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On closed abelian subgroups of real Lie groups

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Abstract. Let G be a locally compact real Lie group such that all abelian subgroups of G/G_0 are finite. Assume that $A \subseteq G$ is an closed abelian subgroup. Then A is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^k \times F$ where F is a finite abelian group. If $C \subseteq G$ is a closed solvable subgroup, then C is compactly generated.

1. Introduction

The structure of locally compact abelian groups is well known. However, there seems to be no citable information on the explicit structure of closed abelian subgroups of locally compact real Lie groups with finitely many connected components. (It is clear that we have to restrict ourselves to locally compact Lie groups.) In the following we provide such a reference. Moreover, as an application we can prove that in such Lie groups all closed solvable subgroups are compactly generated.

In the whole note, G denotes a real Lie group. Following [3], we call a locally compact abelian group *elementary* when it is isomorphic to $\mathbb{R}^n \times \mathbb{T}^m \times \mathbb{Z}^k \times F$ with a finite abelian group F and $n, m, k \in \mathbb{N}_0$. Our aim is to prove that each closed abelian subgroup A of G, where all abelian subgroups of G/G_0 are finitely generated, is elementary. First, we restrict ourselves to a connected Lie group G. We shall consider a representation $\rho\colon G \to \mathrm{Gl}(n, \mathbb{R})$ with elementary kernel. Finally, we show that this implies our hypothesis.

M. Moskowitz has shown that a locally compact abelian group is isomorphic to $\mathbb{R}^n \times \mathbb{Z}^k \times C$ with a compact abelian group C if and only if it is *compactly generated*, i.e., there is a compact subset which generates it algebraically ([7, Theorem 2.5]). A Lie group has no small subgroups. A closed subgroup of a Lie group is a Lie group with respect to the induced topology. Thus a compact abelian subgroup of a real Lie group is isomorphic to $\mathbb{T}^m \times F$ with finite abelian group F. In general, the identity-component of a topological group is generated by each 1-neighborhood, thus the identity-component of a locally compact group is com-

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pactly generated. We denote the identity-component of a G by G_0. Summing up these remarks we get the following statement where G is not necessarily connected:

Proposition 1.1. *Let A be a closed abelian subgroup of a locally compact real Lie group G and A_0 its identity component. The following conditions are equivalent:*

1. *A is elementary.*
2. *A is compactly generated.*
3. *A/A_0 is finitely generated.*

In particular, discrete abelian subgroups of a locally compact Lie group G with finitely many connected components are finitely generated. Moreover, it would be sufficient that all discrete abelian subgroups of G are finitely generated. But we will see that this would not simplify the proof. First of all, we consider an example which shows that discrete subgroups, in general, can be far away from being finitely generated.

Example 1.2. We consider the projective special linear group $\mathrm{PSl}(2,\mathbb{R})$. Then $\mathrm{PSl}(2,\mathbb{Z})$ is a discrete subgroup and isomorphic to the free product $\mathbb{Z}(2)*\mathbb{Z}(3)$, where the cosets

$$x=\left[\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}\right] \quad\text{and}\quad y=\left[\begin{pmatrix} -1 & 1\\ -1 & 0\end{pmatrix}\right]$$

are the generators (c.f. [8, p. 187]). Moreover, $\mathrm{PSl}(2,\mathbb{Z})$ contains the free group $F(a,b)$ with the two generators $a=xy$ and $b=xy^2$. The commutator group $F'(a,b)\subseteq F(a,b)\subseteq \mathrm{PSl}(2,\mathbb{R})$ is a free subgroup of infinite rank. Since it is a subgroup of a discrete subgroup of $\mathrm{PSl}(2,\mathbb{R})$, it is discrete itself, but not finitely generated.

Thus our statement cannot be generalized to arbitrary abelian subgroups.

2. Preliminaries

We summarize some classical results which are useful in our context.

Theorem 2.1. *Let A be a locally compact abelian group and B a closed subgroup. Then A is compactly generated if and only if B and A/B are compactly generated.*

One can find a proof of this statement in [7, Theorem 2.6]. We can prove a little bit more. For this, we need the following classical lemmas:

Lemma 2.2. *Let G be a topological group and N a closed normal subgroup. Assume that B is a closed subgroup containing N. Then B/N is closed.*

Lemma 2.3. *Let H_1, H_2 be topological groups and $\rho\colon H_1\to H_2$ a surjective group homomorphism. If H_1 is compactly generated then so is H_2.*

Proposition 2.4. *Let G be a locally compact group and N a closed normal subgroup. If N and G/N are compactly generated then G is compactly generated.*

Proof. Let B be a compact set which generates G/N and q the quotient map $q: G \to G/N$. Now we consider $q^{-1}(B)$. Since G is locally compact, for each $x \in q^{-1}(B)$ there is a compact neighborhood K_x and a open neighborhood $O_x \subset K_x$. Thus we have $B \subseteq \bigcup_{x\in q^{-1}(B)} \pi(O_x)$. Since B is compact we find a finite set $S \subseteq q^{-1}(B)$ such that $B \subseteq \bigcup_{x\in S} \pi(O_x) \subseteq \bigcup_{x\in S} \pi(K_x) = \pi(\bigcup_{x\in S} K_x)$. Thus, $\bigcup_{x\in S} K_x$ is a compact subset of G whose image under q contains B. Since N is compactly generated there is a compact subset C such that $N = \langle C \rangle$. Then $C \cup \bigcup_{x\in S} K_x$ is compact and generates G. ■

A group G is called *polycyclic* if there is a Jordan-Hölder series $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = \{1\}$ such that each factor G_n/G_{n+1} is cyclic. L. Auslander and R. Tolimieri have shown the following Theorem (cf. [1, Theorem 1]):

Theorem 2.5. *Let C be a closed solvable subgroup of $\mathrm{Gl}(n, \mathbb{R})$ then the group C/C_0 is polycyclic.*

Let C be a solvable group, $C^{(0)} = C$, $C^{(m)} = C^{(m-1)'}$ the iterated commutator groups. We denote by the *length* $l(C)$ of C the smallest natural number such that $C^{(n)} = \{0\}$. In [10, Proposition 4.1] is proved the following:

Proposition 2.6. *Let C be a solvable group. Then the following statements are equivalent:*

1. *C is polycyclic.*
2. *C is finitely generated.*
3. *$C^{(k-1)}/C^{(k)}$ is finitely generated for all $k = 1, \ldots, l(G)$.*

A. I. Mal'cev has shown the following ([6, Theorem 8]):

Theorem 2.7. *If every abelian subgroup of a solvable group C is finitely generated then $C^{(k-1)}/C^{(k)}$ is finitely generated for all $k = 1, \ldots, l(G)$.*

Proposition 2.6 and Theorem 2.7 imply:

Corollary 2.8. *If every abelian subgroup of a solvable group C is finitely generated then C is finitely generated itself.*

We know that C_0 is compactly generated and that C/C_0 is finitely generated, in particular compactly generated. By Proposition 2.4, we get the following:

Corollary 2.9. *If $C \subseteq \mathrm{Gl}(m, \mathbb{R})$ is a closed solvable subgroup of C, then C is compactly generated.*

The following proposition is very useful in this context. It was proved by D.Ž. Djoković ([4, Proposition 5]).

Proposition 2.10. *Let U be an analytic subgroup of a* $\mathrm{Gl}(n,\mathbb{R})$. *Then there is a faithful representation* $\omega\colon U \to \mathrm{Gl}(m,\mathbb{R})$ *such that* $\omega(U)$ *is closed in* $\mathrm{Gl}(m,\mathbb{R})$.

If G is a connected real Lie group, $\mathrm{Ad}(G)$ is a not necessarily closed analytic subgroup of $\mathrm{Gl}(\mathfrak{g})$. Since $\mathfrak{g}$ is a real vector space we have $\mathrm{Gl}(\mathfrak{g}) \cong \mathrm{Gl}(n,\mathbb{R})$ for $n = \dim \mathfrak{g}$. If we set $\pi := \omega \circ \mathrm{Ad}$, then $\pi(G)$ is closed, and by the canonical decomposition of homomorphisms, $\pi(G) \cong G/\ker \pi$.

Corollary 2.11. *Let G be a connected real Lie group. Then there is a representation* $\pi\colon G \to \mathrm{Gl}(m,\mathbb{R})$ *such that* $\pi(G)$ *is closed and* $\ker \pi$ *is the center Z of G. Moreover, G/Z is isomorphic to* $\pi(G)$.

If we have a closed subgroup C of G which contains Z, then $\pi(C) \cong C/Z$ is closed by Lemma 2.2. So, it is useful to examine the smallest closed subgroup M of G such that M contains both C and Z. In our context, it is sufficient to suppose that C is solvable.

Proposition 2.12. *Let G be a locally compact real Lie group. Let C be a closed subgroup and Z the center of G. Set* $M := \overline{CZ}$. *Then* $\overline{M'} = \overline{C'}$ *and, moreover,* $\overline{C^{(k)}} = \overline{M^{(k)}}$. *If, in addition, C is solvable then M is a closed solvable subgroup which contains the center of G, and* $l(\overline{M^{(k)}}) = l(\overline{C^{(k)}}) = l(C^{(k)})$.

Proof. Denote by (v,w) for $v,w \in G$ the commutator $vwv^{-1}w^{-1}$. A simple computation yields $(c_1z_1, c_2z_2) = (c_1,c_2)$ for all $c_1,c_2 \in C$ and $z_1,z_2 \in Z$. Note that M' consists of finite products of commutators of M. So, let (m_1,m_2) with $m_1,m_2 \in M$ be an arbitrary commutator of M. Then there are sequences $(c_n)_{n\in\mathbb{N}}$, $(d_n)_{n\in\mathbb{N}}$ with $c_n,d_n \in C$ for all $n \in \mathbb{N}$ and sequences $(z_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ with $z_n,u_n \in Z$ for all $n \in \mathbb{N}$ such that $m_1 = \lim_{n\to\infty} c_nz_n$ and $m_2 = \lim_{n\to\infty} d_nu_n$. Since the commutator is continuous we get $(m_1,m_2) = \lim_{n\to\infty}(c_nz_n, d_nu_n) = \lim_{n\to\infty}(c_n,d_n) \in \overline{C'}$. But $\overline{C'}$ is a subgroup of G. Thus finite products of commutators of M are contained in $\overline{C'}$, too. This implies $M' \subseteq \overline{C'}$, hence $\overline{M'} \subseteq \overline{C'}$.

On the other hand, $C' \subseteq M'$, hence $\overline{C'} \subseteq \overline{M'}$. By [2, III.9.1], we get $\overline{C^{(k)}} = \overline{M^{(k)}}$. The rest follows immediately. ■

We need some information about the structure of the center.

Proposition 2.13. *The center Z of a connected real Lie group is elementary.*

Proof. By [5, Chap. XVI], the center is contained in a connected closed abelian subgroup A of G. Since A is connected it is compactly generated. Since Z is closed in G, hence closed in A, it is compactly generated by Theorem 2.1. ■

¿From Theorem 2.1 we get immediately:

Corollary 2.14. *Each closed subgroup of the center is elementary.*

3. Results

First we restrict ourselves to a connected Lie group G. By Proposition 1.1, we have to prove that each closed ablian subgroup of G is compactly generated.

Theorem 3.1. *Let G be a connected real Lie group and A a closed abelian subgroup. Then A is elementary.*

Proof. Let Z be the center of G. By Corollary 2.11, there is a representation $\pi: G \to \mathrm{Gl}(m, \mathbb{R})$ with kernel Z such that $\pi(G)$ is closed and $G/Z \cong \pi(G)$. Let A be a closed abelian subgroup of G and $A_1 := \overline{AZ}$. Lemma 2.2 implies that $\pi(A_1)$ is a closed abelian subgroup of $\mathrm{Gl}(m, \mathbb{R})$. By Corollary 2.9, we know that $\pi(A_1)$ is compactly generated. Hence, by Theorem 2.1, $\pi(A_1)$ is elementary. We note that $\pi(A_1)$ is isomorphic to A_1/Z. If we apply Theorem 2.1 again and Proposition 2.13, we see that A_1 is elementary, and hence A as a closed subgroup of A_1 is elementary, too. ∎

We note that we can generalize this result to a wider class of Lie groups:

Theorem 3.2. *Let G be a locally compact real Lie group where the abelian subgroups of G/G_0 are finitely generated. Then the closed abelian subgroups of G are elementary.*

Proof. Let A be an abelian subgroup of G. By group theory, we know that $A/(A \cap G_0)$ is isomorphic to $(A/A_0)/((A \cap G_0)/A_0)$. By our assumption, $A/(A \cap G_0)$ is finitely generated. By Theorem 3.1 and Proposition 1.1, $(A \cap G_0)/A_0$ is finitely generated, too. Theorem 2.1 implies that A/A_0 is finitely generated. By Theorem 2.1, and since A_0 is compactly generated, A is elementary. ∎

It is clear that the abelian subgroups of G/G_0 are finitely generated if G/G_0 is finite. Hence we get:

Corollary 3.3. *If G/G_0 is finite then the closed abelian subgroups of G are elementary.*

Note that it is not sufficient to require that G/G_0 is finitely generated. In [9, p. 56], we find the following example:

Example 3.4. Let $H = \left\langle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \subseteq \mathrm{Gl}(2, \mathbb{Q})$. This group is metabelian and contains an abelian subgroup, namely H', which is isomorphic to the additive group of dyadic rationals, hence not finitely generated. If G/G_0 is isomorphic to H together with the discrete topology, we have an example for our claim.

Now we turn to solvable closed subgroups of real Lie groups. First, we assume that G is connected.

Theorem 3.5. *Let G be a connected real Lie group and $C \subseteq G$ a closed solvable subgroup. Then C is compactly generated.*

Proof. We prove the statement by induction on the length of C. If $l(C) = 1$, then C is abelian, hence compactly generated because of Theorem 3.1. Now assume that the claim is true for length n. Suppose that $l(C) = n + 1$. Denote by Z the center of G. We set $M = \overline{CZ}$. By Proposition 2.12, we have $\overline{M'} = \overline{C'}$. We note that $l(\overline{M'}) = l(\overline{C'}) = l(C') = n$. By assumption, $\overline{C'}$ is compactly generated. On the other hand, by Corollary 2.11 there is a representation π of G in a $\mathrm{Gl}(m, \mathbb{R})$ such that $\pi(G)$ is closed and $\ker \pi = Z$. In particular by Lemma 2.2, $\pi(M) \cong M/Z$ is closed solvable in $\mathrm{Gl}(m, \mathbb{R})$. By Corollary 2.9, it is compactly generated. Since, by Proposition 2.13, Z is compactly generated, too, M is compactly generated because of Proposition 2.4. Lemma 2.3 implies that $M/\overline{C'}$ is compactly generated. Furthermore, it is abelian. By Lemma 2.2, $C/\overline{C'}$ is closed in $M/\overline{C'}$. By Proposition 2.1, $C/\overline{C'}$ is compactly generated and by Proposition 2.4, C is compactly generated. ■

Now we will consider the case where G is not connected.

Theorem 3.6. *If G is a locally compact real Lie group such that all abelian subgroups of G/G_0 are finitely generated. Then all closed solvable subgroups of G are compactly generated.*

Proof. Assume that C is solvable. Then $C \cap G_0$ is a solvable closed subgroup of G_0, hence by Theorem 3.5 compactly generated. Moreover, $C/(C \cap G_0) \cong CG_0/G_0 \subseteq G/G_0$ is solvable and by Corollary 2.8 finitely generated. By Proposition 2.4, this implies that C is compactly generated. ■

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