected Lie group G, then G is weakly exponential if and only if N and G/N are weakly exponential. Since connected solvable Lie groups are weakly exponential, the problem of determination of weakly exponential Lie groups is thus reduced to the case of semisimple Lie groups. The result due to A. BOREL, that *a connected real semisimple group G is weakly exponential if and only if all Cartan subgroups of G are connected*, is only published in that paper [39], to the best of our knowledge. The assertion remains intact for all connected reductive real Lie groups as ĐOKOVIĆ and NGUYEN noted in [17]. Indeed this is true more generally. The concept of a Cartan subgroup introduced in Section 2 will assist us in the formulation of a general characterisation of weakly exponential groups. We also need the concept of a regular element of a Lie group. Let $\mathrm{Reg}(G)$ denote the set of those elements $g \in G$ of a Lie group for which the dimension of the nilspace of $\mathrm{Ad}(g) - \mathrm{id}_{\mathfrak{g}}$ equals $\operatorname{rank} \mathfrak{g}$, the dimension of the Cartan subalgebras of $\mathfrak{g}$. This set is always open and dense in G and an element $\exp X \in E_G$ is contained in $\mathrm{Reg}(G)$ if and only if X is a regular element of $\mathfrak{g}$ and exp is regular at X (see e.g. [31], Lemma 4). A topological group G is called *spacious* [39], if there is a nonempty open subset $U \subseteq G$ such that $U^n \cap U^{n+1} = \emptyset$ for all $n \in \mathbb{N}$. These concepts enter the following Characterisation Theorem in which the conditions equivalent to (1) are taken from various sources: (2) from HOFMANN [31] (Corollary 18), (3) from NEEB [65], (4) and (5) from JAWORSKI [42] (Theorem 12), (6) from HOFMANN and MUKHERJEA [39].

Theorem 3.1. *For a connected Lie group G the following statements are equivalent:*

(1) *G is weakly exponential.*
(2) $\mathrm{Reg}(G) \subseteq E_G$.
(3) *All Cartan subgroups are connected.*
(4) *G is not spacious.*
(5) *The minimal parabolic subgroups are connected.*
(6) $G/\mathrm{Rad}(G)$ *is weakly exponential.* ∎

MITTENHUBER [55] calls a connected Lie group G *completely spacious* if it contains an open subsemigroup S satisfying $S \cap E_G = \emptyset$. An example is furnished by the simply connected covering group of $\mathrm{SL}(2,\mathbb{R})$. He shows that *a connected Lie group is completely spacious if the minimal parabolic subgroups have infinitely many components.*

An explicit identification of all weakly exponential semisimple Lie groups has been achieved recently by ĐOKOVIĆ and NGUYEN [18] and by NEEB [65]. Specifically, we have the following information. Let us say with NEEB that a Lie algebra $\mathfrak{g}$ is *weakly exponential* if there is a weakly exponential Lie group G with Lie algebra (isomorphic to) $\mathfrak{g}$ (equivalently, if the adjoint group of $\mathfrak{g}$ is weakly expoential), and that it is *completely weakly exponential* if the simply connected Lie group G with Lie algebra $\mathfrak{g}$ is weakly exponential. NEEB offers the following Catalog Theorem.

Theorem 3.2. *Among the noncompact real forms of the complex simple Lie algebras, only the ones listed below are weakly exponential; it is specified whether or not they are completely weakly exponential:*

(A I) $\mathfrak{sl}(2,\mathbb{R})$ *is not completely weakly exponential.*
(A II) $\mathfrak{su}^*(2n) \cong \mathfrak{sl}(n,\mathbb{H})$, $n \geq 2$, *is completely weakly exponential.*
(A III, IV) $\mathfrak{su}(p,q)$, $1 \leq q \leq p$, *is completely weakly exponential if and only if* $q < p$.
(B II, D II) $\mathfrak{so}(n,1)$, $n > 3$, *is completely weakly exponential.*
(C II) $\mathfrak{sp}(p,q)$, $p \geq q \geq 1$, $p \geq 2$, *is completely weakly exponential.*
(D I) $\mathfrak{so}(2p,2)$, $p \geq 3$, *is not completely weakly exponential.*
(D III) $\mathfrak{so}^*(2n)$, $n \geq 4$, *is completely weakly exponential if and only if* n *is odd.*
(E III) $\mathfrak{e}_{(6,-14)}$ *is completely weakly exponential.*
(E IV) $\mathfrak{e}_{(6,-26)}$ *is completely weakly exponential.*
(E VII) $\mathfrak{e}_{(7,-25)}$ *is not completely weakly exponential.*
(F II) $\mathfrak{f}_{(4,-20)}$ *is completely weakly exponential.* ■

This theorem does not yet answer the question which *semisimple* connected real Lie groups are weakly exponential. In [39] (2.2) it was already observed that $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{C})/\{(\mathbf{1},\mathbf{1}),(-\mathbf{1},-\mathbf{1})\}$ is weakly exponential while $\mathrm{SL}(2,\mathbb{R})$ is not. In general, let G be a connected semisimple real Lie group and $\mathfrak{g} = \mathfrak{L}(G) = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ its Lie algebra with its unique simple ideals $\mathfrak{s}_j$. Let $\widetilde{G} = \widetilde{S_1} \times \cdots \times \widetilde{S_n}$ be the simply connected covering group of G with the simply connected Lie groups $\widetilde{S_j}$ satisfying $\mathfrak{L}(\widetilde{S_j}) = \mathfrak{s}_j$. The Cartan subalgebras of $\mathfrak{g}$ are of the form $\mathfrak{h} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_n$ with Cartan subalgebras $\mathfrak{h}_j$ of $\mathfrak{s}_j$, and the Cartan subgroups of $\widetilde{G}$ are of the form

$$H^* = H_1^* \times \cdots \times H_n^*, \quad H_j^* = Z(\mathfrak{h}_j, \widetilde{S_j}), \quad j = 1, \ldots, n,$$

where, as is usual, for a subset $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ of a Lie group G we write $Z(\mathfrak{h}, G) = \{g \in G : (\forall X \in \mathfrak{h})\ \mathrm{Ad}(g)(X) = X\}$. We may identify G with a quotient $\widetilde{G}/D$, where

$$D \subseteq Z(\widetilde{G}) = Z(\widetilde{S}_1) \times \cdots \times Z(\widetilde{S}_n),$$

and accordingly we identify the Cartan subalgebras H of G with the quotients H^*/D. If G is weakly exponential, then $\mathrm{Ad}(G)$ is weakly exponential, and, accordingly, because of $\mathfrak{L}\big(\mathrm{Ad}(G)\big) \cong \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$, all $\mathfrak{s}_j$ are weakly exponential and thus have to come from the catalog of Theorem 3.2. If not all $\widetilde{S}_j$ are weakly exponential, some of the $\mathfrak{s}_j$ fail to be completely weakly exponential. ĐOKOVIĆ and NGUYEN clarify the question of weak exponentiality of G in [18]. Their main result is the following Classification Theorem in whose formulation we continue the notation and the identifications just introduced; we observe first that every semisimple real Lie algebra has maximally split Cartan subalgebras; these are unique up to conjugacy.

Theorem 3.3. *For a connected real semisimple Lie group G choose for each $j = 1, \ldots, n$ a Cartan subalgebra $\mathfrak{h}_j$ of $\mathfrak{s}_j$ to be maximally split. Then the following conditions are equivalent:*

(i) *G is weakly exponential.*

(ii) *All simple Lie algebras $\mathfrak{s}_j$ are weakly exponential and*

$$Z(\widetilde{G}) = D\cdot\big((Z(\widetilde{S_1}) \cap H_1^*) \times \cdots \times (Z(\widetilde{S_n}) \cap H_n^*)\big). \qquad \blacksquare$$

For a connected real Lie group S with a simple Lie algebra $\mathfrak{s}$, and for a maximally split Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$, the group $Z(\widetilde{S}) \cap Z(\mathfrak{h}, \widetilde{S})$ is an invariant of $\mathfrak{s}$. These invariants were classified in [18] for all weakly exponential simple Lie algebras $\mathfrak{s}$; this is relevant for Condition 3.3(ii) above. We refer to this source for further details.

4. Exponential Lie groups

Let G be a connected Lie group. Assume that $\exp_G$ is injective. Then G cannot contain a circle subgroup and thus no compact nontrivial subgroup. Furthermore, if $z \in Z(G)$, then by 2.4, there is an $X \in \mathfrak{g}$ such that $\exp X = z$. Then for any $g \in G$, $\exp X = z = gzg^{-1} = \exp \mathrm{Ad}(g)X$, whence $\mathrm{Ad}(g)X = X$ by the injectivity of $\exp$. Hence $X \in \mathfrak{z}(\mathfrak{g})$. Thus the center $Z(G)$ is connected. This applies to every analytic subgroup, in particular to a Levi factor. Every nontrivial semisimple analytic subgroup contains a three dimensional simple analytic subgroup which must be center-free by what we saw. This leaves $\mathrm{SO}(3)$ and $\mathrm{PSL}(2,\mathbb{R})$. Both of these contain a circle group, a contradiction. Therefore G must be solvable and simply connected.

Consider the following solvable real Lie algebras:

(a) $\mathfrak{a}$ has a basis of elements X, Y, Z such that $[X,Y] = Z$, $[X,Z] = -Y$, and $[Y,Z] = 0$,

(b) $\mathfrak{b}$ has a basis X, Y, Z, U such that $[X,Y] = Z$, $[X,Z] = -Y$, $[Y,Z] = U \in \mathfrak{z}(\mathfrak{b})$.

Proposition 4.1. *For a simply connected solvable Lie group G with exponential function $\exp_G\colon \mathfrak{g} \to G$ the following conditions are equivalent:*

(1) $\exp_G$ *is injective.*
(2) $\exp_G$ *is surjective.*
(3) $\exp_G$ *is bijective.*
(4) $\exp_G$ *is an analytic diffeomorphism.*
(5) $\mathfrak{g}$ *does not contain isomorphic copies of* $\mathfrak{a}$ *or* $\mathfrak{b}$.
(6) $\mathfrak{g}$ *has no quotient with a subalgebra isomorphic to* $\mathfrak{a}$.
(7) *Each root of* $\mathfrak{g}$ *is of the form* $(1+i\lambda)\cdot\omega$ *with a real number* λ *and a linear form* $\omega\colon\mathfrak{g}\to\mathbb{R}$.
(8) *For all* $X\in\mathfrak{g}$ *one has* $\operatorname{Spec}\operatorname{ad}X\cap i\cdot\mathbb{R}=\{0\}$.
(9) *For all* $g\in G$ *one has* $\operatorname{Spec}\operatorname{Ad}(g)\cap\mathbb{S}^1\subseteq\{1\}$.
(10) *If* $X, Y\in\mathfrak{g}$ *and if* $\exp_G X$ *and* $\exp_G Y$ *commute, then* $[X,Y]=0$. ∎

These and further equivalent statements can be found in papers by Dixmier [12] and Saito [73]; (see also Bourbaki [3], Chap. III, §9, Ex. 17). From Dixmier's results in [12] one can also derive that every connected solvable Lie group is weakly exponential. The equivalence of Condition (10) with the other conditions is due to Corwin and Moskowitz [9] and is much more recent. At the end of the section we shall offer some additional comments on domains of injectivity of the exponential function (see 4.7ff.).

The status of the theory of weakly exponential groups as summarized in Section 3 is much more satisfactory than that of exponential groups. We do not exactly know today in general terms how we should characterize the class of exponential groups within the class of weakly exponential ones. Nevertheless, substantial results on certain special subclasses are available. Let us begin with solvable groups for which a complete characterisation of exponentiality is available through the work of Wüstner [75, 78]. The concept of a near-Cartan subalgebra enters the following Characterisation Theorem for the exponentiality of solvable Lie groups.

Theorem 4.2. (Wüstner [75], IV.2.44) *For a connected solvable real Lie group* G *the following conditions are equivalent.*
(1) G *is exponential.*
(2) *For each Cartan subgroup* H *of* G *and each* $h\in H$ *there is an* $X\in\mathfrak{h}$, *the Lie algebra of* H, *such that* $h=\exp X$ *and* $\exp$ *is regular at* X.
(3) *For each Cartan subgroup* H *of* G *and each* $x\in G$, *the centralizer* $Z(x,H)$ *is connected.*
(4) *For each Cartan subgroup* H *of* G *and each* ad*-nilpotent* $X\in\mathfrak{g}$, *the centralizer* $Z(X,H)=\{h\in H:\operatorname{Ad}(h)(X)=X\}$ *is connected.*
(5) *For each* ad*-nilpotent* $X\in\mathfrak{g}$, *the centralizer* $Z(X,G)=\{g\in G:\operatorname{Ad}(g)(X)=X\}$ *is exponential.*
(6) *For each* $x\in G$ *there is an* $X\in\mathfrak{g}$ *such that* $x=\exp X$ *and* $\exp$ *is regular at* X.
(7) *The near-Cartan subgroups of* G *are connected.* ∎

Proposition 4.1 says (among other things) that, under the hypothesis of simple connectivity of G, condition 4.1(10) is sufficient for G to be exponential. In Theorem 4.2 it is shown without the hypothesis of simple connectivity of

G that condition 4.2(3) (or 4.2(4)) is sufficient for G to be exponential. But Condition 4.1(10) is quickly seen to imply conditions 4.2(3) and 4.2(4). In this sense some implications of Wüstner's Theorem 4.2 are stronger than similar implications of 4.1.

Solvable connected Lie groups occupy one end of the spectrum of general real connected Lie groups, the other one being settled by semisimple connected Lie groups. A full theory is not yet available for these, but a good deal of information on special classes is known.

In [75], WÜSTNER characterizes exponentiality in the class of splittable linear complex Lie groups (which includes the class of affine complex algebraic groups). Recall that a subgroup G of $\mathrm{GL}(V)$ for a vector space over a field is called *splittable*, if every element $g \in G$ has a unique multiplicative Jordan decomposition $g = g_s g_u = g_u g_s$ with a semisimple element $g_s \in G$ and a unipotent element $g_u \in G$.

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Then the diagonal 2×2 matrix group over $\mathbb{K}$ with diagonal entries e^t, $e^{\pi t}$, $t \in \mathbb{K}$ is splittable (over $\mathbb{K}$) but is not an algebraic subgroup of $\mathrm{GL}(2,\mathbb{K})$. WÜSTNER proves the following Characterisation-Classification Theorem.

Theorem 4.3. *Let G be a connected complex Lie group and assume that G is a splittable subgroup of $\mathrm{GL}(V)$ for some complex vector space V. Then the following conditions are equivalent:*

(1) *G is exponential.*

(2) *The centralizer $Z(X,G) = \{g \in G : \mathrm{Ad}(g)(X) = X\}$ of each nilpotent $X \in \mathfrak{g}$ is connected.*

If, in addition, G is semisimple, then these conditions are equivalent to

(3) *$G \cong \prod_{j=1}^{N} \mathrm{PSL}(n_j,\mathbb{C})$ for a suitable N-tuple $(n_1,\ldots,n_N)$ of natural numbers $n_j \geq 2$.* ∎

Proof of the last assertion. (3)⇒(1): This follows from Theorem 1.5. (1)⇒(3): Let $x \in Z(G)$ be arbitrary and let $u \in G$ be a regular unipotent element. Then $Z(u,G)^\circ$ is a unipotent group. For $x \stackrel{\mathrm{def}}{=} zu$ we have $x_s = z$ and $x_u = u$. Since G is exponential, Theorem 1.3 implies that $z = x_s \in Z(u,G)^\circ$. Thus z is both semisimple and unipotent, and so $z = 1$. ∎

Note that Theorems 1.4 and 1.5 imply immediately that an exponential complex semisimple Lie group is an almost direct product of groups of type A. Condition (3) above is more specific. The equivalence of (1) and (2) was proved in the case of connected affine complex algebraic groups in [15], Theorem 3.2. Since connected complex Lie groups are always weakly exponential, semisimple complex Lie groups which are not isomorphic to those in 4.3(3) provide simple examples of groups which are weakly exponential but not exponential. Other examples will be given in 5.3 and 5.4; connected solvable Lie groups are always weakly exponential and we know from 4.2 which among them are exponential; each connected solvable Lie group which fails to satisfy the conditions of 4.2 provides an example as well.

In the case of real splittable groups, WÜSTNER shows the following Characterisation Theorem [79].

Theorem 4.4. *Let $G \subseteq \mathrm{GL}(V)$ be a splittable connected Lie group. Then the following statements are equivalent:*

(1) *G is exponential.*

(2) *For each $X \in \mathfrak{g}$ which is nilpotent on V and each $g_s \in Z(X,G)$ which is semisimple on V there is an element $X_s \in \mathfrak{z}(X,\mathfrak{g})$ which is semisimple on V with $g_s = \exp X_s$.*

(3) *For each element $X \in \mathfrak{g}$ which is nilpotent on $\mathfrak{gl}(V)$, the centralizer $Z(X,G)$ is weakly exponential.* ∎

The equivalence of (1) and (3) remains intact for any covering group of G; condition (2) need be appropriately modified (see [79], 4.1). Since every semisimple Lie group has a splittable Lie algebra and therefore the associated adjoint group is splittable, the equivalence of conditions (1) and (3) applies, in particular, to all semisimple Lie groups.

In [17], Theorem 2.2, ĐOKOVIĆ and NGUYEN show the following result

Theorem 4.5. *Let G be the identity component, with respect to the Euclidean topology, of a real algebraic matrix group. Then the following conditions are equivalent:*

(1) *G is exponential.*

(2) *For every unipotent element $u \in G$ the centralizer $Z(u,G)$ is weakly exponential.* ∎

MOSKOWITZ proves in [60] the following Characterisation Theorem:

Theorem 4.6. *For a connected reductive complex linear algebraic group G the following conditions are equivalent*

(1) *G is exponential.*

(2) *Every Borel subgroup is exponential.*

Proof. It is instructive to have a quick independent proof. (2)⇒(1): Let B be a Borel subgroup of G. Then $G = \bigcup_{g\in G} gBg^{-1}$. This implies the asserted implication. (1)⇒(2): Let Z denote the center of G. Then G/Z° is semisimple and is exponential by (1). Theorem 4.3 then gives the structure of G/Z° as in 4.3(3). The Borel subgroup of $\mathrm{GL}(n,\mathbb{C})$ is exponential (see Theorem 2.1 [11]). Then the Borel subgroups of $\mathrm{PSL}(n,\mathbb{C})$ as well as those of G/Z° are exponential. Let B be a Borel subgroup of G and let $x \in B$. Then there is an $X \in \mathfrak{L}(B)$ and a $z \in Z^\circ$ such that $x = (\exp_G X)z$. Write $z = \exp_G Y$ with a Y in the Lie algebra $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{b}$ of Z. Hence $x = \exp_G(X+Y)$ and $X+Y \in \mathfrak{b}$. ∎

The exponentiality of solvable Lie groups, however, is accessible through WÜSTNER's Theorem 4.2. MOSKOWITZ shows in [60], Theorem 11, that in any exponential complex linear algebraic group which is either reductive or has a semisimple Levi factor, the center is connected. He conjectures that this is true in general. The remark on p. 28 saying that *a connected noncompact simple Lie group which is the group of real points of a complex algebraic group cannot be exponential if it contains -1 in its center* is incorrect. The group $\mathrm{SL}(n,\mathbb{H})$ is a counterexample.

In [17], ĐOKOVIĆ and NGUYEN recently addressed the special case of the class Σ of real Lie groups G that occur as identity component of the group of real points of an almost simple algebraic group defined over $\mathbb{R}$ and classified all noncompact exponential members of Σ as in the following Catalog Theorem.

Theorem 4.7. *The non-compact exponential groups in Σ are the groups listed below and their quotients by finite central subgroups:*

1. $\mathrm{SL}(n,\mathbb{H})$, $n \geq 2$,
2. $\mathrm{PSU}(p,p) = \mathrm{SU}(p,p)/Z_{2p}$, $p \geq 1$,
3. $\mathrm{SU}(p,q)/Z_d$, *where* $p > q \geq 1$, d *is an odd divisor of* $n = p+q$, *and every odd prime dividing* d, *respectively,* n/d *is* $\leq n/(p-q)$, *respectively,* $> n/(p-q)$,
4. $\mathrm{Spin}(2n,1)$, $n \geq 2$,
5. $\mathrm{Sp}(p,q)$, $p \geq q \geq 1$,
6. $\mathrm{Spin}(2n-1,1)$, $n \geq 3$,
7. $\mathrm{PSO}(2n-2,2)^0$, n *odd* ≥ 3,
8. $\mathrm{Spin}(2n-2,2)/\langle z\rangle$, n *even* ≥ 4,
9. $\mathrm{Spin}^*(2n)$, n *odd* ≥ 3,
10. $\mathrm{SO}^*(2n)$, n *even* ≥ 4,
11. $\mathrm{Spin}^*(2n)/\langle z'\rangle$, n *even* ≥ 4,
12. G^* *of type* E IV. ■

In 11.) above, z' is one of the two central involutions that are mapped to -1 by the double covering map $\mathrm{Spin}^*(2n) \to \mathrm{SO}^*(2n)$. As n is even, there is no automorphism of $\mathrm{Spin}^*(2n)$ interchanging these two involutions. For the precise definition of z' see [16].

In 12.), G^* denotes the group or real points of the simply connected complex Lie group E_6 with real structure of type E IV.

Information on the injectivity of the exponential function was reported in Proposition 4.1. Additional information was provided by LAZARD and TITS [51] who determined for the exponential function $\exp\colon \mathfrak{g} \to G$ of a real Lie group a maximal open domain in $\mathfrak{g}$ on which exp is injective regardless of the structure of $\mathfrak{g}$ or G. Specifically, if a norm $\|\cdot\|$ is selected on $\mathfrak{g}$ such that $\|[x,y]\| \leq \|x\|\cdot\|y\|$, and if we consider the open balls $B_r = \{x \in G : \|x\| < r\}$, then $\exp|B_\pi$ is injective regardless of the structure of $\mathfrak{g}$ and G (and if G is simply connected, then $\exp|B_{2\pi}$ is injective). A similar but more elucidating approach was pointed out by HOFMANN through the following definition [30]:

Definition 4.8. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Define the function $\sigma\colon \mathfrak{g} \to \mathbb{R}^+$ by

$$\sigma(x) = \max\{\mathrm{Im}(\lambda) : \lambda \in \mathrm{Spec}(\mathrm{ad}\, x)\}. \quad ■$$

If x is a *compact element*, i.e., if $\mathrm{Spec}(x) \in i\cdot\mathbb{R}$, then $\sigma(x)$ is the spectral radius of $\mathrm{ad}\, x$. It is easy to see that (i) $\sigma(x) = 0$ iff $\mathrm{Spec}(\mathrm{ad}\, x)$ is real, (ii) $\sigma(r\cdot x) = |r|\sigma(x)$, (iii) $\sigma\big(\alpha(x)\big) = \sigma(x)$ for all automorphisms α of $\mathfrak{g}$.

Furthermore, (iv) if x and y are contained in some solvable subalgebra, then $\sigma(x+y) \leq \sigma(x)+\sigma(y)$, and (v) $\sigma(x+y)=\sigma(x)$ for all $x \in \mathfrak{g}$ and all y in the nilradical of $\mathfrak{g}$ [30]. If $\operatorname{sing}\exp$ denotes the set of all points $x \in \mathfrak{g}$ at which exp fails to be regular, then $\inf \sigma(\operatorname{sing}\exp) \geq 2\pi$. The following proposition is from [30]:

Proposition 4.9. *Let* $\exp\colon \mathfrak{g} \to G$ *be the exponential function of a real Lie group. Set* $U = \{x \in \mathfrak{g} : \sigma(x) < \pi\}$ *and* $V = \exp U \subseteq G$. *Then the following conclusions hold:*

(i) *U is an open neighborhood of 0 at all of whose elements* exp *is regular.*
(ii) $[-1,1]\cdot U = U$, *i.e., U is star shaped and symmetric.*
(iii) *U is invariant under all automorphisms of $\mathfrak{g}$ and V is invariant under all automorphism of G and thus is stable under conjugation.*
(iv) *U contains all $x \in \mathfrak{g}$ such that* $\operatorname{Spec}(\operatorname{ad} x)$ *is real.*
(v) *If $\mathfrak{n}$ denotes the nilradical of $\mathfrak{g}$, then $U + \mathfrak{n} = U$. In particular, U contains the nilradical of $\mathfrak{g}$ and V contains the nilradical of G.*
(vi) $\exp|U\colon U \to V$ *is a diffeomorphism onto a symmetric neighborhood of the identity of G.*
(vii) *If G is simply connected, and if the elements $x, y \in \mathfrak{g}$ satisfy* $\exp x = \exp y$ *and $\sigma(x)+\sigma(y) < 2\pi$, then $x = y$.* ∎

5. Special aspects and conjectures on exponentiality in the general case

As soon as we leave the domain of special classes of Lie groups the information we have on the surjectivity of the exponential function is not coherent yet, and the literature is occasionally a bit shaky. Various authors have observed, notably in the context of semisimple connected Lie groups, that the center plays a role in characterizing exponentiality, but this aspect appears to be less clear cut than was thought in the beginning. For solvable groups, the connectivity of the center is a necessary condition for exponentiality. In [57] MOSKOWITZ clarifies various statements made in [58] regarding centerless groups and proves that *the centerless connected simple Lie groups* $\mathrm{SO}^\circ(n,1)$ *are exponential.* This is a special case of NISHIKAWA's Theorem 1.6 above, and it also follows from the Catalog Theorem 4.6 of ĐOKOVIĆ and NGUYEN in whose list the simply connected covering groups of these occur. MOSKOWITZ also settles the issue of exponentiality of all simple noncompact real rank one groups, all of which are exponential safe the exceptional specimen. That the latter is not exponential emerged from the work of ĐOKOVIĆ and THANG in [17]. MOSKOWITZ [57] uses a geometric argument, taking into account the fact that a centerless rank one simple real Lie group is the connected component of the isometry group of an irreducible rank one symmetric space and studying this action. The articles [57] and [58] should only be consulted jointly.

Information on mixed groups is sparse. But MOSKOWITZ has elucidated this aspect of exponentiality with the methods of [57, 58] on "generalized motion groups," which exemplify some of the technical complications which one encounters in the mixed case, i.e., the case of connected real Lie groups which are neither solvable nor semisimple—even under otherwise special conditions. So we consider, by way of example, a connected real Lie group G whose radical R is a vector group and whose semisimple factor group G/R is compact. By a theorem of IWASAWA [41], every such Lie group G is isomorphic to $V \rtimes C$ where V is n-dimensional real vector group, C a compact connected Lie group, and where the multiplication in G is defined in terms of a representation $\rho: C \to \mathrm{GL}(V)$ by $(v,g)(w,h) = (v+\rho(g)(w), gh)$. Moreover, every compact subgroup is contained in a conjugate of $\{0\} \times C$ under an element of $V \times \{1\}$ (cf. [2], chap. VII, §3, n 2, Proposition 3). For each element $v \in V$ we define the isotropy group of C at v to be $C_v = \{c \in C : \rho(c)(v) = v\} = \{c \in C : v \in \ker(\rho(c) - \mathrm{id}_V)\}$. Note $V \times C_v = Z\big((v,1), V \rtimes C)\big)$.

Proposition 5.1. *Let V be an n-dimensional real vector group and $\rho: C \to \mathrm{GL}(V)$ a representation of a compact connected Lie group C on V. Set $G \stackrel{\mathrm{def}}{=} V \rtimes C$ with multiplication $(v,g)(w,h) = (v+\rho(g)(w), gh)$. For $v \in V$ set $C_v = \{g \in C : \rho(g)(v) = v\}$. Then the following conditions are equivalent.*

(i) *G is exponential.*

(ii) *C_v is connected for all $v \in V$.*

Proof. Every compact Lie group has a faithful real linear representation $\pi: C \to \mathrm{GL}(W)$ and thus is a real algebraic matrix group (see [70], p. 133, Theorem 5, p. 247, Theorem 12). The group G has a real algebraic matrix representation $\alpha: G \to \mathrm{GL}(W \oplus V \oplus \mathbb{R})$ given, in an obvious notation, by

$$\alpha(v,g) = \begin{pmatrix} \pi(g) & 0 & 0 \\ 0 & \rho(g) & v \\ 0 & 0 & 1 \end{pmatrix}.$$

The unipotent elements of G are exactly those of $V \times \{1\}$; the centralizer $Z\big((v,1), G\big)$ is $V \rtimes C_v$. Every element $g \in G$ has a unique multiplicative Jordan decomposition $g = g_s g_u$ with g_s semisimple, commuting with g_u unipotent.

(i)⇒(ii) Let $v \in V$; we must show that C_v is connected. Let $c \in C_v$. Then $(v,c)_s = (0,c)$ and $(v,c)_u = (v,1)$. By (i) there is a one-parameter subgroup $t \mapsto \big(\phi(t), f(t)\big)$ of G such that $\phi(1) = v$ and $f(1) = c$. Since $\big(\phi(t), f(t)\big)$ commutes with (v,c) and $(v,1) = (v,c)_u$, it follows that $\big(\phi(t), f(t)\big)$ and $(v,1)$ commute, i.e., $f(t) \in C_v$. As $f(0) = 1$ and $f(t) = c$, the group C_v is connected.

(ii)⇒(i) Let $g = (v,c) \in G$. We want to show that $g \in E_G$. Let $g = g_s g_u$ be the Jordan decomposition. Then g_s is contained in some compact subgroup, and by the conjugacy part of IWASAWA's Theorem, g_s has a conjugate in $\{0\} \times C$. Since E_G is invariant under conjugation, we may assume that $g_s = (0,c)$. Now $g_u = (v,1)$ with $c \in C_v$. By (ii), the compact Lie group C_v is connected, hence exponential. Thus $g = (v,c) \in \mathbb{R}{\cdot}v \times C_v$, and this direct product is exponential, whence $g \in E_G$. The proposition is proved. ∎

We thank E. B. VINBERG for having directed us towards this direct proof. Considering the conjugacy part of IWASAWA's Theorem, the fact that the connected compact Lie group C is exponential, and the fact that E_G is invariant under conjugation, we observe that $G \setminus E_G$ are exactly those elements g of $G \setminus V \times \{1\}$ which fail to be contained in a compact subgroup, i.e., for which $\langle g \rangle \cong \mathbb{Z}$ (by A. WEIL's Lemma saying that a cyclic subgroup of a locally compact group is either relatively compact or is isomorphic to $\mathbb{Z}$ algebraically and topologically).

Proposition 5.1 is also a consequence of Theorem 4.5. Indeed, if $V \rtimes C_v$ is weakly exponential, then $V \times C_v$ is connected and thus C_v is connected. Conversely, if C_v is connected, since $V \times \{1\}$ is contained in the radical of $V \rtimes C_v$, Condition (6) of Theorem 3.1 is satisfied and thus $V \rtimes C_v$ is weakly exponential. Then the assertion follows from Theorem 4.5.

Proposition 5.1 shows, in particular, that the euclidean motion group $G = \mathbb{R}^n \rtimes \mathrm{SO}(n)$, $n \geq 2$, is exponential. Indeed, all unit vectors in the euclidean space $\mathbb{R}^n$ are conjugate under a rotation to $v = (0, \ldots, 0, 1)$; for this v we have $\big(\mathrm{SO}(n)\big)_v \cong \mathrm{SO}(n-1)$ whence 5.1(ii) is satisfied. This is due to MOSKOWITZ [58]. All proper covering groups of G fail to satisfy 5.1(ii). The group

$$G \overset{\text{def}}{=} \left\{ \begin{pmatrix} e^{2it} & 0 & u \\ 0 & e^{it} & v \\ 0 & 0 & 1 \end{pmatrix}, \quad t \in \mathbb{R},\, u, v \in \mathbb{C} \right\}$$

satisfies the general hypotheses of Proposition 5.1 with a faithful ρ, but G does not satisfy 5.1(ii) and thus is not exponential. In connection with 5.2 and 5.3 below, the content of Proposition 5.1 and its ramifications illustrate quite well some of the intricacies of the subject in circumstances which, on the surface, appear to be very simple. The article [58] contains results on semidirect products $V \rtimes C$, where V is a 2-step nilpotent group and C is from a special class of compact Lie groups; in [62] the MOSKOWITZ and WÜSTNER present results which are more special than Proposition 5.1 insofar as C is assumed to be a torus, but which are more general in as much as V is replaced by any of a class of connected solvable Lie groups including nilpotent and simply connected solvable exponential Lie groups. Because of 5.2 and 5.3 below this is not a generalisation of 5.1; however these results remain a useful tool for yielding potential sufficient conditions for the exponentiality of connected real Lie groups containing such subgroups; an example of such a situation follows in Proposition 5.2 below in the form of the implication (ii)$\Rightarrow$(iii).

Proposition 5.2. *Assume the general hypotheses of* Proposition 5.1. *Consider the following three conditions:*

(i) *If T is a maximal torus of C then the subgroup $V \rtimes T$ is exponential.*

(ii) *For each maximal torus T of C and each $v \in V$ the group $T_v = C_v \cap T$ is connected.*

(iii) *G is exponential.*

Then (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii)$\not\Rightarrow$(i).

Proof. (i)$\Leftrightarrow$ (ii): This is a special case of Proposition 5.1.

(ii)$\Rightarrow$(iii) Let $\mathcal{T}$ denote the set of maximal tori of C. In view of $C = \bigcup \mathcal{T}$ we note $C_v = \bigcup\{T \cap C_v : T \in \mathcal{T}\} = \bigcup\{T_v : T \in \mathcal{T}\}$. By (iii) all

T_v are connected. Hence C_v is connected then Theorem 5.2 shows that G is exponential.

(iii) $\not\Rightarrow$ (ii) This requires an example which we discuss below in 5.3. ■

According to Propsition 5.2, the maximal tori of G cannot tell the full story.

Both of the examples which follow belong to the context of the series of Theorem 4.7.1, and both of them are quite instructive. Let $\mathbb{H}$ denote the division ring of quaternions with identity $\mathbf{1}$. Let $\mathbb{R}$, respectively $\mathbb{C}$ denote the subfields of real, respectively, complex numbers and let $\mathbb{S}^3$ denote the group of unit quaternions which is isomorphic to $\mathrm{SU}(2)$. Set $\mathbb{S}^0 = \mathbb{S}^3 \cap \mathbb{R}$ and $\mathbb{S}^1 = \mathbb{S}^3 \cap \mathbb{C}$. We set $\mathfrak{s}^3 = \mathbb{R}{\cdot}i + \mathbb{R}{\cdot}j + \mathbb{R}{\cdot}k$. The multiplicative group $\mathbb{H}^\times = (\mathbb{H} \setminus \{0\}, \cdot)$ is a Lie group with Lie algebra $(\mathbb{H}, +, [\cdot,\cdot])$ and the standard exponential function $\exp\colon \mathbb{H} \to \mathbb{H}^\times$, $\exp x = \mathbf{1} + x + \frac{1}{2!}{\cdot}x^2 + \cdots$; then $\mathfrak{s}^3 = \mathfrak{L}(\mathbb{S}^3)$. For $h \in \mathbb{H}$ let $Z(h,\mathbb{H}) = \{x \in \mathbb{H} : xh = hx\}$ denote the centralizer of $h \in \mathbb{H}$. If $h \notin \mathbb{R}$ there is a unique maximal commutative subfield $\mathbb{C}_h$ containing h, namely, $\mathbb{C}_h = \mathbb{R}{\cdot}\mathbf{1} + \mathbb{R}{\cdot}h \cong \mathbb{C}$. Then

$$Z(h,\mathbb{H}) = \begin{cases} \mathbb{H} & \text{if } h \in \mathbb{R}, \\ \mathbb{C}_h \cong \mathbb{C} & \text{if } h \notin \mathbb{R}. \end{cases}$$

For different $\mathbb{C}_h$ and $\mathbb{C}_{h'}$ we have $\mathbb{C}_h \cap \mathbb{C}_{h'} = \mathbb{R}$.

Example 5.3. We define

$$G = \left\{ \begin{pmatrix} u & h \\ 0 & u \end{pmatrix} \in \mathrm{GL}(2,\mathbb{H}) : h \in \mathbb{H},\ u \in \mathbb{S}^3 \right\},$$
$$\mathfrak{g} = \left\{ \begin{pmatrix} s & h \\ 0 & s \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{H}) : h \in \mathbb{H},\ s \in \mathfrak{s}^3 \right\}.$$

Let C consist of all diagonal matrices $u{\cdot}\mathbf{1}_2$, $\mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Clearly $C \cong \mathbb{S}^3$. Let N denote the normal subgroup of unitriangular matrices. Then N is a vector group isomorphic to $\mathbb{R}^4$. A quick calculation shows that for $h \in \mathbb{H}$ and $v = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ we have

$$C_v = \{u{\cdot}\mathbf{1}_2 : u \in \mathbb{S}^3 \text{ and } hu = uh\} \cong Z(h,\mathbb{H}) \cap \mathbb{S}^3 = \begin{cases} \mathbb{S}^3 & \text{if } h \in \mathbb{R}, \\ \mathbb{C}_h \cap \mathbb{S}^3 \cong \mathbb{S}^1 & \text{if } h \notin \mathbb{R}. \end{cases}$$

This group is connected. Hence G is exponential by Proposition 5.1 and Condition 5.2(iii) holds. A maximal torus of C is given by $T = \{e^{ti}{\cdot}\mathbf{1}_2 : t \in \mathbb{R}\}$. Then

$$C_v \cap T \cong Z(h,\mathbb{H}) \cap \mathbb{S}^1 = \begin{cases} \mathbb{S}^1 & \text{if } h \in \mathbb{C}, \\ \mathbb{S}^0 & \text{if } h \notin \mathbb{C}. \end{cases}$$

Thus some isotropy groups of T acting on the vector group N are disconnected. Hence the group NT is a maximal connected solvable subgroup of the exponential group G and NT is not exponential since condition 5.1(ii) is not satisfied.

The element $X \stackrel{\text{def}}{=} \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}$ is ad-nilpotent. We compute its centralizer $Z(X,G) = \{g \in G : \mathrm{Ad}(g)X = X\}$ in G as

$$\left\{ \begin{pmatrix} u & x \\ 0 & u \end{pmatrix} : u \in Z(h,\mathbb{H}) \cap \mathbb{S}^3 \text{ and } x \in \mathbb{H} \right\} \cong \mathbb{H} \rtimes \begin{cases} \mathbb{S}^3 & \text{if} \quad h \in \mathbb{R}, \\ \mathbb{C}_h \cap \mathbb{S}^3 \cong \mathbb{S}^1 & \text{if} \quad h \notin \mathbb{R} \end{cases} .$$

If $h \notin \mathbb{C}$ then $Z(h,\mathbb{H}) \cap \mathbb{S}^3$ is a maximal torus of $\mathbb{S}^3$ and thus is conjugate to T, and then $Z(X,G)$ is conjugate to NT; therefore $Z(X,G)$ is *not exponential.* Hence *there are* ad*-nilpotent elements* $X \in \mathfrak{g}$ *such that* $Z(X,G)$ *is not exponential. On the other hand,* $Z(X,G)$, *being connected and solvable, is weakly exponential.*

Since G is exponential, the Cartan subgroups of G are connected by 3.1 and thus each of them is of the form $\exp \mathfrak{h}$ for a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. One Cartan subalgebra $\mathfrak{h}$ and the Cartan subgroup it generates are given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} ti & z \\ 0 & ti \end{pmatrix} : t \in \mathbb{R},\, z \in \mathbb{C} \right\} \cong \mathbb{R}^3,$$

$$H = \left\{ \begin{pmatrix} e^{ti} & z \\ 0 & e^{ti} \end{pmatrix} : t \in \mathbb{R},\, z \in \mathbb{C} \right\} = (H \cap N)T \cong \mathbb{R}^2 \rtimes \mathbb{S}^1.$$

If $h \notin \mathbb{C}$, then

$$Z(X,H) = (H \cap N)(C_v \cap T) \cong \mathbb{R}^2 \times \mathbb{S}^0.$$

Hence *there exists a Cartan subgroup* H *and an* ad*-nilpotent* $X \in \mathfrak{g}$ *such that* $Z(X,H)$ *is disconnected.* ∎

This example belongs to the circle of ideas of the following

Example 5.4. Let $G = \mathrm{SL}(n,\mathbb{H})$, $n \geq 2$. This group is a real form of the complex algebraic group $\mathrm{SL}(2n,\mathbb{C})$. The group G consists of all $n \times n$ quaternionic matrices with Dieudonné determinant 1. By Theorem 4.6.1 all groups G are exponential.

All Cartan subgroups of G are conjugate (see e.g. [16] or [53]). One of them consists of all complex diagonal matrices whose determinant has absolute value 1.

All minimal parabolic subgroups of G are conjugate. The group P of upper triangular matrices in G is one of them. Let L be the reductive subgroup consisting of all diagonal matrices in G and N the subgroup of all upper unitriangular matrices. Then $P = NL$ is a Levi decomposition and N is the unipotent radical of P. Thus P is an example of a "mixed" group.

All maximal connected solvable subgroups of G are conjugate (see [53]). One of them is the subgroup S of all upper triangular matrices with complex diagonal entries. Then S is also a maximal connected solvable subgroup of P. The center of S has order 2. Hence S is not exponential as a consequence of Theorem 4.2. ∎

We are unable to decide whether or not P is exponential. This illustrates the deficiencies of our knowlege of necessary conditions for the exponentiality of mixed groups.

The preceding results lead to speculations on potential Characterisation Theorems for exponential Lie groups. We formulate some of them in the form of problems. The idea which is suggested by the classification Theorem 3.1 for weakly exponential Lie groups, is to point out classes of subgroups (such as Cartan subgroups or minimal parabolic subgroups) which might serve as test for exponentiality. This is the reason why, apart from preparing for Theorem 3.1, we dwelled at some length on such classes in Section 2. Every element of a connnected real Lie group is contained in a near-Cartan subgroup [32]. Every near-Cartan subgroup is nilpotent. Hence *if all near-Cartan subgroups of a connected real Lie group are connected then it is exponential.* This leads to

Problem 5.5. Prove or disprove the following conjecture: *If a connected real Lie group G is exponential then all of its near-Cartan subgroups are connected.* ∎

WÜSTNER's Theorems 4.2, 4.4 and Theorem 4.5 suggest

Problem 5.6. Prove or disprove the following conjecture: *A connected real Lie group G is exponential if an only if for each* ad*-nilpotent element $X \in \mathfrak{g}$ the centralizer $Z(X,G)$ is weakly exponential.* ∎

Example 5.3 exhibits an exponential real Lie group G and an ad-nilpotent element such that the centralizer $Z(X,G)$ is weakly exponential but not exponential, and it has a Cartan subgroup H such that $Z(X,H) = Z(X,G) \cap H$ is disconnected.

MOSKOWITZ' Corollary 13 of [60] and JAWORSKI's work suggest

Problem 5.7. Prove or disprove the following conjecture: *A connected real Lie group G is exponential if and only if minimal parabolic subgroups of G are exponential.* ∎

Failing that one might start with the following special case.

Problem 5.7a. Prove or disprove that P in Example 5.4 is exponential. ∎

All of these conjectures are supported by the case of solvable groups through Theorem 4.2. The more speculative Conjecture in Problem 5.7 may be somewhat supported by one equivalence for weak exponentiality in Theorem 3.1.

Theorem 4.6 shows that within special classes, Borel subgroups serve as a good class of test subgroups. However, far reaching conjectures concerning maximal connected solvable subgroups are inappropriate after Examples 5.3 and 5.4 where it is shown that exponential Lie groups can contain maximal connected solvable subgroups which fail to be exponential.

Problem 5.8. Clarify the role played by maximal connected solvable subgroups in the context of the surjectivity of the exponential function. ∎

In view of our knowledge of the exponentiality of connected solvable Lie groups through Theorem 4.2 any information in the direction of 5.8 would be welcome.

6. Divisibility and the exponential function

The algebraic concept of divisibility in a group is, on Lie groups, intimately related with the exponential function.

Definition 6.1. An element g in a group G is called *divisible* if for each natural number $n \in \mathbb{N}$ there is an $x \in G$ such that $x^n = g$. A group G itself is called *divisible* if each $g \in G$ is divisible. ■

This topic has been widely discussed in group theory and in the theory of topological groups although definitive results in Lie groups are recent. Clearly the additive group $\mathbb{Q}$ of rational numbers is divisible and is indeed a prototype. In a divisible group G, for each element $g \in G$ by an elementary argument of picking *successive* roots one finds a homomorphism $f\colon \mathbb{Q} \to G$ such that $f(1) = g$, and conversely, if every element in G is so embeddable in a homomorphic image of $\mathbb{Q}$, then G is divisible. In particular, this applies to Lie groups. Thus every exponential Lie group is clearly divisible. Consider the (additively written) free abelian group F generated by the countable sequence of elements $\{e_1, e_2, \ldots\}$, and let S be the subgroup generated by the subset $\{2{\cdot}e_2 - e_1, 3{\cdot}e_3 - e_1, \ldots\}$. The quotient group $G \overset{\text{def}}{=} F/S$ is an infinite abelian group which does not contain any divisible subgroup, but it contains a nonzero element which is divisible, namely, $e_1 + S \in G = F/S$. Moreover, the factor group of G modulo its torsion subgroup is isomorphic to $\mathbb{Q}$. In $\mathrm{SL}(2, \mathbb{R})$, or alternatively, in the three dimensional solvable group

$$\left\{ \begin{pmatrix} e^{it} & z \\ 0 & e^{it} \end{pmatrix} : t \in \mathbb{R},\, z \in \mathbb{C} \right\},$$

there is a sequence of elements x_n converging to ∞ (i.e., eventually leaving every compact subset) satisfying $(x_n)^n = -\mathbf{1}$. (Pictures illustrating the geometry of one-parameter groups in the group of motions of the euclidean plane and its covering groups as well as in $\mathrm{SL}(2, \mathbb{R})$ and its universal covering group may be found in [40], Figures 2 (p. 16) and 4 (p. 18); another frequently useful picture of $\mathrm{SL}(2, \mathbb{R})$ is reproduced in [26], p. 429, Figure 17.) As a consequence, in the *divisible* group $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\mathbf{1}, -\mathbf{1}\}$ we find a sequence of elements ξ_n converging to ∞ such that $(\xi_n)^n = \mathbf{1}$. Thus one must reject the possible belief that a sequence of higher and higher roots of an element in a Lie group would have to cluster, let alone converge to the identity. However, while the example F/S above shows that divisible elements need not be contained in divisible subgroups even in the case of abelian groups, inside connected real Lie groups the situation is much better, even though there may be no obvious reason visible at the outset. Indeed McCrudden proved the following theorem.

Theorem 6.2. *In a connected real Lie group, the set of divisible elements is precisely* $E_G = \operatorname{im}\exp$. ■

Put differently: *An element in a connected real Lie group is divisible if and only if it lies on a one-parameter subgroup.* In particular, this shows that in a weakly exponential but not exponential group the closure of the set of divisible elements contains elements which fail to be divisible. As a consequence of Theorem 6.2, *a connected real Lie group is divisible if and only if it is exponential.* (This corollary permits easier proofs [35].) Moreover, McCrudden's Theorem allows us to understand the structure of homomorphic images of the group $\mathbb{Q}$ in a connected Lie group; the following proposition is taken from [40].

Proposition 6.3. *Let $f\colon \mathbb{Q} \to G$ be a group homomorphism into a connected Lie group. Then $\overline{f(\mathbb{Q})}$ is singleton, or is a torus, or is isomorphic to the direct product of a torus (possibly of dimension 0) and a line $\mathbb{R}$. In particular, $\overline{f(\mathbb{Q})}$ is connected.* ∎

Indeed every Lie group of the form $\mathbb{T}^n \times \mathbb{R}^d$, $d \in \{0,1\}$ contains a dense copy of $\mathbb{Q}$. By 6.3, any homomorphic image of $\mathbb{Q}$ in a connected Lie group is contained in E_G. Therefore, the closure of a divisible subgroup of a connected Lie group is weakly exponential and thus is, in particular, connected.

The question of passing one parameter subsemigroups through elements having arbitrary roots originates, in spirit, from probability theory where it is called the embeddability problem in the following sense: Probability measures on a locally compact group G form a semigroup under convolution, whose identity is the point mass at the identity. A probability measure μ on G is called (infinitely) divisible if for each $n \in \mathbb{N}$ there is a probability measure ν such that $\underbrace{\nu * \cdots * \nu}_{n\text{ times}} = \mu$. It is called *embeddable* if there is a (weakly) continuous one parameter semigroup $t \mapsto \mu_t$ of probability measures with $\mu_0 = \delta_{\mathbf{1}}$, the point mass in $\mathbf{1}$, and $\mu_1 = \mu$. Any Gauss measure on $\mathbb{R}$ is embeddable. Embeddability implies divisibility trivially; the investigation of the converse implication has vexed probabilists for a long time. It was shown by DANI and MCCRUDDEN that *on every connected Lie group which has a finite dimensional continuous linear representation with a discrete kernel a divisible probability measure is embeddable* [10]. The embedding problem motivated MCCRUDDEN's Theorem 6.2 in the first place; for more references on the widely and throughly studied embeddability problem in probability theory see e.g. [10].

7. The exponential function of subsemigroups of Lie groups

The theory of Lie semigroups, sometimes also called "Geometric Semigroup Theory," has emerged in close connection with such fields of mathematics as geometric control theory, holomorphic representation theory, causality and chronogeometry on pseudo-Riemannian manifolds (see e.g., [26, 27, 34, 36, 38, 64]). To every closed subsemigroup S in a Lie group G there is attached a tangent object $W = \mathfrak{L}(S)$ in the Lie algebra $\mathfrak{g}$ of G, defined by

$$\mathfrak{L}(S) = \{X \in \mathfrak{g} \mid \exp \mathbb{R}^+ \cdot X \subseteq S\}.$$

Definition 7.1. A subsemigroup S of a Lie group G is called *exponential*, respectively, *weakly exponential* if $S = \exp \mathfrak{L}(S)$, respectively, $S = \overline{\exp \mathfrak{L}(S)}$. ∎

This definition is a straightforward extension of Definition 0.1. In order to properly distinguish subsemigroups from subgroups we shall say that a subsemigroup is *reduced* if S does not contain a closed normal subgroup N (which would allow us to pass to $S/N \subseteq G/N$) and if S algebraically generates G. (Why this is not a restriction of generality is explained in [40].) In a striking contrast with the group situation in which many questions remain open, HOFMANN and RUPPERT completely classified reduced weakly exponential semigroups and showed that these are exponential. (Not true for weakly exponential subsemigroups which are not reduced!) In order to discuss the essential aspects of this result we need some background concepts.

The set $\mathfrak{L}(S)$ is a so-called Lie wedge. Indeed, a *wedge* W, i.e., an additively and topologically closed convex subset of $\mathfrak{g}$, is called a *Lie wedge* if

(A) $(\forall X \in W \cap -W) \quad e^{\operatorname{ad} X} W \subseteq W$.

If S is a closed *subgroup* then $\mathfrak{L}(S)$ is exactly the Lie algebra of S. We say that a subsemigroup S of a Lie group G is a *Lie semigroup* if it is closed and $S = \overline{\langle \exp \mathfrak{L}(S) \rangle}$, i.e., the subsemigroup $\langle \exp \mathfrak{L}(S) \rangle$, which is algebraically generated by the exponential image of $\mathfrak{L}(S)$, is dense in S. It is an immediate consequence of the definition that every Lie semigroup is connected. A subgroup S of G is a Lie semigroup if and only if it is a closed connected Lie subgroup, or, equivalently, a closed connected subgroup of G. A wedge W in a Lie algebra $\mathfrak{g}$ is called a *Lie semialgebra* if it satisfies the following condition.

(B) There is an open convex neighborhood B of 0 in $\mathfrak{g}$ on which the Campbell-Hausdorff-Dynkin multiplication $*$ is defined and satisfies $(W \cap B) * (W \cap B) \subseteq W$.

Each Lie semialgebra is a Lie wedge. A wedge W in a Lie algebra $\mathfrak{g}$ is called *invariant*, if

(C) $(\forall X \in \mathfrak{g}) \quad e^{\operatorname{ad} X} W \subseteq W$.

All invariant wedges are Lie semialgebras. Invariant wedges have been characterized and classified (cf. e.g. [21, 26, 27]). Suppose that the following condition is satisfied for a wedge W:

(D) There is a hyperplane subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that the boundary of W is $\mathfrak{h}$.

Then W is a semialgebra, called *halfspace semialgebra.* Hyperplane subalgebras are a classical theme; a final classification for the purpose of also classifying all intersections of halfspace semialgebras was given by HOFMANN in [29]. Lie semialgebras have been classified [19, 34, 25]. In [19], EGGERT showed that *every Lie semialgebra is an intersection of semialgebras belonging to one of the two main types: invariant wedges and half-space semialgebras.* It was shown in a non-trivial process in [40] that *the Lie wedge $\mathfrak{L}(S)$ of any weakly exponential reduced subsemigroup S of a Lie group G is a Lie semialgebra in $\mathfrak{L}(G)$.* This emerged as a consequence of a complete classification. Not even every invariant Lie wedge is the Lie wedge of an exponential Lie semigroup. In order to first exhibit the structure of those Lie groups G which contain an exponential reduced subsemigroup, we say that a Lie algebra $\mathfrak{d}$ is *diagonally metabelian* if $[\mathfrak{d},\mathfrak{d}]$ is abelian and if there is a Cartan algebra $\mathfrak{h}$ such that each of the operators $\operatorname{ad} m$, $m \in \mathfrak{h}$, is diagonalizable over the reals. The structure of diagonally metabelian Lie algebras is completely known ([29]).

Theorem 7.2. *If a connected Lie group G contains a weakly exponential reduced subsemigroup then there are ideals $\mathfrak{s}_j$, $j = 1,\dots,k$, all isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, a diagonally metabelian and centerfree ideal $\mathfrak{d}$, and a compact ideal $\mathfrak{k}$ such that $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \oplus \mathfrak{d} \oplus \mathfrak{k}$.*

Conversely, if G is a simply connected Lie group whose Lie algebra $\mathfrak{g}$ is of this type then G contains a weakly exponential reduced subsemigroup. ∎

We note that the radical of $\mathfrak{g}$ is $\mathfrak{d} \oplus \mathfrak{z}(\mathfrak{k})$ and that $\mathfrak{g}$ has a unique Levi complement $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \oplus \mathfrak{k}'$, $\mathfrak{k}'$ denotes the commutator algebra $[\mathfrak{k},\mathfrak{k}]$ of $\mathfrak{k}$. All subalgebras isomorphic with $\mathfrak{sl}(2,\mathbb{R})$ are contained in the ideal $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k \cong \big(\mathfrak{sl}(2,\mathbb{R})\big)^k$. Once more, this emphasizes the very special role of $\mathfrak{sl}(2,\mathbb{R})$ in Lie theory.

Once the structure of $\mathfrak{g}$ is known we can formulate the Classification Theorem on the Lie wedges of exponential reduced subsemigroups of real Lie groups.

Theorem 7.3. *Let G be a connected Lie group containing a reduced weakly exponential subsemigroup S with Lie wedge W. Then $S = \exp W$ and, in the notation of* Theorem 7.2, *the following conclusions hold:*

(i) *$W = (\mathfrak{s}_1 \cap W) \oplus \cdots \oplus (\mathfrak{s}_k \cap W) \oplus W_0$, where the intersections $\mathfrak{s}_j \cap W$ are intersection algebras, and $W_0 = W \cap (\mathfrak{d} \oplus \mathfrak{k})$.*

(ii) *W_0 is described as follows:*
Set $W_{\rm inv} = \overline{W_0 + \mathfrak{d}'}$. Then the wedge $W_{\rm inv}$ is the smallest invariant wedge containing W_0. There is an intersection algebra $W_{\rm sec}$ containing $\mathfrak{k}'$ such that $W_0 = W_{\rm inv} \cap W_{\rm sec}$.

(iii) *$W \cap -W$ is a metabelian subalgebra of $\mathfrak{g}$ with $W \cap -W \cap \mathfrak{k} = \{0\}$. More specifically, $W \cap -W = \mathfrak{a} \oplus \mathfrak{m}$, where $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{s}_1 \oplus \mathfrak{s}_2 \oplus \cdots \oplus \mathfrak{s}_k$, and $\mathfrak{m}$ is a subalgebra of $\mathfrak{d} \oplus \mathfrak{z}(\mathfrak{k})$.*

(iv) *The group of invertible elements in S is exponential, that is, $S \cap S^{-1} = \exp(W \cap -W)$.* ∎

Let $p\colon \widetilde{G} \to G$ be the universal covering homomorphism of G and $\widetilde{S} = \exp_{\widetilde{G}} W$.

Theorem 7.4. *Under the hypotheses of* Theorem 7.3, *the set $\widetilde{S}$ is a closed exponential semigroup and $S = p(\widetilde{S})$.* ∎

In other words, exponential closed subsemigroups of G can be lifted to the universal covering group $\widetilde{G}$, a fact which a priori is not clear at all. In many discussions concerning exponential semigroups this permits us to assume that G is simply connected. For a detailed discussion see [40].

Example 7.5. The universal covering group G of the group of euclidean motions may be realized as $\mathbb{C} \rtimes \mathbb{R}$, $(c,r)(d,s) = (c + e^{ir}d, r+s)$. The half- space subsemigroup $G_+ \stackrel{\text{def}}{=} \mathbb{C} \times \mathbb{R}_+$, $\mathbb{R}_+ = [0, \infty[$ is weakly exponential but not exponential. It is not reduced, because it contains the normal subgroup $\mathbb{C} \times \{0\}$. Other fairly natural closed proper subsemigroups of G_+ are described in [26], p. 409, Example V.4.14. These are not weakly exponential, the subsemigroup generated by their one parameter subsemigroups is dense. ■

Problem 7.6. *Describe the structure of a (not necessarily reduced) weakly exponential and exponential closed subsemigroup S of a connected real Lie group G.* ■

One may, of course, assume that $G = \overline{\langle S \cup S^{-1} \rangle}$, and one has a largest closed normal subgroup N of G contained in S. Then $S/N \subseteq G/N$ is the reduced situation described in the structure theorems 7.2, 7.3, and 7.4. But example 7.5 illustrates that other phenomena occur in the general situation which are not covered by the existing theory.

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