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Stability and control in spacecraft dynamics

Mircea Puta

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Abstract. An optimal control problem for the spacecraft dynamics is discussed and some of its properties are pointed out.

1. Introduction

Recent work in nonlinear control has drawn attention to drift–free systems with fewer controls than state variables. These arise in problems of motion planning for wheeled robots subject to nonholonomic controls [12], [13], models of kinematic drift effects in space systems subject to appendage vibrations or articulations [5], [6], models of self-propulsion of paramecia at low Reynolds numbers [16] and autonomous underwater vehicle dynamics [8].

The goal of our paper is to discuss a similar problem for the spacecraft dynamics. The case with drift can be found in [11].

2. The spacecraft as a drift-free left invariant system on $\mathbf{SO}(3)$

Let us consider a spacecraft free to move in $\mathbb{R}^3$. Let (b_1, b_2, b_3) be an orthonormal frame fixed on the body and let (r_1, r_2, r_3) define an inertial frame with the origin coincident with the origin of the body-fixed frame. Then we define $X(t) \in \mathrm{SO}(3)$ ($=$ the special orthogonal group, i.e., $\mathrm{SO}(3) = \{A \in \mathcal{M}_{3\times 3}(\mathbb{R}) \mid A^t \cdot A = I_3,\ \det(A) = 1\}$) such that $r_i = X(t) b_i$, i.e., $X(t)$ determines the attitude of the spacecraft at time t. Let $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$. Define

$$\hat{}\ :\ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \mapsto \hat{x} = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \in \mathfrak{so}(3)$$

and

$$A_i = \hat{e}_i; \quad i = 1,\, 2,\, 3.$$

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Then $\{A_1, A_2, A_3\}$ is the standard basis for $\mathfrak{so}(3)$ and $X(t)$ satisfies:

$$\dot{X} = X\hat{\Omega}; \quad \hat{\Omega} = \sum_{i=1}^{3} \Omega_i(t)A_i, \tag{2.1}$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)^T$ is the angular velocity of the spacecraft in the body-fixed coordinates. If we let

$$u_i = \Omega_i; \quad i = 1,\ 2,\ 3,$$

i.e., if we interpret the components of the angular velocity as our controls, then (2.1) takes the form

$$X = X\left(\sum_{i=1}^{3} u_i A_i\right). \tag{2.2}$$

We shall be most interested in the case when only two components of the angular velocity can be controlled. For example, if we can control the angular velocity about the b_1 and b_2 axes, then $X(t)$ satisfies

$$\dot{X} = X(u_1A_1 + u_2A_2).$$

This realization of the spacecraft dynamics is due to Leonard [7].

Theorem 2.1. *The system* (2.3) *is controllable and it is a single bracket one.*■

Let us observe that the above theorem tells us in fact that we can reorient the spacecraft as desired by controlling only two of the three angular velocity components (e.g. roll and pitch velocities).

Remark 2.1. Similar results hold for the controls about:

(i) b_1 and b_2 axes;

(ii) b_2 and b_3 axes.

3. An optimal control problem for the spacecraft dynamics

Let J be the cost function given by

$$J(u_1, u_2) = \frac{1}{2}\int_0^{t_f} [c_1u_1^2(t) + c_2u_2^2(t)]dt; \quad c_1 > 0,\ c_2 > 0. \tag{3.1}$$

Then we can prove:

Theorem 3.1. *The controls that minimize J and steer the system* (2.3) *from $X=0$ at $t=0$ to $X=X_f$ at $t=t_f$ are given by:*

$$u_1 = \frac{1}{c_1}P_1; \quad u_2 = \frac{1}{c_2}P_2,$$

where the functions P_i are solutions of

$$\begin{cases} \dot{P}_1 = -\dfrac{1}{c_2}P_2P_3 \\ \dot{P}_2 = \dfrac{1}{c_1}P_1P_3 \\ \dot{P}_3 = \left(\dfrac{1}{c_2} - \dfrac{1}{c_1}\right)P_1P_2\,. \end{cases} \tag{3.2}$$

Proof. Simply apply Krishnaprasad's theorem [4]. It follows that the optimal Hamiltonian is given by

$$h = \frac{1}{2c_1}P_1^2 + \frac{1}{2c_2}P_2^2. \tag{3.3}$$

It is in fact the controlled Hamiltonian H given by

$$H = P_1u_1 + P_2u_2 - \frac{1}{2}(c_1u_1^2 + c_2u_2^2),$$

which is reduced to $(\mathfrak{so}(3))^\star_-$ via the Poisson reduction. Here $(\mathfrak{so}(3))^\star_-$ is $(\mathfrak{so}(3))^\star$ together with the minus Lie–Poisson structure

$$\Pi_{RB} = \begin{bmatrix} 0 & -P_3 & P_2 \\ P_3 & 0 & -P_1 \\ -P_2 & P_1 & 0 \end{bmatrix}. \tag{3.4}$$

Then the optimal controls are given by

$$u_1 = \frac{1}{c_1}P_1; \quad u_2 = \frac{1}{c_2}P_2,$$

where the functions P_i are solutions of the reduced HAMILTON's equations (or momentum equations) given by

$$[\dot{P}_1, \dot{P}_2, \dot{P}_3] = \Pi_{RB}\cdot\nabla h,$$

which are nothing else but the required equations. ∎

Remark 3.1. The function C given by

$$C = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) \tag{3.5}$$

is a Casimir of our configuration $((\mathfrak{so}3))^\star, \Pi_{RB}) \simeq (\mathbb{R}^3, \Pi_{RB})$, i.e.,

$$(\nabla C)^T \cdot \Pi_{RB} = 0.$$

Remark 3.2. The phase space curves of our system (3.2) are the intersections of the elliptic cylinders

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = 2h$$

with the spheres

$$P_1^2 + P_2^2 + P_3^2 = 2C.$$

Theorem 3.2. *The dynamics* (3.2) *is equivalent to the pendulum dynamics.*

Proof. Indeed, h is a constant of motion, so

$$\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2} = l^2.$$

Let us take now

$$\begin{cases} P_1 = l\sqrt{c_1}\cos\theta \\ P_2 = l\sqrt{c_2}\sin\theta\,. \end{cases}$$

Then

$$\begin{aligned} \dot{P}_1 &= -\sqrt{c_1}\sin\theta\cdot\dot{\theta} \\ &= -l\sqrt{\frac{c_1}{c_2}}\sqrt{c_2}\sin\theta\cdot\dot{\theta} \\ &= -\sqrt{\frac{c_1}{c_2}}P_2\dot{\theta}, \end{aligned}$$

or equivalently,

$$\begin{aligned} \dot{\theta} &= -\sqrt{\frac{c_1}{c_2}}\frac{\dot{P}_1}{P_2} \\ &= -\sqrt{\frac{c_1}{c_2}}\left(-\frac{P_2P_3}{P_2}\right)\frac{1}{c_2} \\ &= \frac{1}{\sqrt{c_1c_2}}P_3. \end{aligned}$$

Differentiating again, we get

$$\ddot{\theta} = \frac{l^2}{\sqrt{c_1c_2}}\left(\frac{1}{c_2} - \frac{1}{c_1}\right)c_1c_2\sin 2\theta,$$

or equivalently

$$\ddot{\theta} = \frac{l^2}{2}\sqrt{c_1c_2}(c_1 - c_2)\sin 2\theta. \tag{3.6}$$

Thus, pendulum dynamics as required. ■

In the particular case $c_1 = c_2 = 1$ we refined BAILLIEUL's theorem [2], namely:

Theorem (BAILLIEUL). *The controls which minimize*

$$J(u_1, u_2) = \frac{1}{2}\int_0^{t_f}(u_1^2 + u_2^2)dt$$

and steer the system (2.3) *from* $X = 0$ *at* $t = 0$ *to* $X = X_f$ *at* $t = t_f$ *are given by sinusoids.*

Proof. Indeed, let us take in (3.6) $c_1 = c_2 = 1$. Then $\theta = k_1t + k_2$ and the optimal controls are given via theorems 3.1 and 3.2 by

$$\begin{aligned} u_1 &= l\cos(k_1t + k_2) \\ u_2 &= l\sin(k_1t + k_2) \end{aligned}$$

as required. ■

Theorem 3.4. *The system* (3.2) *may be realized as a* HAMILTON–POISSON *system in an infinite number of different ways, i.e., there exist infinitely many different (in general nonisomorphic) Poisson structures on* $\mathbb{R}^3$ *such that the system* (3.2) *is induced by an appropriate Hamiltonian .*

Proof. Indeed, to begin with, let us observe that our system can be put in an equivalent form:

$$\dot{P} = \nabla C \times \nabla h,$$

where $P = [P_1, P_2, P_3]^T$ and C, h are respectively given by (3.5) and (3.3). Now, an easy computation shows us that the system (3.2) may be realized as a HAMILTON–POISSON system with the phase space $\mathbb{R}^3$, the Poisson bracket $\{\cdot,\cdot\}_{ab}$ given by

$$\{f, g\}_{ab} = -\nabla C' \cdot (\nabla f \times \nabla g),$$

where a, $b \in \mathbb{R}$ and

$$C' = \frac{a}{2}C + \frac{b}{2}h,$$

and the Hamiltonian h' defined by

$$h' = cC + dh,$$

where c, $d \in \mathbb{R}$, $ad - bc = 1$. ∎

Finally, we shall discuss the integrability of the equations (3.2) via elliptic functions. More precisely, we have

Theorem 3.5. *The equations* (3.2) *may be explicitely integrated by elliptic functions.*

Proof. It is known that

$$P_1^2 c_2 + P_2^2 c_1 = l, \quad l = 2hc_1c_2$$

and

$$P_1^2 + P_2^2 + P_3^2 = 2C$$

are constants of motion. Then an easy computation shows us that

$$P_2^2 = \frac{c_2}{c_2 - c_1}\left[\frac{2Cc_2 - l}{c_2} - P_3^2\right]$$

and

$$P_1^2 = \frac{c_1}{c_1 - c_2}\left[\frac{2Cc_1 - l}{c_1} - P_3^2\right].$$

Using now the third equation from (3.2) we get

$$(\dot{P}_3)^2 = \frac{1}{c_1c_2}\left(P_3^2 - \frac{2Cc_2 - l}{c_2}\right)\left(\frac{2Cc_1 - l}{c_1} - P_3^2\right),$$

that is

$$t = \int_{P_3(0)}^{P_2} \frac{dt}{\sqrt{\frac{1}{c_1c_2}\left(P_3^2 - \frac{2Cc_2-l}{c_2}\right)\left(\frac{2Cc_1-l}{c_1} - P_3^2\right)}},$$

which shows that P_3, and hence P_1, P_2 are elliptic functions of time. ∎

4. Numerical integration of the equation (3.2)

In this section we shall discuss the numerical integration of the equations (3.2) via the Lie–Trotter formula and the midpoint rule, and we shall point out some of their geometric properties.

To begin with, let us observe that the Hamiltonian vector field X_h splits as follows:

$$X_h = X_{h_1} + X_{h_2},$$

where

$$h_1 = \frac{1}{2c_1}P_1^2 \quad h_2 = \frac{1}{2c_2}P_2^2.$$

The integral curves of X_{h_1} and X_{h_2} are given by

$$P(t) = \exp(tX_{h_1}) \cdot P(0) = \Phi_1(t, P(0))$$

and respectively,

$$P(t) = \exp(tX_{h_2}) \cdot P(0) = \Phi_2(t, P(0)).$$

Now, following [10] (see also [15]), the Lie–Trotter formula gives rise to an explicit integrator of the equation (3.2), namely:

$$P^{k+1} = \Phi_1(t, \Phi_2(t, P^k))$$

or explicitely:

$$\begin{cases} P_1^{k+1} = P_1^k \cos\dfrac{P_2(0)}{c_2}t - P_3^k \sin\dfrac{P_2(0)}{c_2}t \\ P_2^{k+1} = P_1^k \sin\dfrac{P_1(0)}{c_1}t \sin\dfrac{P_2(0)}{c_2}t + P_2^k \cos\dfrac{P_1(0)}{c_1}t + P_3^k \sin\dfrac{P_1(0)}{c_1}t \cos\dfrac{P_1(0)}{c_1}t \\ P_3^{k+1} = P_1^k \cos\dfrac{P_1(0)}{c_1}t \sin\dfrac{P_2(0)}{c_2}t - P_2^k \sin\dfrac{P_1(0)}{c_1}t + P_3^k \cos\dfrac{P_1(0)}{c_1}t \cos\dfrac{P_2(0)}{c_2}t \end{cases} \tag{4.1}$$

Some of its properties are sketched in the following theorem:

Theorem 4.1.

(i) *The numerical integrator* (4.1) *preserves the Poisson structure* (3.4)*.*

(ii) *The numerical integrator* (4.1) *preserves the Casimirs of our configuration* $(\mathbb{R}^3, \Pi_{RB})$*.*

(iii) *Its restriction to each coadjoint orbit* ($C = \frac{1}{2}k_2$, $\omega_k = \frac{1}{k}(P_2 dP_1 \wedge dP_3 - P_3 dP_1 \wedge dP_2 - P_1 dP_2 \wedge dP_3)$) *gives rise to a symplectic integrator.*

(iv) *The numerical integrator* (4.1) *does not preserve the Hamiltonian* h *given by* (3.3)*.*

Proof. The items (i) – (iii) hold because Φ_1 and Φ_2 are flows of some Hamiltonian vector fields, hence they are Poisson maps.

Item (iv) is essentially due to the fact that $\{h_1, h_2\} \neq 0$. ∎

An alternative way for the numerical integration of the equations (3.2) is given by the midpoint rule of Gauss–Legendre integrator. In our case it is given by the following implicit formula:

$$\begin{cases} P_1^{k+1} = P_1^k - \dfrac{h}{4c_2}(P_2^{k+1} + P_2^k)(P_3^{k+1} + P_3^k) \\ P_2^{k+1} = P_2^k + \dfrac{h}{4c_1}(P_1^{k+1} + P_1^k)(P_3^{k+1} + P_3^k) \\ P_3^{k+1} = P_3^k + \dfrac{h(c_1 - c_2)}{4c_1c_2}(P_1^{k+1} + P_1^k)(P_2^{k+1} + P_2^k) \end{cases} \tag{4.2}$$

Using now the same arguments as in [1] with obvious modifications we can prove

Theorem 4.2.

(i) *The numerical integrator* (4.3) *preserves the Hamiltonian* (3.3) *and the Casimir* (3.5).

(ii) *The numerical integrator* (4.3) *does not preserve the Poisson structure* (3.4).

(iii) *The numerical integrator* (4.3) *does not preserve the symplectic structure:*

$$\omega_k = \frac{1}{k}[P_2 dP_1 \wedge dP_3 - P_3 dP_1 \wedge dP_2 - P_1 dP_2 \wedge dP_3],$$

hence its restriction to each coadjoint orbit $\left(C = \frac{1}{2}k^2, \omega_k\right)$ *is not a symplectic integrator.* ■

5. Stability

It is not hard to see that the equilibrium states of our system (3.2) are

$$e_1 = (M, 0, 0); \quad e_2 = (0, M, 0); \quad e_3 = (0, 0, M),$$

where $M \in \mathbb{R}$. Now we shall discuss their nonlinear stability. Recall that an equilibrium point p is nonlinearly stable if trajectories starting close to p stay close to p. In other words, a neighbourhood of p must be flow invariant.

First consider the system linearized about c_1. Its eigenvalues are given by solutions of the equation

$$\lambda\left(\lambda^2 - \frac{c_1 - c_2}{c_1^2 c_2}M^2\right) = 0.$$

If $c_1 > c_2$, then a root of the characteristic polynomial has positive real part, thus e_1 is unstable. If $c_1 < c_2$, then the characteristic polynomial has two imaginary eigenvalues and one zero eigenvalue. Is the system stable? We shall prove that it is, via the energy–Casimir method.

Recall that the energy–Casimir method (see [3], [9] or [14]) requires finding a constant of motion for the system, say H, usually the energy, and a family of constants of motion $\mathcal{C}$ such that for some $C \in \mathcal{C}$, $C + H$ has a critical point at the equilibrium of interest. C's are often taken to be Casimirs. Definiteness of $\delta^2(H + C)$, the second variation of $H + C$ and the critical point is sufficient to prove the stability if the phase space of the system is finite dimensional. We use this method to prove the following

Theorem 5.1. *The equilibrium state* $e_1 = (M, 0, 0)$ *is nonlinear stable if* $c_1 < c_2$.

Proof. Consider the energy–Casimir function

$$h_\varphi = \frac{1}{2}\left(\frac{P_1^2}{c_1} + \frac{P_2^2}{c_2}\right) + \varphi\left(\frac{1}{2}(P_1^2 + P_2^2 + P_3^2)\right),$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth real valued function defined on $\mathbb{R}$. Let φ', φ'' denote its first and second derivatives. Now, the first variation of h_φ is given by

$$\delta h_\varphi = \frac{P_1}{c_1}\delta P_1 + \frac{P_2}{c_2}\delta P_2 + \varphi'(\cdot)(P_1\delta P_1 + P_2\delta P_2 + P_3\delta P_3).$$

This equals zero at the equilibrium of interest if and only if

$$\varphi'\left(\frac{1}{2}M^2\right) = -\frac{1}{c_1}. \tag{5.1}$$

The second variation of h_φ at the equilibrium of interest is given via (5.1) by

$$\delta^2 h_\varphi(c_1) = \varphi''\left(\frac{1}{2}M^2\right)M^2(\delta P_1)^2 + \frac{c_1 - c_2}{c_1 c_2}(\delta P_2)^2 - \frac{1}{c_1}(\delta P_3)^2.$$

Since $c_1 < c_2$; c_1, $c_2 > 0$ and having chosen φ such that

$$\varphi''\left(\frac{1}{2}M^2\right) < 0,$$

we can conclude that the second variation at the equilibrium of interest is negative definite and thus e_1 is nonlinear stable. ∎

In a similar manner we can prove

Theorem 5.2. *The equilibrium state* $e_2 = (0, M, 0)$ *is*

(i) *unstable, if* $c_1 < c_2$;

(ii) *nonlinear stable, if* $c_1 > c_2$. ∎

Theorem 5.3. *The equilibrium state* $e_3 = (0, 0, M)$ *is nonlinear stable.* ∎

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Seminarul de Geometrie–Topologie,
University of Timişoara,
B-dul V. Pârvan 4, 1900,
Timişoara,
Romania
Fax: 0040-56-194002
puta@tim1.uvt.ro

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