Journal of Lie Theory
Volume **8** (1998) 139–152

Complex Structures Contained in Classical Groups

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Communicated by K.-H. Neeb

Abstract. For any classical group G let $\mathcal{C}(G)=\{J\in G \mid J^2=-\mathbf{1}\}$ denote the space of complex structures in G. $\mathcal{C}(G)$ is a symmetric space with finitely many connected components, which are described explicitly. In particular these components are flag domains in the sense of J. A. Wolf. Moreover it is shown that the components of $\mathcal{C}(G)$ are parameter spaces for nondegenerate complex tori with certain endomorphism structure.

Let V be a real vector space of finite dimension. A complex structure on V is an endomorphism J of V with $J^2 = -\mathrm{id}_V$. If V admits a complex structure J, then necessarily V is of even dimension, say $2n$. The pair $V_J := (V, J)$ is a complex vector space of dimension n with respect to the scalar multiplication $\mathbb{C} \times V \to V, (x+iy, v) \mapsto xv + yJ(v)$. It is well-known that the set of all complex structures on $V \simeq \mathbb{R}^{2n}$ is the symmetric space

$$\mathcal{C}(GL(V)) \quad \simeq \quad GL_{2n}(\mathbb{R})/GL_n(\mathbb{C}).$$

Suppose the complex vector space $V_J := (V, J)$ admits a hermitian scalar product H. Then with respect to a suitable basis for $V = \mathbb{R}^{2n}$ the hermitian form H is given by the matrix ${}^tJ\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right) + i\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)$. Since $\operatorname{Re} H$ is symmetric and positive definite, the complex structure J is contained in the symplectic group $Sp_{2n}(\mathbb{R})$ and ${}^tJ\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)$ is positive definite. Thus the set of hermitian complex vector spaces of dimension n can be identified with the space $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ of complex structures on $\mathbb{R}^{2n}$ contained in $Sp_{2n}(\mathbb{R})$ such that ${}^tJ\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right) > 0$. It is well-known that $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ is the symmetric space

$$\mathcal{C}_0(Sp_{2n}(\mathbb{R})) \quad \simeq \quad Sp_{2n}(\mathbb{R})/U_n(\mathbb{C}).$$

The space $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ admits a further interpretation: it is isomorphic to the Siegel upper half space and as such parametrizes families of polarized abelian varieties. Quotients of $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ by suitable arithmetic subgroups of $Sp_{2n}(\mathbb{R})$ are moduli spaces of abelian varieties with certain level structures.
It is the aim of this note to generalize these results to arbitrary classical groups, i.e. the groups

ISSN 0949–5932 / \$2.50

$$GL_n(\mathbb{R}), GL_n(\mathbb{C}), GL_n(\mathbb{H}), O_{p,q}(\mathbb{R}), O_n(\mathbb{C}), U_{p,q}(\mathbb{C}), U_{p,q}(\mathbb{H}), Sp_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{C})$$

and the antiunitary quaternionic group $\alpha U_n(\mathbb{H})$ [1] (here $\mathbb{H} = \mathbb{C} + j\mathbb{C}$ denotes the skew field of Hamiltonian quaternions). Let G be any classical group of the above list. The set of all complex structures contained in G is denoted by

$$\mathcal{C}(G) := \{J \in G |\, J^2 = -\mathbf{1}\}.$$

The group G acts on $\mathcal{C}(G)$ by conjugation. It will be shown that for $G = GL_{2n}(\mathbb{R}), GL_n(\mathbb{H}), O_{2p,2q}(\mathbb{R}), O_{2n}(\mathbb{C}), U_{p,q}(\mathbb{H})$ and $Sp_{2n}(\mathbb{C})$ the set $\mathcal{C}(G)$ is a G-orbit. To be more precise

Theorem 1. *a) The action of G on $\mathcal{C}(G)$ induces isomorphisms*

$$\begin{aligned}
\mathcal{C}(GL_{2n}(\mathbb{R})) &\simeq GL_{2n}(\mathbb{R})/GL_n(\mathbb{C}) \\
\mathcal{C}(GL_n(\mathbb{H})) &\simeq GL_n(\mathbb{H})/GL_n(\mathbb{C}) \\
\mathcal{C}(O_{2p,2q}(\mathbb{R})) &\simeq O_{2p,2q}(\mathbb{R})/U_{p,q}(\mathbb{C}) \\
\mathcal{C}(O_{2n}(\mathbb{C})) &\simeq O_{2n}(\mathbb{C})/GL_n(\mathbb{C}) \\
\mathcal{C}(U_{p,q}(\mathbb{H})) &\simeq U_{p,q}(\mathbb{H})/U_{p,q}(\mathbb{C}) \\
\mathcal{C}(Sp_{2n}(\mathbb{C})) &\simeq Sp_{2n}(\mathbb{C})/GL_n(\mathbb{C})
\end{aligned}$$

b) The spaces $\mathcal{C}(GL_{2n+1}(\mathbb{R})), \mathcal{C}(O_{p,q}(\mathbb{R}))$ with p or q odd, and $\mathcal{C}(O_{2n+1}(\mathbb{C}))$ are empty.

For the embeddings of the respective subgroups into G see §1 Remark 1.1.
If G is one of the remaining groups: $GL_n(\mathbb{C}), U_{p,q}(\mathbb{C}), Sp_{2n}(\mathbb{R})$ or $\alpha U_n(\mathbb{H})$, the space $\mathcal{C}(G)$ splits up into finitely many orbits characterized by some index and/or signature condition. For this denote

$$\mathcal{C}_k(Sp_{2n}(\mathbb{R})) := \{J \in \mathcal{C}(Sp_{2n}(\mathbb{R}))|\, \mathrm{ind}_{\mathbb{R}}\left(\left(\begin{smallmatrix} 0 & -\mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right) J\right) = 2k\}$$ [2]

$$\mathcal{C}_r(GL_n(\mathbb{C})) := \{J \in \mathcal{C}(GL_n(\mathbb{C}))|\, \mathrm{sign}(J) = (r, n-r)\}$$ [3]

$$\mathcal{C}_{k,r}(U_{p,q}(\mathbb{C})) := \left\{ J \in \mathcal{C}(U_{p,q}(\mathbb{C})) \,\middle|\, \begin{array}{l} \mathrm{ind}_{\mathbb{C}}\left(\left(\begin{smallmatrix} i\mathbf{1}_p & 0 \\ 0 & -i\mathbf{1}_q \end{smallmatrix}\right) J\right) = k \\ \mathrm{sign}(J) = (r, n-r) \end{array} \right\}$$ [4]

$$\mathcal{C}_k(\alpha U_n(\mathbb{H})) := \{J \in \mathcal{C}(\alpha U_n(\mathbb{H}))|\, \mathrm{ind}_{\mathbb{H}}(Ji) = k\}.$$ [5]

These spaces have the following structure

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[1] $\alpha U_n(\mathbb{H}) = \{M \in GL_n(\mathbb{H}) | M(i\mathbf{1}_n)^t\overline{M} = i\mathbf{1}_n\}$, where $\overline{M}$ denotes quaternionic conjugation of the matrix M. Some authors use the notation $SO^*(2n)$ for this group.

[2] $\mathrm{ind}_{\mathbb{R}}$ is the number of negative eigenvalues of a nondegenerate symmetric real matrix.

[3] $\mathrm{sign}(J) = (r, s)$ if and only if $J = A^{-1}\left(\begin{smallmatrix} i\mathbf{1}_r & 0 \\ 0 & -i\mathbf{1}_s \end{smallmatrix}\right)A$ for some $A \in GL_n(\mathbb{C})$.

[4] $\mathrm{ind}_{\mathbb{C}}$ is the number of negative eigenvalues of a nondegenerate hermitian matrix.

[5] For $M \in GL_n(\mathbb{H})$ with ${}^t\overline{M} = M$ define $\mathrm{ind}_{\mathbb{H}} M = k$ if ${}^tAM\overline{A} = \left(\begin{smallmatrix} \mathbf{1}_{n-k} & 0 \\ 0 & -\mathbf{1}_k \end{smallmatrix}\right)$ for some $A \in GL_n(\mathbb{H})$.

Theorem 2. *a)* $\mathcal{C}(Sp_{2n}(\mathbb{R})) = \bigcup_{k=0}^{n} \mathcal{C}_k(Sp_{2n}(\mathbb{R}))$ *and*

$$\mathcal{C}_k(Sp_{2n}(\mathbb{R})) \quad \simeq \quad Sp_{2n}(\mathbb{R})/U_{n-k,k}(\mathbb{C})$$

b) $\mathcal{C}(GL_n(\mathbb{C})) = \bigcup_{r=0}^{n} \mathcal{C}_r(GL_n(\mathbb{C}))$ *and*

$$\mathcal{C}_r(GL_n(\mathbb{C})) \quad \simeq \quad GL_n(\mathbb{C})/(GL_r(\mathbb{C}) \times GL_{n-r}(\mathbb{C}))$$

c) $\mathcal{C}(U_{p,q}(\mathbb{C})) = \bigcup_{r=0}^{p+q} \quad \bigcup_{k=0,k\equiv r-q(mod\,2)}^{p+q} \mathcal{C}_{k,r}(U_{p,q}(\mathbb{C}))$ *and*

$$\mathcal{C}_{r-q+2m,r}(U_{p,q}(\mathbb{C})) \simeq U_{p,q}(\mathbb{C})/(U_{r-q+m,q-m}(\mathbb{C}) \times U_{p+q-r-m,m}(\mathbb{C}))$$

d) $\mathcal{C}(\alpha U_n(\mathbb{H})) = \bigcup_{k=0}^{n} \mathcal{C}_k(\alpha U_n(\mathbb{H}))$ *and*

$$\mathcal{C}_k(\alpha U_n(\mathbb{H})) \quad \simeq \quad \alpha U_n(\mathbb{H})/U_{n-k,k}(\mathbb{C})$$

For the embeddings of the respective subgroups see again §1 Remark 1.1. Within this note the notation $\mathcal{C}_*(G)$ refers to any of the spaces of Theorems 1 or 2.

In some cases, namely for $G = GL_n(\mathbb{R}), GL_n(\mathbb{C}), O_{2n}(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$, the results of Theorems 1 and 2 are well-known, at least for $\mathcal{C}_0(G)$. They are included here for the sake of completeness.

The symmetric spaces $\mathcal{C}_*(G)$ can be interpreted as open subspaces of proper subvarieties of certain Grassmannian varieties. In these terms the theorems seem to be a consequence of Witt's Theorem. However it turns out that this approach does not apply in every case (see [1]). Within this note we restrict ourself to the point of view of complex structures, in order to unify the procedure.
As an immediate consequence of the theorems one obtains that every complex structure admits a *normal form* (see Section 2). In Section 2 also two examples of less immediate applications to Linear Algebra are presented.
As a further result it will be shown in Section 3 that the connected components of the symmetric spaces $\mathcal{C}_*(G)$ are flag domains in the sense of J. A. Wolf. This implies for example that they are simply connected submanifolds of smooth projective varieties and that there are certain vanishing theorems for the cohomology of coherent sheaves on $\mathcal{C}_*(G)$.

As mentioned above $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ is isomorphic to the Siegel upper half space $\mathcal{H}_n$. Similarly all spaces $\mathcal{C}_*(G)$ can be interpreted as parameter spaces for nondegenerate complex tori (X, H) with certain level structures. (For the definition of nondegenerate complex tori see Section 4.) This level structure consists of the prescription of an endomorphism structure for (X, H). In this way every nondegenerate complex torus belongs to one of the spaces $\mathcal{C}_*(G)$ or a product of these.

Notation:

$$\begin{array}{lcll} \mathbf{1}_n & = & \text{unit} & n \times n - \text{matrix}, \\ \mathbf{0}_n & = & \text{zero} & n \times n - \text{matrix}, \\ \mathbf{0}_{s,r} & = & \text{zero} & s \times r - \text{matrix}, \\ I_{p,q} & := & \left(\begin{smallmatrix} \mathbf{1}_p & 0 \\ 0 & -\mathbf{1}_q \end{smallmatrix}\right) & \end{array}$$

$O_{2n}(\mathbb{C})$ is the orthogonal group for the symmetric bilinear form $\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ \mathbf{1}_n & 0 \end{smallmatrix}\right)$. $O_{2p,2q}(\mathbb{R})$ is the orthogonal group for the symmetric bilinear form $\left(\begin{smallmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)$.

1. The Proofs of Theorems 1 and 2

As already mentioned in the introduction the results of Theorems 1 and 2 are well-known for $\mathcal{C}(GL_{2n}(\mathbb{R})), \mathcal{C}_r(GL_n(\mathbb{C})), \mathcal{C}(O_{2n}(\mathbb{R}))$ and $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$.

Proof. (of Theorem 1 b)): Suppose first $G = GL_{2n+1}(\mathbb{R})$ or $O_{2n+1}(\mathbb{C})$. The determinant of any $J \in \mathcal{C}(G)$ is ± 1. On the other hand the eigenvalues of J are $\pm i$, so $\det J = i^r \cdot (-i)^{2n+1-r} = (-1)^{n+r+1} i$ for some $r = 0, \dots, 2n+1$, a contradiction. Moreover for $G = O_{p,q}(\mathbb{R})$ one uses the fact that every $J \in \mathcal{C}(O_{p,q}(\mathbb{R}))$ defines a nondegenerate hermitian form $H = I_{p,q} + iI_{p,q}J$ on the complex vector space $V_J = (\mathbb{R}^{p+q}, J)$. But then $2\,\mathrm{ind}_{\mathbb{C}} H = \mathrm{ind}_{\mathbb{R}} \mathrm{Re}\, H = \mathrm{ind}_{\mathbb{R}} I_{p,q} = q$, so q is even. Since $p+q = 2\dim_{\mathbb{C}} V_J$ is even, p is also even. ∎

Proof. (of Theorem 1, case $\mathcal{C}(GL_n(\mathbb{H}))$): This is a consequence of results of Louise Wolf. Recall the complex representation $\rho : GL_n(\mathbb{H}) \hookrightarrow GL_{2n}(\mathbb{C}), A + Bj \mapsto \left(\begin{smallmatrix} A & B \\ -\bar{B} & \bar{A} \end{smallmatrix}\right)$. In [6] it is shown that quaternionic matrices are similar if and only if their complex representations are similar, and that the eigenvalues of a matrix in $\mathrm{im}\{\rho : GL_n(\mathbb{H}) \hookrightarrow GL_{2n}(\mathbb{C})\}$ appear as complex conjugate pairs. This implies that every complex structure $J \in \mathcal{C}(GL_n(\mathbb{H}))$ is conjugate to $i\mathbf{1}_n \in GL_n(\mathbb{H})$. Finally $M^{-1}(i\mathbf{1}_n)M = i\mathbf{1}_n$ in $GL_n(\mathbb{H})$ if and only if M commutes with i, or equivalently if $M \in GL_n(\mathbb{C}) \subset GL_n(\mathbb{H})$. This implies the assertion for $\mathcal{C}(GL_n(\mathbb{H}))$. ∎

The proofs of the remaining cases follow more or less the same pattern. Therefore we present here only the proofs of the most complicated cases $G = U_{p,q}(\mathbb{C})$ and $\alpha U_m(\mathbb{H})$. The other proofs are just variations of these.

Proof. (of Theorem 2 c)): Denote $n = p+q$. By definition one has

$$\mathcal{C}(U_{p,q}(\mathbb{C})) = \bigcup_{r=0}^{n} \bigcup_{k=0}^{n} \mathcal{C}_{k,r}(U_{p,q}(\mathbb{C})).$$

The index k meets the following requirements:

STEP I: $\mathcal{C}_{k,r}(U_{p,q}(\mathbb{C})) = \emptyset$ *unless* $k \equiv r - q \pmod 2$
Suppose $J \in \mathcal{C}_{k,r}(U_{p,q}(\mathbb{C}))$. By definition J is conjugate to $iI_{r,s}$ where $s = n - r$, i.e.

$$J = N^{-1} i I_{r,s} N$$

with some $N \in GL_n(\mathbb{C})$. On the other hand J being unitary of type (p,q) implies that ${}^t\overline{N}^{-1} I_{p,q} N^{-1}$ commutes with $iI_{r,s}$. This means that ${}^t\overline{N}^{-1} I_{p,q} N^{-1}$ is a block matrix of the form

$${}^t\overline{N}^{-1} I_{p,q} N^{-1} = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$$

with $\alpha \in GL_r(\mathbb{C})$ and $\beta \in GL_s(\mathbb{C})$. In particular α and β are hermitian and

$$\mathrm{ind}_{\mathbb{C}} \alpha + \mathrm{ind}_{\mathbb{C}} \beta = \mathrm{ind}_{\mathbb{C}} I_{p,q} = q. \tag{1}$$

By definition we have

$$\begin{aligned} k &= \mathrm{ind}_{\mathbb{C}}(iI_{p,q}J) = \mathrm{ind}_{\mathbb{C}}(iI_{p,q}N^{-1}(iI_{r,s})N) \\ &= \mathrm{ind}_{\mathbb{C}}(-{}^t\overline{N}^{-1}I_{p,q}N^{-1}I_{r,s}) = \mathrm{ind}_{\mathbb{C}}(-\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)I_{r,s}) \\ &= \mathrm{ind}_{\mathbb{C}}\left(\begin{smallmatrix}-\alpha & 0\\ 0 & \beta\end{smallmatrix}\right) = \mathrm{ind}_{\mathbb{C}}(-\alpha) + \mathrm{ind}_{\mathbb{C}}\beta \\ &= r - \mathrm{ind}_{\mathbb{C}}\alpha + \mathrm{ind}_{\mathbb{C}}\beta \end{aligned} \tag{2}$$

Combining (1) and (2) gives

$$k = r - q + 2\,\mathrm{ind}_{\mathbb{C}}\,\beta \equiv r - q (mod\, 2).$$

Step II: *The homomorphism*

$$\varphi : U_{p,q}(\mathbb{C}) \to \mathcal{C}_{k,r}(U_{p,q}(\mathbb{C})), M \mapsto M^{-1}\left(\begin{smallmatrix} i\mathbf{1}_r & 0\\ 0 & -i\mathbf{1}_s\end{smallmatrix}\right)M$$

is surjective.

According to Step I, we may assume that $k = r - q + 2m$ for some integer m. Moreover, we may assume that $U_{p,q}(\mathbb{C})$ is the unitary group for the symmetric bilinear form $I := \begin{pmatrix} \mathbf{1}_{r-q+m} & & \\ & -\mathbf{1}_q & \\ & & \mathbf{1}_{s-m}\end{pmatrix}$ on $\mathbb{C}^{p+q}$. Suppose $J \in \mathcal{C}_{k,r}(U_{p,q}(\mathbb{C}))$. Then as in Step I, replacing $I_{p,q}$ by I there is an $N \in GL_n(\mathbb{C})$ with $J = N^{-1}iI_{r,s}N$ and ${}^t\overline{N}^{-1}I\,N^{-1} = \left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$ with hermitian matrices $\alpha \in GL_r(\mathbb{C})$ and $\beta \in GL_s(\mathbb{C})$. Equations (1) and (2) imply that $\mathrm{ind}_{\mathbb{C}}\beta = m$ and $\mathrm{ind}_{\mathbb{C}}\alpha = q - m$. Hence there are matrices $A \in GL_r(\mathbb{C})$ and $B \in GL_s(\mathbb{C})$ such that

$${}^t\overline{A}\alpha A = \begin{pmatrix} \mathbf{1}_{r-q+m} & 0\\ 0 & -\mathbf{1}_{q-m}\end{pmatrix},\ {}^t\overline{B}\beta B = \begin{pmatrix} -\mathbf{1}_m & 0\\ 0 & \mathbf{1}_{s-m}\end{pmatrix}.$$

An immediate computation shows that the matrix $M := \left(\begin{smallmatrix} A & 0\\ 0 & B\end{smallmatrix}\right)^{-1}N$ is an element of $U_{p,q}(\mathbb{C})$ and that $\varphi(M) = M^{-1}iI_{r,s}M = J$.

Step III: $\varphi^{-1}(iI_{r,s}) = U_{r-q+m,q-m}(\mathbb{C}) \times U_{s-m,m}(\mathbb{C})$.

The stabilizer of $iI_{r,s}$ in $U_{p,q}(\mathbb{C})$ is the group

$$\begin{aligned} \varphi^{-1}(iI_{r,s}) &= \{M \in U_{p,q}(\mathbb{C}) \mid M\,I_{r,s} = I_{r,s}M\} \\ &= \left\{ M = \left(\begin{smallmatrix} M_1 & 0\\ 0 & M_2\end{smallmatrix}\right) \in U_{p,q}(\mathbb{C}) \,\middle|\, \begin{array}{l} M_1 \in GL_r(\mathbb{C}) \\ M_2 \in GL_s(\mathbb{C})\end{array} \right\} \end{aligned}$$

Since we still assume that $U_{p,q}(\mathbb{C})$ is the unitary group for the hermitian form $I = \left(\begin{smallmatrix} I_{r-q+m,q-m} & 0\\ 0 & -I_{m,s-m}\end{smallmatrix}\right)$, this implies the assertion of Step III.

Combining Steps I to III implies the assertion of Theorem 2 c). ∎

Proof. (of Theorem 2 d)): Obviously

$$\mathcal{C}(\alpha U_n(\mathbb{H})) = \bigcup_{k=0}^{n} \mathcal{C}_k(\alpha U_n(\mathbb{H})).$$

Notice that $J_0 = -iI_{n-k,k}$ is an element of $\mathcal{C}_k(\alpha U_n(\mathbb{H}))$.

STEP I: $\varphi : \alpha U_n(\mathbb{H})) \to \mathcal{C}_k(\alpha U_n(\mathbb{H})), M \mapsto M^{-1}J_0M$ *is surjective.*

Suppose $J \in \mathcal{C}_k(\alpha U_n(\mathbb{H}))$. Obviously $\mathcal{C}_k(\alpha U_n(\mathbb{H}))$ is a subset of $\mathcal{C}(GL_n(\mathbb{H}))$. Hence J is conjugate to $i\mathbf{1}_n$, i.e.,

$$J = N^{-1}(i\mathbf{1}_n)N \quad \text{for some} \quad N \in GL_n(\mathbb{H}).$$

Now J being an element of $\alpha U_n(\mathbb{H}) = \{M \in GL_n(\mathbb{H}) | M(i\mathbf{1}_n)\,{}^t\overline{M} = i\mathbf{1}_n\}$ implies that $N(i\mathbf{1}_n)\,{}^t\overline{N}$ commutes with $i\mathbf{1}$. So necessarily $N(i\mathbf{1}_n)\,{}^t\overline{N}$ is a matrix with complex entries:

$$N(i\mathbf{1}_n)\,{}^t\overline{N} \in GL_n(\mathbb{C}).$$

Moreover, by definition, $N(i\mathbf{1}_n)\,{}^t\overline{N}$ is skew hermitian. So $AN(i\mathbf{1})\,{}^t\overline{N}\,{}^t\overline{A} = -iI_{n-a,a}$ for some $A \in GL_n(\mathbb{C})$ and an integer $a, 0 \le a \le n$. Using the fact that A commutes with i one sees that

$$\begin{aligned} k = \mathrm{ind}_{\mathbb{H}}(Ji) &= \mathrm{ind}_{\mathbb{H}}(N^{-1}(i\mathbf{1})N(i\mathbf{1})) \\ &= \mathrm{ind}_{\mathbb{H}}((i\mathbf{1})N(i\mathbf{1})\,{}^t\overline{N}) \\ &= \mathrm{ind}_{\mathbb{H}}((i\mathbf{1})AN(i\mathbf{1})\,{}^t\overline{N}\,{}^t\overline{A}) \\ &= \mathrm{ind}_{\mathbb{H}} I_{n-a,a} = a, \end{aligned}$$

i.e. $k = a$. The matrix

$$M := \begin{pmatrix} j\mathbf{1}_{n-k} & 0 \\ 0 & \mathbf{1}_k \end{pmatrix} AN$$

is an element of $\alpha U_n(\mathbb{H})$, because

$$\begin{aligned} M(i\mathbf{1})\,{}^t\overline{M} &= \begin{pmatrix} j\mathbf{1}_{n-k} & 0 \\ 0 & \mathbf{1}_k \end{pmatrix} AN(i\mathbf{1})\,{}^t\overline{N}\,{}^t\overline{A} \begin{pmatrix} -j\mathbf{1}_{n-k} & 0 \\ 0 & \mathbf{1}_k \end{pmatrix} \\ &= \begin{pmatrix} jij\mathbf{1}_{n-k} & 0 \\ 0 & i\mathbf{1}_a \end{pmatrix} = i\mathbf{1}_n. \end{aligned}$$

Moreover,

$$\begin{aligned} \varphi(M) = M^{-1}J_0M &= N^{-1}A^{-1}\begin{pmatrix} -j\mathbf{1}_{n-k} & \\ & \mathbf{1}_k \end{pmatrix}\begin{pmatrix} -i\mathbf{1}_{n-k} & 0 \\ 0 & i\mathbf{1}_k \end{pmatrix}\begin{pmatrix} j\mathbf{1}_{n-k} & \\ & \mathbf{1}_k \end{pmatrix} AN \\ &= N^{-1}(i\mathbf{1})N = J. \end{aligned}$$

This implies the assertion.
STEP II: $\varphi^{-1}(J_0) \simeq U_{n-k,k}(\mathbb{C})$.
Note that

$$\varphi^{-1}(J_0) = \{M \in \alpha U_n(\mathbb{H}) | J_0M = MJ_0\} = \alpha U_n(\mathbb{H}) \cap U_{n-k,k}(\mathbb{H}).$$

But $\alpha U_n(\mathbb{H}) \cap U_{n-k,k}(\mathbb{H})$ is isomorphic to $U_{n-k,k}(\mathbb{C})$. To see this write $U \in U_{n-k,k}(\mathbb{C})$ in the form $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \in M_{n-k}(\mathbb{C}), b$ and ${}^tc \in M((n-k)\times k, \mathbb{C})$ and $d \in M_k(\mathbb{C})$. Then the assignment

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \overline{a} & -jb \\ j\overline{c} & d \end{pmatrix}$$

defines an isomorphism $U_{n-k,k}(\mathbb{C}) \to \alpha U_n(\mathbb{H}) \cap U_{n-k,k}(\mathbb{H})$. ∎

Remark 1.1. In the proof of Theorem 2 d) above it is shown that $\mathcal{C}_k(\alpha U_n(\mathbb{H}))$ is isomorphic to the symmetric space $\alpha U_n(\mathbb{H})/U_{n-k,k}(\mathbb{C})$ where $U_{n-k,k}(\mathbb{C})$ embeds into $\alpha U_n(\mathbb{H})$ by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \overline{a} & -jb \\ j\overline{c} & d \end{smallmatrix}\right)$ with $a \in M_{n-k}(\mathbb{C})$, b, ${}^tc \in M((n-k)\times k, \mathbb{C})$ and $d \in M_k(\mathbb{C})$. Here the respective embeddings for the other cases are collected

$$\mathcal{C}(GL_{2n}(\mathbb{R})) \;:\; GL_n(\mathbb{C}) \hookrightarrow GL_{2n}(\mathbb{R}), A+iB \mapsto \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$$

$$\mathcal{C}(GL_n(\mathbb{H})) \;:\; GL_n(\mathbb{C}) \hookrightarrow GL_n(\mathbb{H}) \text{ induced by } \mathbb{C} \hookrightarrow \mathbb{H}$$

$$\mathcal{C}(O_{2p,2q}(\mathbb{R})) \;:\; U_{p,q}(\mathbb{C}) \hookrightarrow O_{2p,2q}(\mathbb{R}), A+iB \mapsto \left(\begin{smallmatrix} A & -BI_{p,q} \\ I_{p,q}B & I_{p,q}AI_{p,q} \end{smallmatrix}\right),$$

where $O_{2p,2q}(\mathbb{R})$ is the group of isometries for the symmetric bilinear form $\left(\begin{smallmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)$.

$$\mathcal{C}(O_{2n}(\mathbb{C})) \;:\; GL_n(\mathbb{C}) \hookrightarrow O_{2n}(\mathbb{C}),\ \alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{smallmatrix}\right)$$

$$\mathcal{C}(U_{p,q}(\mathbb{H})) \;:\; U_{p,q}(\mathbb{C}) \hookrightarrow U_{p,q}(\mathbb{H}), \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \overline{\alpha} & -j\beta \\ j\overline{\gamma} & \delta \end{smallmatrix}\right)$$

(the same embedding as for $\alpha U_n(\mathbb{H})$).

$$\mathcal{C}(Sp_{2n}(\mathbb{C})) \;:\; GL_n(\mathbb{C}) \hookrightarrow Sp_{2n}(\mathbb{C}),\ \alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{smallmatrix}\right)$$

$$\mathcal{C}_k(Sp_{2n}(\mathbb{R})) \;:\; U_{n-k,k}(\mathbb{C}) \hookrightarrow Sp_{2n}(\mathbb{R}), A+iB \mapsto \left(\begin{smallmatrix} A & -BI_{n-k,k} \\ I_{n-k,k}B & I_{n-k,k}AI_{n-k,k} \end{smallmatrix}\right)$$

$$\mathcal{C}_r(GL_n(\mathbb{C})) \;:\; GL_r(\mathbb{C}) \times GL_s(\mathbb{C}) \hookrightarrow GL_n(\mathbb{C})$$

the natural embedding

$$\mathcal{C}_{k,r}(U_{p,q}(\mathbb{C})) \;:\; U_{r-q+m,q-m}(\mathbb{C}) \times U_{s-m,m}(\mathbb{C}) \hookrightarrow U_{p,q}(\mathbb{C})$$

the natural embedding.

2. Applications

An immediate consequence of Theorems 1 and 2 is the fact that there exist normal forms for complex structures contained in classical groups. To be more precise, for every space $\mathcal{C}_*(G)$ there exists a *normal form* $J_0 \in \mathcal{C}_*(G)$ such that any other complex structure in $\mathcal{C}_*(G)$ is conjugate to J_0 in G. Some normal forms J_0 are listed in the following table

$\mathcal{C}_*(G)$	J_0	$\mathcal{C}_*(G)$	J_0
$\mathcal{C}(GL_{2n}(\mathbb{R}))$	$\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)$	$\mathcal{C}_{k,r}(U_{p,q}(\mathbb{C}))$	$iI_{r,p+q-r}$
$\mathcal{C}_r(GL_n(\mathbb{C}))$	$iI_{r,n-r}$	$\mathcal{C}(U_{p,q}(\mathbb{H}))$	$iI_{p,q}$
$\mathcal{C}(GL_n(\mathbb{H}))$	$i\mathbf{1}_n$	$\mathcal{C}_k(Sp_{2n}(\mathbb{R}))$	$\left(\begin{smallmatrix} 0 & I_{n-k,k} \\ -I_{n-k,k} & 0 \end{smallmatrix}\right)$
$\mathcal{C}(O_{2p,2q}(\mathbb{R}))$	$\left(\begin{smallmatrix} 0 & I_{p,q} \\ -I_{p,q} & 0 \end{smallmatrix}\right)$	$\mathcal{C}(Sp_{2n}(\mathbb{C}))$	$iI_{n,n}$
$\mathcal{C}(O_{2n}(\mathbb{C}))$	$iI_{n,n}$	$\mathcal{C}_k(\alpha U_n(\mathbb{C}))$	$iI_{k,n-k}$

As mentioned already in the introduction, Theorems 1 and 2 admit also applications of purely linear algebraic nature. Here we present only the following two applications, various other results could be deduced in a similar way.

Corollary 2.1. *For $M \in O_{p+q}(\mathbb{C})$ we have*

$$\det \operatorname{Im} \left(M \begin{pmatrix} i\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{pmatrix} \right) = \begin{cases} 0 & q \equiv 1 \pmod 2 \\ \neq 0 & q \equiv 0 \pmod 2). \end{cases}$$

Corollary 2.2. *Let $\tau \in M_{p+q}(\mathbb{C})$ such that $I_{p,q}\tau$ is symmetric. Then $I_{p,q} + {}^t\tau I_{p,q}\overline{\tau}$ is a hermitian form with at most p positive (respectively q negative) eigenvalues, i.e.*

$$I_{p,q} + {}^t\tau I_{p,q}\overline{\tau} = {}^t\alpha I_{p,q}\overline{\alpha} \quad \text{for some} \quad \alpha \in M_{p+q}(\mathbb{C}).$$

The significance of these corollaries lies in the fact that the spaces $\mathcal{C}_*(G)$ can be interpreted as open subspaces of certain Grassmannian varieties. In these terms the statements of the above corollaries are in fact equivalent to the results of Theorem 1 for the groups $O_{p,q}(\mathbb{R})$ and $U_{p,q}(\mathbb{H})$. It seems difficult to prove these corollaries directly.

Proof. (of Corollary 2.1): Write $n = p+q$. Consider the following subvariety of the Grassmannian $\mathrm{Gr}_n(\mathbb{C}^{2n})$ of n-dimensional subvector spaces of $\mathbb{C}^{2n}$:

$$S = \left\{ V \in \mathrm{Gr}_n(\mathbb{C}^{2n}) \;\middle|\; \begin{array}{l} V \text{ is isotropic w.r.t.} \\ \text{the symmetric form } \left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right) \end{array} \right\}.$$

Every $V \in \mathrm{Gr}_n(\mathbb{C}^{2n})$ is of the form $V = \pi(\mathbb{C}^n)$ with $\pi \in M(2n \times n, \mathbb{C})$. Denote $\overline{V} := \overline{\pi}(\mathbb{C}^n)$ the complex conjugate subspace. This definition does not depend on the choice of the matrix π. Let S^0 denote the subset

$$S^0 = \{ V \in S \,|\, V \cap \overline{V} = \{0\} \}.$$

Suppose $V = \pi(\mathbb{C}^n) \in S^0$. Then $V \oplus \overline{V} \simeq \mathbb{C}^{2n}$ and the complex structure of V induces a complex structure J_V on $\mathbb{C}^{2n}$ by :

$$\begin{array}{ccc} V \oplus \overline{V} & \xrightarrow{(\pi,\overline{\pi})} & \mathbb{C}^{2n} \\ \left(\begin{smallmatrix} i\mathbf{1}_n & 0 \\ 0 & -i\mathbf{1}_n \end{smallmatrix}\right) \downarrow & & \downarrow J_V \\ V \oplus \overline{V} & \xrightarrow{(\pi,\overline{\pi})} & \mathbb{C}^{2n}. \end{array}$$

The following computation shows that J_V is a real matrix.

$$\begin{aligned} J_V &= (\pi,\overline{\pi})\left(\begin{smallmatrix} i\mathbf{1}_n & 0 \\ 0 & -i\mathbf{1}_n \end{smallmatrix}\right)(\pi,\overline{\pi})^{-1} \\ &= (\pi,\overline{\pi})\left(\begin{smallmatrix} \mathbf{1}_n & -i\mathbf{1}_n \\ \mathbf{1}_n & i\mathbf{1}_n \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \mathbf{1}_n & -i\mathbf{1}_n \\ \mathbf{1}_n & i\mathbf{1}_n \end{smallmatrix}\right)^{-1}(\pi,\overline{\pi})^{-1} \\ &= (\operatorname{Re}\pi, \operatorname{Im}\pi)\left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & o \end{smallmatrix}\right)(\operatorname{Re}\pi, \operatorname{Im}\pi)^{-1}. \end{aligned}$$

On the other hand, using the fact that V is isotropic with respect to $\left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)$, i.e. ${}^t\pi\left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)\pi = 0$, it is easy to see that J_V is orthogonal with respect to the symmetric bilinear form $\left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)$. So $J_V \in \mathcal{C}(O_{2n-q,q}(\mathbb{R}))$. The assignment $V \to J_V$ defines a map $S^0 \to \mathcal{C}(O_{2n-q,q}(\mathbb{R}))$ which is equivariant with respect to the natural action of $O_{2n-q,q}(\mathbb{R})$ on both spaces. Obviously the map $S^0 \to \mathcal{C}(O_{2n-q,q}(\mathbb{R}))$ is injective. Hence Theorem 1 says that

$$S^0 \simeq \begin{cases} \mathcal{C}(O_{2n-q,q}(\mathbb{R})) & q \equiv 0 (mod 2) \\ & \text{if} \\ \emptyset & q \equiv 1 (mod 2). \end{cases}$$

Now suppose $M \in O_n(\mathbb{C})$. Then M defines an element $V_M \in S$ by

$$V_M := \pi_M(\mathbb{C}^n) \quad \text{with} \quad \pi_M := \begin{pmatrix} M\left(\begin{smallmatrix} i\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{smallmatrix}\right) \\ \mathbf{1}_n \end{pmatrix}.$$

It is easy to see that $V_M \in S^0$ if and only if $(\pi_M, \overline{\pi_M})$ or equivalently if the matrix $(\operatorname{Re} \pi_M, \operatorname{Im} \pi_M)$ is invertible. But

$$(\operatorname{Re} \pi_M, \operatorname{Im} \pi_M) = \begin{pmatrix} \operatorname{Re}\left(M\left(\begin{smallmatrix} i\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{smallmatrix}\right)\right) & \operatorname{Im}\left(M\left(\begin{smallmatrix} i\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{smallmatrix}\right)\right) \\ \mathbf{1}_n & 0 \end{pmatrix}.$$

So $V_M \in S^0$ if and only if $\det \operatorname{Im}(M\left(\begin{smallmatrix} i\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{smallmatrix}\right)) \neq 0$. This implies the assertion. ■

Proof. (of Corollary 2.2). Write $n = p + q$. Consider the following subvariety of the Grassmannian $\operatorname{Gr}_n(\mathbb{C}^{2n})$:

$$T := \left\{ V \in \operatorname{Gr}_n(\mathbb{C}^{2n}) \;\middle|\; \begin{array}{l} V \quad \text{is isotropic w. r. t.} \\ \text{the alternating form } \left(\begin{smallmatrix} 0 & I_{p,q} \\ -I_{p,q} & 0 \end{smallmatrix}\right) \end{array} \right\}.$$

Suppose $V = \pi(\mathbb{C}^n) \in \operatorname{Gr}_n(\mathbb{C}^{2n})$, with $\pi \in M(2n \times n, \mathbb{C})$. Define $V^0 \in \operatorname{Gr}_n(\mathbb{C}^{2n})$ by $V^0 := \left(\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)\overline{\pi}(\mathbb{C}^n)$. Note that this definition does not depend on the choice of π. The set

$$T^0 := \{ V \in T \mid V \cap V^0 = \{0\} \}$$

is an open subset of T.

STEP I. $T^0 \simeq \mathcal{C}(U_{p,q}(\mathbb{H}))$.

Suppose $V = \pi(\mathbb{C}^n) \in T^0$ and write $\pi = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)$. Since $V \oplus V^0 \simeq \mathbb{C}^{2n}$, the matrix $(\pi, \left(\begin{smallmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{smallmatrix}\right)\overline{\pi}) = \left(\begin{smallmatrix} u & -\overline{v} \\ v & \overline{u} \end{smallmatrix}\right)$ is invertible. Recall the natural complex representation of quaternion matrices

$$\rho : GL_n(\mathbb{H}) \hookrightarrow GL_{2n}(\mathbb{C}), \quad A + Bj \mapsto \left(\begin{smallmatrix} A & B \\ -\overline{B} & \overline{A} \end{smallmatrix}\right).$$

This shows that $u + vj = {}^t({}^tu + {}^tvj)$ is an element of $GL_n(\mathbb{H})$. Define $J_V := (u + vj)(i\mathbf{1})(u + vj)^{-1}$. The fact that V is isotropic with respect to $\left(\begin{smallmatrix} 0 & I_{p,q} \\ -I_{p,q} & 0 \end{smallmatrix}\right)$ means in terms of u and v that ${}^tuI_{p,q}v - {}^tvI_{p,q}u = 0$. Using this one easily sees that ${}^tJ_VI_{p,q}\overline{J}_V = I_{p,q}$, i.e., $J \in U_{p,q}(\mathbb{H})$. The assignment $V \mapsto J_V$ defines a map $T^0 \to \mathcal{C}(U_{p,q}(\mathbb{H}))$. Note that

$$\rho(U_{p,q}(\mathbb{H})) = \left\{ M = \left(\begin{smallmatrix} A & B \\ -\overline{B} & \overline{A} \end{smallmatrix}\right) \in SL_{2n}(\mathbb{C}) | M\left(\begin{smallmatrix} 0 & I_{p,q} \\ -I_{p,q} & 0 \end{smallmatrix}\right){}^tM = \left(\begin{smallmatrix} 0 & I_{p,q} \\ -I_{p,q} & 0 \end{smallmatrix}\right) \right\}$$

So $U_{p,q}(\mathbb{H})$ acts on T via the representation ρ. By definition the map $T^0 \to \mathcal{C}(U_{p,q}(\mathbb{H}))$ is equivariant with respect to the action of $U_{p,q}(\mathbb{H})$ on both spaces. Moreover it is easy to see that $T^0 \to \mathcal{C}(U_{p,q}(\mathbb{H}))$ is injective. Since by Theorem 1 the action of $U_{p,q}(\mathbb{H})$ on $\mathcal{C}(U_{p,q}(\mathbb{H}))$ is transitive, $T^0 \to \mathcal{C}(U_{p,q}(\mathbb{H}))$ is also surjective.

STEP II: Consider the hermitian form $H := \left(\begin{smallmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{smallmatrix}\right)$ on $\mathbb{C}^{2n}$. For every $V = \pi(\mathbb{C}^n) \in T^0$ the pull back of H to $V \oplus V^0$ is nondegenerate implying that $H|V = {}^t\pi H \overline{\pi}$ is nondegenerate. By Step I the set T^0 is a connected homogeneous space, so the map $T^0 \to \mathbb{Z}$, $V \mapsto \operatorname{ind}(H|V)$ is constant. Note that $V_0 = \left(\begin{smallmatrix} \mathbf{1} \\ 0 \end{smallmatrix}\right)\mathbb{C}^n = \mathbb{C}^n \times \{0\}^n$ is an element of T^0 and that $H|V_0$ is given by the matrix $I_{p,q}$. Hence $\operatorname{ind}(H|V) = q$ for all $V \in T_0$. Suppose $\tau \in M_n(\mathbb{C})$ with $\tau I_{p,q}$ symmetric. Then $V_\tau = \left(\begin{smallmatrix} \mathbf{1} \\ \tau \end{smallmatrix}\right)\mathbb{C}^n$ is an element of T and

$$H|V_\tau = I_{p,q} + {}^t\tau I_{p,q}\overline{\tau}.$$

Since T^0 is an open subset of T this implies the assertion. ∎

3. Flag Domains

Let $G_{\mathbb{C}}$ be a connected reductive complex Lie group. Given a parabolic subgroup P of $G_{\mathbb{C}}$ the projective variety $G_{\mathbb{C}}/P$ is called *complex flag manifold.* Any real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ acts on $G_{\mathbb{C}}/P$ in a natural way. There exist only finitely many $G_{\mathbb{R}}$-orbits in $G_{\mathbb{C}}/P$. The connected components of the open $G_{\mathbb{R}}$-orbits are called *flag domains* (see [3]). Flag domains are used in Representation Theory in order to describe certain representations of $G_{\mathbb{R}}$. To be more precise, the Dolbeault cohomology groups of certain homogeneous line bundles on flag domains of $G_{\mathbb{R}}$ yield the discrete series of representations of $G_{\mathbb{R}}$. The geometric properties of flag domains were studied by J. A. Wolf, W. Schmid and others. For example flag domains are always simply connected. Moreover a flag domain $X = G_{\mathbb{R}}/H$ is $(s+1)$-complete in the sense of Andreotti and Grauert, where s is the maximum of the dimensions of all compact complex subvarieties of X. This is shown in [2] in the case of compact H, if H is reductive this is proved in [4]. In [5] J. A. Wolf presents a proof of the general case. The $(s+1)$-completeness implies in particular that $H^q(X, \mathcal{F}) = 0$ for all $q > s$ and every coherent sheaf $\mathcal{F}$ on X.

The classical groups G of above are not necessarily connected. They consist of one (for $G = GL_n(\mathbb{C})$, $GL_n(\mathbb{H})$, $U_{p,q}(\mathbb{C})$, $U_{p,q}(\mathbb{H})$, $Sp_{2n}(\mathbb{R})$, $Sp_{2n}(\mathbb{C})$ or $\alpha U_n(\mathbb{H})$), two (for $G = GL_n(\mathbb{R}), O_n(\mathbb{R}), O_n(\mathbb{C})$) respectively four (for $G = O_{p,q}(\mathbb{R})$) components. Denote by n_G the number of connected components of G. The group G can be considered as a real form of a connected reductive complex Lie group $G_{\mathbb{C}}$. Using Theorems 1 and 2 we show

Proposition 3.1. *The spaces $\mathcal{C}_*(G)$ of Theorems 1 and 2 are disjoint unions of n_G flag domains for the real Lie group G.*

Proof. Proof for $G = GL_{2n}(\mathbb{R})$: $GL_{2n}(\mathbb{R})$ consists of two components. Accordingly the symmetric space $\mathcal{C}(GL_{2n}(\mathbb{R}))$ has two connected components, both isomorphic to $SL_{2n}(\mathbb{R})/GL_n(\mathbb{C})\cap SL_{2n}(\mathbb{R})$. The group $GL_{2n}(\mathbb{R})$ is a real form of the connected reductive complex Lie group $GL_{2n}(\mathbb{C})$. Consider the parabolic subgroup $P = \{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_n & D \end{smallmatrix}\right) \in GL_{2n}(\mathbb{C})\}$. An immediate computation shows that the embedding $GL_{2n}(\mathbb{R}) \hookrightarrow GL_{2n}(\mathbb{C}), M \mapsto \left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ i\mathbf{1}_n & \mathbf{1}_n \end{smallmatrix}\right) M \left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ i\mathbf{1}_n & \mathbf{1}_n \end{smallmatrix}\right)^{-1}$ induces an embedding of $GL_{2n}(\mathbb{R})/GL_n(\mathbb{C})$ into $GL_{2n}(\mathbb{C})/P$. Since both manifolds are of dimension n^2, the image of $GL_{2n}(\mathbb{R})/GL_n(\mathbb{C})$ in $GL_{2n}(\mathbb{C})/P$ is an open $GL_{2n}(\mathbb{R})$-orbit, i.e., a flag domain. ∎

As for the proofs of the other cases we only give the following list of corresponding embeddings $G \hookrightarrow G_{\mathbb{C}}$, flag manifolds, and flag domains:

$\underline{\mathcal{C}_r(GL_n(\mathbb{C}))\ :}$

$GL_n(\mathbb{C}) \hookrightarrow GL_n(\mathbb{C}) \times GL_n(\mathbb{C}), M \mapsto (M, {}^t\overline{M}^{-1})$,
Flag manifold: $GL_n(\mathbb{C})^2/\{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_{s,r} & D \end{smallmatrix}\right) \in GL_n(\mathbb{C})\}^2$,
Flag domain: $\mathcal{C}_r(GL_n(\mathbb{C})) \simeq GL_n(\mathbb{C})/(GL_r(\mathbb{C}) \times GL_s(\mathbb{C}))$.

$\underline{\mathcal{C}(GL_n(\mathbb{H}))\ :}$

$GL_n(\mathbb{H}) \hookrightarrow GL_{2n}(\mathbb{C}),\ A + Bj \mapsto \left(\begin{smallmatrix} A & B \\ -\overline{B} & \overline{A} \end{smallmatrix}\right)$,
Flag manifold: $GL_{2n}(\mathbb{C})/\{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_n & D \end{smallmatrix}\right) \in GL_{2n}(\mathbb{C})\}$,
Flag domain: $\mathcal{C}(GL_n(\mathbb{H})) \simeq GL_n(\mathbb{H})/GL_n(\mathbb{C})$.

$\underline{\mathcal{C}(O_{2p,2q}(\mathbb{R}))\ :}$

$O_{2p,2q}(\mathbb{R}) \hookrightarrow O_{2(p+q)}(\mathbb{C}), M \mapsto M_0^{-1} M M_0$ with
$M_0 := \left(\begin{smallmatrix} \frac{1}{2}X & X \\ \frac{i}{2}X & -iX \end{smallmatrix}\right)$, and $X := \left(\begin{smallmatrix} i\mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q \end{smallmatrix}\right)$,
Flag manifold: $O_{2(p+q)}(\mathbb{C})/\left\{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_{p+q} & D \end{smallmatrix}\right) \in O_{2(p+q)}(\mathbb{C})\right\}$,
Flag domains: the two (resp. four) components of $\mathcal{C}(O_{2p,2q}(\mathbb{R}))$.

$\underline{\mathcal{C}(O_{2n}(\mathbb{C}))\ :}$

$O_{2n}(\mathbb{C}) \hookrightarrow O_{2n}(\mathbb{C}) \times O_{2n}(\mathbb{C}), M \mapsto (M, {}^t\overline{M}^{-1})$,
Flag manifold: $O_{2n}(\mathbb{C})^2/\left\{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_n & D \end{smallmatrix}\right) \in O_{2n}(\mathbb{C})\right\}^2$,
Flag domains: the two components of $\mathcal{C}(O_{2n}(\mathbb{C}))$.

$\underline{\mathcal{C}_{k,r}(U_{p,q}(\mathbb{C}))\ :}$

$U_{p,q}(\mathbb{C}) \hookrightarrow GL_{p+q}(\mathbb{C}), M \mapsto M$,
Flag manifold: $GL_{p+q}(\mathbb{C})/\left\{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_{s,r} & D \end{smallmatrix}\right) \in GL_{p+q}(\mathbb{C})\right\}$,
Flag domain: $\mathcal{C}_{k,r}(U_{p,q}(\mathbb{C})) \simeq U_{p,q}(\mathbb{C})/(U_{r-q+m,q-m}(\mathbb{C}) \times U_{s-m,m}(\mathbb{C}))$.

$\underline{\mathcal{C}(U_{p,q}(\mathbb{H}))\ :}$

$U_{p,q}(\mathbb{H}) \hookrightarrow Sp_{2(p+q)}(\mathbb{C}), A + Bj \mapsto P^{-1}\left(\begin{smallmatrix} A & BI_{p,q} \\ -I_{p,q}\overline{B} & I_{p,q}\overline{A}I_{p,q} \end{smallmatrix}\right)\ P$ with

$$P = \begin{pmatrix} 0 & 0 & \mathbf{1}_p & 0 \\ 0 & \mathbf{1}_q & 0 & 0 \\ \mathbf{1}_p & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_q \end{pmatrix}$$

Flag manifold: $Sp_{2(p+q)}(\mathbb{C})/\left\{\left(\begin{smallmatrix} A & B \\ \mathbf{0}_{p+q} & D \end{smallmatrix}\right) \in Sp_{2(p+q)}(\mathbb{C})\right\}$,
Flag domain: $\mathcal{C}(U_{p,q}(\mathbb{H})) \simeq U_{p,q}(\mathbb{H})/U_{p,q}(\mathbb{C})$.

$\underline{\mathcal{C}_k(Sp_{2n}(\mathbb{R})):}$

$Sp_{2n}(\mathbb{R}) \hookrightarrow Sp_{2n}(\mathbb{C}), M \mapsto \left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ iI_{p,q} & \mathbf{1}_n \end{smallmatrix}\right) M \left(\begin{smallmatrix} \mathbf{1}_n & 0 \\ iI_{p,q} & \mathbf{1}_n \end{smallmatrix}\right)^{-1}$,
Flag manifold: $Sp_{2n}(\mathbb{C}) / \left\{ \left(\begin{smallmatrix} A & B \\ \mathbf{0}_n & D \end{smallmatrix}\right) \in Sp_{2n}(\mathbb{C}) \right\}$,
Flag domain: $\mathcal{C}_k(Sp_{2n}(\mathbb{R})) \simeq Sp_{2n}(\mathbb{R})/U_{n-k,k}(\mathbb{C})$.

$\underline{\mathcal{C}(Sp_{2n}(\mathbb{C})):}$

$Sp_{2n}(\mathbb{C}) \hookrightarrow Sp_{2n}(\mathbb{C}) \times Sp_{2n}(\mathbb{C}), M \mapsto (M, {}^t\overline{M}^{-1})$,
Flag manifold: $Sp_{2n}(\mathbb{C})^2 / \left\{ \left(\begin{smallmatrix} A & B \\ \mathbf{0}_n & D \end{smallmatrix}\right) \in Sp_{2n}(\mathbb{C}) \right\}^2$.
Flag domain: $\mathcal{C}(Sp_{2n}(\mathbb{C})) \simeq Sp_{2n}(\mathbb{C})/GL_n(\mathbb{C})$.

$\underline{\mathcal{C}_k(\alpha U_n(\mathbb{H})):}$

$\alpha U_n(\mathbb{H}) \hookrightarrow O_{2n}(\mathbb{C}), A + Bj \mapsto P^{-1} \left(\begin{smallmatrix} A & B \\ -\overline{B} & \overline{A} \end{smallmatrix}\right) P$ with

$$P = \begin{pmatrix} 0 & 0 & \mathbf{1}_k & 0 \\ 0 & \mathbf{1}_{n-k} & 0 & 0 \\ \mathbf{1}_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-k} \end{pmatrix}$$

Flag manifold: $O_{2n}(\mathbb{C}) / \left\{ \left(\begin{smallmatrix} A & B \\ \mathbf{0}_n & D \end{smallmatrix}\right) \in O_{2n}(\mathbb{C}) \right\}$.
Flag domain: $\mathcal{C}_k(\alpha U_n(\mathbb{H})) = \alpha U_n(\mathbb{H})/U_{n-k,k}(\mathbb{C})$.

4. Families of Nondegenerate Complex Tori Associated to Spaces of Complex Structures

A *polarized abelian variety* of dimension n is a pair (X, H) with a complex torus $X = \mathbb{C}^n/\Lambda$ (Λ is a lattice in $\mathbb{C}^n$) and a positive definite hermitian form H on $\mathbb{C}^n$ such that $\operatorname{Im} H(\Lambda \times \Lambda) \subseteq \mathbb{Z}$. More generally a *nondegenerate complex torus of index k* is a pair (X, H) with a complex torus $X = \mathbb{C}^n/\Lambda$ and a nondegenerate hermitian form H on $\mathbb{C}^n$ with $\operatorname{Im} H(\Lambda \times \Lambda) \subseteq \mathbb{Z}$ such that $\operatorname{ind}_{\mathbb{C}}(H) = k$ (for more details see [1]). It is well-known that the space $\mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ of complex structures of index 0 in the real symplectic group parametrizes polarized abelian varieties of dimension n:
For $J \in \mathcal{C}_0(Sp_{2n}(\mathbb{R}))$ the pair $V_J := (\mathbb{R}^{2n}, J)$ is a complex vector space of dimension n and $X_J := V_J/\mathbb{Z}^{2n}$ is a complex torus. The matrix ${}^tJ \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right) + i \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)$ defines a positive definite hermitian form H_J on V_J such that $\operatorname{Im} H_J$ is integral valued on the lattice $\mathbb{Z}^{2g}$. Thus (X_J, H_J) is a polarized abelian variety. Obviously the same construction applied to $J \in \mathcal{C}_k(Sp_{2n}(\mathbb{R}))$ gives a nondegenerate complex torus (X_J, H_J) of index k. It can be shown that conversely every nondegenerate complex torus of index k and dimension n is of the form (X_J, H_J) (see [1]).
In a similar way any of the spaces $\mathcal{C}_*(G)$ parametrizes nondegenerate complex tori (X, H) with some extra condition on the endomorphism algebra of (X, H). We exhibit this construction for $G = Sp_{2n}(\mathbb{C})$ and $U_{p,q}(\mathbb{H})$. A similar but in the notation considerably more complicated construction applies to every space $\mathcal{C}_*(G)$.

$\underline{\mathcal{C}(Sp_{2n}(\mathbb{C}))}$**:**

Suppose $J \in \mathcal{C}(Sp_{2n}(\mathbb{C}))$. Considering $\mathbb{C}^{2n}$ as a real vector space the pair $V_J := (\mathbb{C}^{2n}, J)$ is again a complex vector space of dimension $2n$. Let H_J denote the $\mathbb{R}$-bilinear form

$$H_J : \begin{cases} \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}, \\ (v, w) \mapsto tr_{\mathbb{C}/\mathbb{R}}({}^tv\, {}^tJ \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right) w) + i\, tr_{\mathbb{C}/\mathbb{R}}({}^tv \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right) w). \end{cases}$$

It is easy to see that H_J is in fact a nondegenerate hermitian form on the complex vector space V_J. According to Theorem 1 the space $\mathcal{C}(Sp_{2n}(\mathbb{C}))$ is connected and thus the index of H_J equals the index of H_{J_0}, where $J_0 = iI_{n,n}$ (see Section 2). Using[6] this shows that $\mathrm{ind}_{\mathbb{C}} H_J = \mathrm{ind}_{\mathbb{C}} H_{J_0} = \frac{1}{2}\mathrm{ind}_{\mathbb{R}}\mathrm{Re}\, H_{J_0} = n$.
Suppose F is an imaginary quadratic field and let $\mathcal{M} \subset F^{2n}$ be a $\mathbb{Z}$-module of rank $4n$ with $tr_{F/\mathbb{Q}}({}^t a \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right) b) \in \mathbb{Z}$ for all $a, b \in \mathcal{M}$. Identify $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{C}^2$, then $\mathcal{M} \subset (F \otimes_{\mathbb{Q}} \mathbb{R})^{2n} = \mathbb{C}^{2n}$ is a lattice and

$$(X_J = V_J/\mathcal{M}, H_J)$$

is a nondegenerate complex torus of dimension $2n$ and index n. In particular by construction the endomorphism algebra $\mathrm{End}(X_J) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains F, i.e., (X_J, H_J) *admits multiplication in* F (for the definition of endomorphism algebra and multiplication in F see [1].) □

$\mathcal{C}(U_{p,q}(\mathbb{H}))$**:**
Suppose $J \in \mathcal{C}(U_{p,q}(\mathbb{H}))$. Considering $\mathbb{H}^n = \mathbb{H}^{p+q}$ as a real vector space the pair $V_J := (\mathbb{H}^n, J)$ is a complex vector space of dimension $2n$. Consider the anti-involution $q \mapsto q' = i\overline{q}i^{-1}$ on $\mathbb{H}$. Then the $\mathbb{R}$-bilinear form

$$H_J : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{C}, H_J(v, w) := tr_{\mathbb{H}/\mathbb{R}}({}^t v({}^t J i I_{p,q})w') + i tr_{\mathbb{H}/\mathbb{R}}({}^t v(iI_{p,q})w')$$

defines a nondegenerate hermitian form on the complex vector space V_J. According to Theorem 1 the space $\mathcal{C}(U_{p,q}(\mathbb{H}))$ is connected and thus the index of H_J equals the index of H_{J_0} with normal form $J_0 = iI_{p,q}$. Now an immediate computation shows that $\mathrm{ind}_{\mathbb{C}} H_{J_0} = n$.
Let F be a definite quaternion algebra over $\mathbb{Q}$ with anti-involution $'$. It is easy to see that F can be embedded into $\mathbb{H}$ in such a way that the anti-involution $'$ extends either to quaternion conjugation on $\mathbb{H}$ or the anti-involution $q' = i\overline{q}i^{-1}$ of above. Let us assume the latter case. Let $\mathcal{M} \subset F^n$ be a free $\mathbb{Z}$-submodule of rank $4n$ and $T \in M_n(F)$ a nondegenerate matrix satisfying ${}^t T' = -T$ such that $tr_{F/\mathbb{Q}}({}^t v T w') \in \mathbb{Z}$ for all $v, w \in \mathcal{M}$. For simplicity we assume here that $T = iI_{p,q}$. Then $\mathcal{M} \subset F^n \subset \mathbb{H}^n$ is a lattice and

$$(X_J := V_J/\mathcal{M},\ H_J)$$

is a nondegenerate complex torus of dimension $2n$ and index n. Moreover by construction the endomorphism algebra $\mathrm{End}(X_J) \otimes_{\mathbb{Z}} \mathbb{Q}$ contains F such that the Rosati-involution (see [1]) restricts to the anti-involution $'$ on F, i.e., (X_J, H_J) admits *multiplication in* $(F, ')$. □

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[6] For $M = A + iB \in M_n(\mathbb{C})$ and $v = v_1 + iv_2, w = w_1 + iw_2 \in \mathbb{C}^n$ one has $tr_{\mathbb{C}/\mathbb{R}}({}^t v M w) = 2\,{}^t\left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)\left(\begin{smallmatrix} A & -B \\ -B & -A \end{smallmatrix}\right)\left(\begin{smallmatrix} w_1 \\ w_2 \end{smallmatrix}\right)$.

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Received February 20, 1998
and in final form October 22, 1998