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Projective subspaces in the variety of normal sections and tangent spaces to a symmetric space

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Abstract. In the present article we continue the study of the variety $X[M]$ associated to pointwise planar normal sections of a natural imbedding for a flag manifold M.

When $M = G/T$ is the manifold of complete flags of a compact simple Lie group G, we obtain two results about subspaces of the tangent space $T_{[T]}(M)$, invariant by the torus action, which give rise to real projective spaces in $X[M]$. The first result determines their maximal dimension. While the other one characterizes those of maximal dimension as tangent spaces to the inner symmetric space G/K (the one of largest dimension for the group G) at a fixed point of the natural action of the torus T.

The last section contains a nice application of these results.

1. Introduction

In the present article we continue our study of the variety $X[M]$ of directions of pointwise planar normal sections for a manifold of complete flags M. The nature of this variety gives information about the extrinsic geometry of a natural imbedding of a flag manifold. For instance, as we saw in [5], this variety happens to be a projective space if and only if M is an extrinsic symmetric submanifold.

In previous papers [5] and [6] we have gotten some results that, we fell, help to understand the nature of these varieties. In [5] we found a characterization of the set of pointwise planar normal sections in terms of the tensors α and D on a general flag manifold M which indicates that this set is in fact a real algebraic variety. In that paper we also computed its Euler-Poincaré characteristic showing that it depends only on the dimension and hence gives little information about the geometric nature of the flag manifold M.

In [6] we studied the presence of projective subspaces in the variety $X[M]$, for a manifold of complete flags $M = G_u/T$. The existence of these subspaces indicates that the variety $X[M]$ is rather special. The main result of that paper

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describes a natural family of maximal projective subspaces contained in $X[M]$. On the other hand [6, Th. 4.2] (Theorem 2.2 in the present paper) characterizes those subspaces of the tangent space $T_{[T]}(M)$, invariant by the torus action, which give rise to real projective spaces contained in $X[M]$. This result motivated our interest in a deeper study of these subspaces and gave rise to the present paper.

Here we present the following two theorems on the variety $X[M]$, for $M = G_u/T$ *manifold of complete flags*, and an interesting consequence of them. In both of them G_u denotes a *compact simply connected simple Lie group.*

The meaning of the name *presymmetric set* is explained at the end of the next section where all our notation is presented. Our first result identifies the largest possible cardinality for these sets of positive roots and the second one shows that each one of them gives the tangent space, at a specific point, of the inner symmetric space G_u/K (the one of largest dimension $d(G_u)$ for the group G_u). These results show an interesting connection of the variety $X[M]$ and the geometry of the natural fibration $G_u/T \to G_u/K$. The result obtained as application of these facts (Theorem 4.1) is a nice expression of this connection.

Theorem 1.1. *For each G_u, the number $\frac{1}{2}d(G_u)$ is the largest possible cardinality for a presymmetric set.*

This means that if $\widetilde{\mathfrak{p}} \subset T_{[T]}(M)$ is invariant by the natural action of the torus T and gives rise to a projective subspace of $X[M]$ then $\dim \widetilde{\mathfrak{p}} \leq d(G_u)$.

Theorem 1.2. *For each G_u and each presymmetric set $\widetilde{\Delta}$ whose cardinality is $\frac{1}{2}d(G_u)$, the subspace $\widetilde{\mathfrak{p}} = \sum_{\gamma\in\widetilde{\Delta}} \mathfrak{m}_\gamma$ is tangent to the inner symmetric space G_u/K at a fixed point of the action of the torus T.*

It is clear that every tangent space to G_u/K at a fixed point of the torus action is of this form, i.e. comes from a presymmetric set of roots. So the last Theorem characterizes the tangent spaces to the inner symmetric space G_u/K at those points.

Notice that the hypothesis of Theorem 1.2 mean that $\widetilde{\mathfrak{p}} \subset T_{[T]}(M)$ is invariant by the natural action of the torus T, its dimension is $d(G_u)$ and it gives rise to a projective subspace of $X[M]$.

This paper is organized as follows. In the next section we include the basic facts obtained in [5] and [6] as well as notations used throughout the paper. In Section 3 we study presymmetric sets of roots and obtain the proofs of Theorems 1.1 and 1.2. In Section 4 we present the mentioned application of these results.

2. Basic facts

In the present section we introduce some of the basic notation to be used throughout the paper. All unexplained notation will have the same meaning as in [5].

Let G be a simply connected, complex, simple Lie group and let $\mathfrak{g}$ be its Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. We may write

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\gamma\in\Delta^+} \mathfrak{g}_\gamma \oplus \mathfrak{g}_{-\gamma}$$

where Δ^+ indicates the set of positive roots with respect to some order.

Let us consider in $\mathfrak{g}$ the Borel subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\gamma \in \Delta^+} \mathfrak{g}_{-\gamma}.$$

Let B be the analytic subgroup of G corresponding to the subalgebra $\mathfrak{b}$. B is closed and its own normalizer in G. The quotient space $M = G/B$ is a complex homogeneous space called the manifold of complete flags of G.

Let $\pi = \{\alpha_1, ..., \alpha_n\} \subset \Delta^+$ be a system of simple roots. We may take in $\mathfrak{g}$ a Weyl basis [11, III, 5] $\{X_\gamma : \gamma \in \Delta\}$ and $\{H_\beta : \beta \in \pi\}$. The following set of vectors provides a basis of a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$.

$$\begin{cases} U_\gamma = \frac{1}{\sqrt{2}} (X_\gamma - X_{-\gamma}) & \gamma \in \Delta^+ \\ U_{-\gamma} = \frac{i}{\sqrt{2}} (X_\gamma + X_{-\gamma}) & \gamma \in \Delta^+ \\ iH_\beta & \beta \in \pi. \end{cases} \tag{1}$$

We shall denote by $\mathfrak{h}_u$ the real vector space generated by $\{iH_\beta : \beta \in \pi\}$ and by $\mathfrak{m}_\gamma$ that of $\{U_\gamma, U_{-\gamma}\}$. Then we may write $\mathfrak{g}_u = \mathfrak{h}_u \oplus \sum_{\gamma \in \Delta^+} \mathfrak{m}_\gamma = \mathfrak{h}_u \oplus \mathfrak{m}$.

Let G_u be the analytic subgroup of G corresponding to $\mathfrak{g}_u$. G_u is compact and acts transitively on M which can be written as

$$M = G_u/T$$

where the subgroup $T = G_u \cap B = \exp \mathfrak{h}_u$ is a maximal torus in G_u. The manifold M is then a compact simply connected complex manifold and it is well known that it is the orbit of a regular element $E \in \mathfrak{g}_u$ by the adjoint action of G_u on $\mathfrak{g}_u$. Then we have a natural embedding of M on $\mathfrak{g}_u$ which we may assume isometric by taking in $\mathfrak{g}_u$ the inner product given by the opposite of the Killing form.

For this embedding we consider the algebraic variety $X[M]$ of directions of pointwise planar normal sections of the flag manifold M which was introduced in [5].

Associated to each simple group G_u we have its family of symmetric spaces of type I [11, p. 518] and among them, those which are inner (i.e. the spaces in which the symmetry at each point belongs to the group G_u). These are, among all symmetric spaces, the only ones that are related with our algebraic variety $X[M]$, as we saw in [6]. They are those of the form G_u/K, where K is a subgroup of maximal rank in G_u. The ones which are not inner in the list in [11, p. 518] are AI, AII, BDI $(p + q = 2n,\ p\ odd,\ 1 \leq p \leq n)$, EI and EIV.

By conjugating K if necessary, we may assume that K contains T.

Let $\mathfrak{k}$ be the Lie algebra of K and write $\mathfrak{g}_u = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form. Then $\mathfrak{h}_u \subset \mathfrak{k}$ and $\mathfrak{p} \subset \mathfrak{m}$.

Now we recall a few results from [6] which we need in the present paper. We shall denote by $RP(\mathfrak{q})$ the real projective space associated to a real vector space $\mathfrak{q}$.

Proposition 2.1. *[6, 4.1] The tangent space $\mathfrak{p}$ of the inner symmetric space G_u/K at $[K]$ gives rise to a projective space $RP(\mathfrak{p})$ contained in $X[M]$.*

The subspaces $\widetilde{\mathfrak{p}} \subset \mathfrak{m}$ invariant by the natural action of the torus T (see [5]) are those of the form $\widetilde{\mathfrak{p}} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma$ where $\widetilde{\Delta}$ is any subset of Δ^+ (an example of this subspace is $\mathfrak{p}$ of above proposition). For these subspaces we have the following results.

Theorem 2.2. *[6, 4.2] If $\widetilde{\mathfrak{p}} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma$ with $\widetilde{\Delta} \subset \Delta^+$ then*

$$RP(\widetilde{\mathfrak{p}}) \subset X[M] \Longleftrightarrow \left(\varepsilon,\ \rho \in \widetilde{\Delta} \Rightarrow \varepsilon + \rho \notin \widetilde{\Delta}\right).$$

The condition $\varepsilon,\ \rho \in \widetilde{\Delta} \Rightarrow \varepsilon + \rho \notin \widetilde{\Delta}$ is weaker than the one usually given in the definition of symmetric space. For this reason, in order to simplify our notation, a subset $\widetilde{\Delta} \subset \Delta^+$ with this property will be called a *presymmetric set.*

For a set A, $\#(A)$ will denote its cardinality.

Theorem 2.3. *[6, 4.3] Let $\mathfrak{p}$ be the tangent space of the inner symmetric space G_u/K at $[K]$. Then $RP(\mathfrak{p})$ is maximal among the projective spaces $RP(\widetilde{\mathfrak{p}})$ contained in $X[M]$, with $\widetilde{\mathfrak{p}}$ of the form $\widetilde{\mathfrak{p}} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma$ for $\widetilde{\Delta} \subset \Delta^+$.*

3. Presymmetric sets

The list of irreducible symmetric spaces [11, p. 518] indicates that the irreducible inner symmetric spaces of maximal dimension are those included in the following table, with their respective dimensions.

Table I

$\mathfrak{g}$	name	$N = G_u/K$		dim$N$
$\mathfrak{a}_l$	AIII	$SU(l+1)/S(U(k+1) \times U(k))$	$l = 2k$	$\frac{1}{2}l(l+2)$
		$SU(l+1)/S(U(k+1) \times U(k+1))$	$l = 2k+1$	$\frac{1}{2}(l+1)^2$
$\mathfrak{b}_l$	BDI	$SO(2l+1)/SO(l+1) \times SO(l)$		$l(l+1)$
$\mathfrak{c}_l$	CI	$Sp(l)/U(l)$		$l(l+1)$
$\mathfrak{d}_l$	BDI	$SO(2l)/SO(l) \times SO(l)$	l *even*	l^2
		$SO(2l)/SO(l+1) \times SO(l-1)$	l *odd*	$l^2 - 1$
$\mathfrak{e}_6$	EII	$E_6/SU(6)Sp(1)$		40
$\mathfrak{e}_7$	EV	$E_7/(SU(8)/Z_2)$		70
$\mathfrak{e}_8$	EVIII	$E_8/(Spin(16)/Z_{2)}$		128
$\mathfrak{f}_4$	FI	$F_4/Sp(3)Sp(1)$		28
$\mathfrak{g}_2$	G	$G_2/SO(4)$		8

We shall denote by $d(G_u)$ the dimension $\dim(G_u/K)$ indicated in the table.

From Proposition 2.1 and Theorem 2.2 we conclude that if $\mathfrak{p} = \sum_{\gamma \in \widetilde{\Delta}} \mathfrak{m}_\gamma$ is the tangent space at $[K]$, for each one of the spaces in the table, then $\widetilde{\Delta}$ is a presymmetric set. It is also clear that the cardinality of $\widetilde{\Delta}$ is $\#\left(\widetilde{\Delta}\right) = \frac{1}{2}d(G_u)$.

We denote by Γ *the set of positive roots of odd height* [13, p. 47] with respect to the system of simple roots π.

In order to give the proof of Theorem 1.1 we need the following four lemmas.

Lemma 3.1. $\#(\Gamma) = \frac{1}{2} d\,(G_u)$.

Proof. It is clear that the statement is true for the exceptional algebras (see [10, pp. 527-531]). We give a proof for each family of classical simple Lie algebras.

$\mathfrak{a}_l$. For each $h \geq 1$, the roots of height h are of the form $\sum_{k \leq i \leq k+h-1} \alpha_i$ with $1 \leq k \leq l-h+1$.

If $l = 2k$, there are $2\,(k-r)$ positive roots of height $2r+1$ with $0 \leq r \leq k-1$. Therefore

$$\#(\Gamma) = \sum_{0 \leq r \leq k-1} 2\,(k-r) = \frac{1}{4} l\,(l+2)\,.$$

If $l = 2k+1$, there are $2\,(k-r)+1$ positive roots of height $2r+1$ with $0 \leq r \leq k$. Therefore

$$\#(\Gamma) = \sum_{0 \leq r \leq k} 2\,(k-r) + k + 1 = \frac{1}{4}\,(l+1)^2\,.$$

$\mathfrak{b}_l$ (and $\mathfrak{c}_l$). For each positive root α let $j\,(\alpha) = j$ be the number of coefficients equal to 2 of α with respect to the system π. Clearly $0 \leq j \leq l-1$ and the set of roots in Γ with $j = 0$ has as many elements as in the algebra $\mathfrak{a}_l$, that is $\frac{1}{4} l\,(l+2)$ or $\frac{1}{4}\,(l+1)^2$ according to wether l is even or odd.

Let us assume first that $l = 2k$. In that set, the amount of odd positive roots with $1 \leq j \leq l-1$ is $k-s$ if $j = 2s$ or $j = 2s+1$. Then

$$\begin{aligned} \#(\Gamma) &= \tfrac{1}{4} l\,(l+2) + \textstyle\sum_{1 \leq s \leq k-1} (k-s) + \sum_{0 \leq s \leq k-1} (k-s) \\ &= \tfrac{1}{2} l\,(l+1)\,. \end{aligned}$$

Now, if $l = 2k+1$ we have $k-s+1$ or $k-s$ odd positive roots when $j = 2s$ or $j = 2s+1$ respectively. Thus

$$\begin{aligned} \#(\Gamma) &= \tfrac{1}{4}\,(l+1)^2 + \textstyle\sum_{1 \leq s \leq k} (k-s+1) + \sum_{0 \leq s \leq k-1} (k-s) \\ &= \tfrac{1}{2} l\,(l+1)\,. \end{aligned}$$

$\mathfrak{d}_l$. The number of elements of Γ such that the coefficients of α_{l-1} and α_l coincide is the same as for the case $\mathfrak{b}_{l-2}$ with the addition of $\frac{1}{2} l - 1$ if l is even and with the addition of $\frac{1}{2}\,(l-1)$ if l is odd. On the other hand, the number of elements such that the coefficients of α_{l-1} and α_l are different is l or $l-1$ according to wether l is even or odd.

$$\begin{array}{lclcll} \#\,(\Gamma) &=& \frac{1}{2}\,(l-1)\,(l-2) + \frac{1}{2} l - 1 + l &=& \frac{1}{2} l^2 & l \text{ even} \\ \#\,(\Gamma) &=& \frac{1}{2}\,(l-1)\,(l-2) + \frac{1}{2}\,(l-1) + l - 1 &=& \frac{1}{2}\,(l^2-1) & l \text{ odd} \end{array}$$

■

Lemma 3.2. *For each simple Lie algebra* $\mathfrak{g}$, *if* $\widetilde{\Delta} \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$ *is a presymmetric set and satisfies*

i) $\#\left(\widetilde{\Delta}\right) = \frac{1}{2} d\,(G_u)$,

ii) $\pi \subset \widetilde{\Delta}$;

then $\widetilde{\Delta} = \Gamma$.

Proof. Let Λ be the subset of $\widetilde{\Delta}$ of those roots that have even height. If $\Lambda = \phi$ the result follows from Lemma 3.1. Let us assume now that $\Lambda \neq \phi$.

Consider first the case in which $\mathfrak{g}$ is of classical type. We make the following claim: $\Psi : \Lambda \to \Gamma$ defined by $\Psi(\gamma) = \gamma - \alpha_{j_\gamma}$ where α_{j_γ} is the first simple root such that $\gamma - \alpha_{j_\gamma}$ is a root of $\mathfrak{g}$, is injective.

If β and γ are two roots in Λ such that $\Psi(\gamma) = \Psi(\beta)$ i.e. $\gamma - \alpha_{j_\gamma} = \beta - \alpha_{j_\beta}$ and either $j_\gamma = j_\beta$ or the coefficients of γ and β,with respect to π, are all different from 2 then it is easy to see that $\gamma = \beta$. This proves that the claim is true for the algebras of type $\mathfrak{a}_l$.

Assume now that the algebra $\mathfrak{g}$ is of type $\mathfrak{b}_l$ or $\mathfrak{d}_l$. If our $\beta \in \Lambda$ has some of its coefficients, say $k_i(\beta)$, equal to 2 then β has an even number of coefficients $k_t(\beta)$ equal to 1 for $t < i$. Then j_β is the index corresponding to the first coefficient equal to 1 of β. Under this condition on β the root γ must have also a coefficient equal to 2 and (for analogous reasons to the previous case) j_γ corresponds also to the first coefficient equal to 1 in the root γ. Then it is clear that $\gamma = \beta$.

Let us consider now the case $\mathfrak{c}_l$. If our $\beta \in \Lambda$ has some of its coefficients, say $k_i(\beta)$, equal to 2 then β has an odd number of coefficients $k_t(\beta)$ equal to 1 for $t < i$. Then j_β is the index corresponding to the first coefficient equal to 1 of β. Under this condition on β the root γ must have also a coefficient equal to 2 and (for analogous reasons to the previous case) j_γ corresponds also to the first coefficient equal to 1 in the root γ. Then it is clear that $\gamma = \beta$.

For the exceptional Lie algebras one may construct an injective function Ψ by choosing for each $\gamma \in \Lambda$ a convenient simple root $\alpha_{j(\gamma)}$. This is seen by checking the roots of even height of these algebras. (see [10, pp. 528-530]).

This proves our claim.

Now we observe that if the algebra $\mathfrak{g}$ is not of type $\mathfrak{a}_l$ with l even, then the maximal root μ has odd height. In this case if ν is of maximal height in Λ, then there is a simple root α_j such that $\nu + \alpha_j \in \Gamma$. Since $\widetilde{\Delta}$ is a presymmetric set and due to hypothesis (ii) we have that $\Psi(\Lambda) \subset \left(\Gamma - \widetilde{\Delta}\right)$ and $\nu + \alpha_j \in \Gamma - \widetilde{\Delta}$.

Therefore

$$\#\left(\widetilde{\Delta}\right) = \#(\Lambda) + \#\left(\Gamma \cap \widetilde{\Delta}\right) = \#(\Psi(\Lambda)) + \#\left(\Gamma \cap \widetilde{\Delta}\right) < \#(\Gamma)$$

and by (i) this is a contradiction.

To finish our proof it remains to reach the same contradiction for the algebras of type $\mathfrak{a}_l$ with l even. If $\mu \notin \widetilde{\Delta}$ the proof is the same as above. Now if $\mu \in \widetilde{\Delta}$ then $\mu - \alpha_l \in \Gamma - \left(\Psi(\Lambda) \cup \widetilde{\Delta}\right)$ and so it also follows that

$$\#\left(\widetilde{\Delta}\right) < \#(\Gamma).$$

Our proof is now complete. ■

Lemma 3.3. *If $\widetilde{\Delta} \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$ is a presymmetric set and the maximal root μ is in $\widetilde{\Delta}$ then $\#\left(\widetilde{\Delta}\right) \leq \frac{1}{2} d(G_u)$.*

Proof. We indicate first some notation that will be used in this proof.

If $\mathfrak{g}$ is not of type $\mathfrak{a}_l$ there is only one root $\alpha_j \in \pi$ such that $\mu - \alpha_j \in \Delta$ and μ is the unique positive root for which $k_j(\mu) = 2$. In this case set $\Delta_0 = \{\gamma \in \Delta^+ : k_j(\gamma) = 0\}$, $\Delta_1 = \{\gamma \in \Delta^+ : k_j(\gamma) = 1\}$.

When $\mathfrak{g}$ is of type $\mathfrak{a}_l$, $\mu - \alpha_j \in \Delta$ for $j = 1,\ l\ (l \geq 2)$ and we set $\Delta_0 = \{\gamma \in \Delta^+ : k_1(\gamma) = k_l(\gamma) = 0\}$, $\Delta_1 = \{\gamma \in \Delta^+ : k_1(\gamma) + k_l(\gamma) = 1\}$.

Then, for any $\mathfrak{g}$, we have $\Delta^+ = \Delta_0 \cup \Delta_1 \cup \{\mu\}$ and by inspection, we observe that the set Δ_1 is union of pairs of positive roots $\gamma_j$and β_j such that $\gamma_j + \beta_j = \mu$.

Let now $\widetilde{\Delta} \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$ be a presymmetric set containing μ. It is clear that $\#\left(\widetilde{\Delta}\right) = \#\left(\widetilde{\Delta} \cap \Delta_0\right) + \#\left(\widetilde{\Delta} \cap \Delta_1\right) + 1$ and $\#\left(\widetilde{\Delta} \cap \Delta_1\right) \leq \frac{1}{2}\#(\Delta_1)$.

To study $\#\left(\widetilde{\Delta} \cap \Delta_0\right)$ and $\#(\Delta_1)$ (and therefore estimate $\#\left(\widetilde{\Delta}\right)$) we must separate the proof into different cases according to the type of the simple Lie algebra in question. The proof in each case follows the same pattern but the numbers involved are different. In the first place one does it, by induction, for the Lie algebras of classical type and then one considers each exceptional algebra using the result obtained for the classical ones. For sake of brevity we will do only one case since we think that the reader will have no difficulty in reconstructing the proof in the other cases.

Let $\mathfrak{g}$ be an algebra of type $\mathfrak{a}_l$. The lemma is immediate for $l = 1, 2$. We proceed with the proof by induction on l. Let $l \geq 3$. With Δ_0 and Δ_1 indicated above for this case we have $\#(\Delta_1) = 2(l-1)$, $(\widetilde{\Delta} \cap \Delta_0)$ is a presymmetric set for an algebra of type $\mathfrak{a}_{l-2}$ $(\alpha_2, ..., \alpha_{l-1}$ are its simple roots) and by the inductive hypothesis its cardinality is

$$\#\left(\widetilde{\Delta} \cap \Delta_0\right) \leq \begin{cases} \frac{1}{4}l(l-2) & l \textit{ even} \\ \frac{1}{4}(l-1)^2 & l \textit{ odd.} \end{cases}$$

Then

$$\#\left(\widetilde{\Delta}\right) \leq \begin{cases} \frac{1}{4}l(l-2) + (l-1) + 1 = \frac{1}{4}l(l+2) & l \textit{ even} \\ \frac{1}{4}(l-1)^2 + (l-1) + 1 = \frac{1}{4}(l+1)^2 & l \textit{ odd.} \end{cases}$$

■

Lemma 3.4. *For each simple Lie algebra $\mathfrak{g}$, if $\widetilde{\Delta} \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$ is a presymmetric set such that $\#\left(\widetilde{\Delta}\right) = \frac{1}{2}d(G_u)$ then there exist elements γ in $\widetilde{\Delta}$ and σ in the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{h})$ such that $\sigma(\gamma)$ is the maximal root μ.*

Proof. Since in each simple Lie algebra the maximal root μ is long, it suffices to find a long root in $\widetilde{\Delta}$ (see [13, p. 53]). This is clear when all roots have the same length. It is necessary to consider only the algebras of type $\mathfrak{b}_l$, $\mathfrak{c}_l$, $\mathfrak{f}_4$ and $\mathfrak{g}_2$.

If the algebra $\mathfrak{g}$ is if type $\mathfrak{b}_l$ then it has only l positive short roots . Since $\frac{1}{2}d(SO(2l+1)) = \frac{1}{2}l(l+1) > l$ there is at least a long root γ in $\widetilde{\Delta}$.

For the algebras $\mathfrak{f}_4$ and $\mathfrak{g}_2$ the conclusion is reached similarly.

If $\mathfrak{g}$ is of type $\mathfrak{c}_l$ there are $l(l-1)$ positive short roots while $\frac{1}{2}d(G_u) = \frac{1}{2}l(l+1)$. In this case we will prove the lemma, by induction on l, showing that if $\widetilde{\Delta}$ is a set of positive short roots then its cardinality is less than $\frac{1}{2}l\,(l+1)$. For $l=3$ this is true because the set of positive short roots is

$$\{\alpha_1, \alpha_2, \alpha_1+\alpha_2, \alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3, \alpha_1+2\alpha_2+\alpha_3\}$$

and clearly is not a presymmetric set. For $l=4$ the set of positive short roots has twelve elements and it is easy to find a partition into four subsets of the form $\{\beta, \gamma, \beta+\gamma\}$. Then any presymmetric set of short root has at most eight elements and since $\frac{1}{2}l\,(l+1)$ is 10 in this case, the assertion follows.

Let now $l \geq 5$. It is easy to see that ([13, p. 64])

$$\nu = \alpha_1 + \sum_{1<j<l} 2\alpha_j + \alpha_l$$

is a short root.

Let us assume first that $\nu \in \widetilde{\Delta}$ and consider the following three subsets

$$\widetilde{\Delta}_r = \left\{\gamma \in \widetilde{\Delta} : \; k_2(\gamma) = r\right\} \qquad r = 0, 1, 2.$$

Let us notice that:

i) Since for each short root of the form $\sum_{1\leq j<l} k_j\alpha_j + \alpha_l$ the first nonzero coefficient k_j $(1 \leq j < l)$ is equal to 1 it follows that $\widetilde{\Delta}_2 = \{v\}$.

ii) Set $\Delta_1 = \{\gamma \in \Delta^+ : \; k_2(\gamma) = 1\}$. The cardinality of Δ_1 is $4l-8$ and by inspection we can see that the set Δ_1 is the union of pairs of positive short roots γ_j, β_j such that $\gamma_j + \beta_j = \nu$. Therefore $\#\left(\widetilde{\Delta}_1\right) = \#\left(\widetilde{\Delta} \cap \Delta_1\right) \leq 2l-4$.

iii) $\widetilde{\Delta}_0 - \{\alpha_1\}$ is a presymmetric set of positive short roots for a subalgebra of type $\mathfrak{c}_{l-2}$ and then, by the inductive hypothesis, $\#\left(\widetilde{\Delta}_0 - \{\alpha_1\}\right)$
$< \frac{1}{2}(l-2)(l-1)$.

Then, since $\widetilde{\Delta}_0$, $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_2$ conform a partition of $\widetilde{\Delta}$, we have

$$\#\left(\widetilde{\Delta}\right) \leq \frac{1}{2}(l-2)(l-1) + 1 + 2l - 4 + 1 < \frac{1}{2}l\,(l+1).$$

Assume now that $\nu \notin \widetilde{\Delta}$. By taking any short root γ in $\widetilde{\Delta}$ there is a σ in the Weyl group such that $\sigma(\gamma) = \nu$. Now $\sigma\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right) \cap \Delta^+$ is a presymmetric set of short roots which contains ν and therefore its cardinality is less than $\frac{1}{2}l\,(l+1)$. Then so does $\widetilde{\Delta}$. This completes the proof of the lemma. ■

Proof of Theorem 1.1. If the maximal root $\mu \in \widetilde{\Delta}$ then conclusion follows from Lemma 3.3. Let assume now that $\mu \notin \widetilde{\Delta}$ and $\#\left(\widetilde{\Delta}\right) > \frac{1}{2}d\,(G_u)$ and consider $\Omega \subset \widetilde{\Delta}$ such that $\#(\Omega) = \frac{1}{2}d\,(G_u)$. By Lemma 3.4 there exists a σ in the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{h})$ such that $\mu \in \sigma(\Omega)$. Since $\mu \in \sigma\left(\widetilde{\Delta}\right)$, $\sigma\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right) \cap \Delta^+$ is a presymmetric set and its cardinality is $\#\left(\widetilde{\Delta}\right)$, we reach a contradiction with Lemma 3.3. This finishes the proof. ■

Our next objective is the proof of Theorem 1.2 and to that end we give the following

Lemma 3.5. *For each simple Lie algebra $\mathfrak{g}$, if $\widetilde{\Delta} \subset \Delta^+(\mathfrak{g}, \mathfrak{h})$ is a presymmetric set with $\#\left(\widetilde{\Delta}\right) = \frac{1}{2} d(G_u)$, then there exists an element σ in the Weyl group of $\Delta(\mathfrak{g}, \mathfrak{h})$ such that $\sigma\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right) \cap \Delta^+$ is the set Γ of positive roots of odd height.*

Proof. We separate the proof into different cases according to the type of the simple Lie algebra in question. We keep the notation from Lemma 3.3 and in the same manner that in that Lemma, the proof in each case follows the same pattern but the numbers involved are different. For sake of brevity we will do only the proof for types $\mathfrak{a}_l$ and $\mathfrak{b}_l$, because we think that the reader will have no difficulty in reconstructing it in the other cases.

Let $\mathfrak{g}$ be an algebra of type $\mathfrak{a}_l$. We proceed with the proof by induction on l. The lemma is immediate for $l = 1, 2$. Set $l \geq 3$. By Lemma 3.4 we may assume that $\mu \in \widetilde{\Delta}$. Due to Lemma 3.2, the assertion that we want to prove is equivalent to find an element σ in the Weyl group such that the system of simple roots π is contained in $\sigma\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$.With Δ_0 and Δ_1 indicated in Lemma 3.3 for the algebra $\mathfrak{a}_l$, we have $\#(\Delta_1) = 2(l-1)$ and $(\widetilde{\Delta} \cap \Delta_0)$ is a presymmetric set for an algebra of type $\mathfrak{a}_{l-2}$ $(\alpha_2, ..., \alpha_{l-1}$ are its simple roots). By Theorem 1.1, its cardinality is

$$\#\left(\widetilde{\Delta} \cap \Delta_0\right) \leq \begin{cases} \frac{1}{4} l(l-2) & l \text{ even} \\ \frac{1}{4}(l-1)^2 & l \text{ odd} \end{cases} \tag{2}$$

and due to the fact that $\mu \in \widetilde{\Delta}$,

$$\#\left(\widetilde{\Delta} \cap \Delta_1\right) \leq l - 1. \tag{3}$$

Now since

$$\#\left(\widetilde{\Delta}\right) = \#\left(\widetilde{\Delta} \cap \Delta_0\right) + \#\left(\widetilde{\Delta} \cap \Delta_1\right) + 1 = \begin{cases} \frac{1}{4} l(l+2) & l \text{ even} \\ \frac{1}{4}(l+1)^2 & l \text{ odd} \end{cases}$$

we get that formulas (2) and (3) are in fact equalities.

Then $(\widetilde{\Delta} \cap \Delta_0)$ is a presymmetric set of maximal cardinality for an algebra of type $\mathfrak{a}_{l-2}$. Thus, by inductive hypothesis, there exists τ_1 in the Weyl group of $\mathfrak{a}_{l-2}$ such that

$$\tau_1\left[\left(\widetilde{\Delta} \cap \Delta_0\right) \cup \left(-(\widetilde{\Delta} \cap \Delta_0)\right)\right] \supset \{\alpha_2, ..., \alpha_{l-1}\}.$$

Let τ be the natural extension of τ_1 to the Weyl group of $\mathfrak{a}_l$. Since τ is product of reflections σ_{α_j} with $2 \leq j \leq l-1$, it is clear that $\tau(\mu) = \mu$ and therefore

$$\{\mu, \alpha_2, \alpha_3, ..., \alpha_{l-1}\} \subset \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right).$$

In order to finish the proof, for the algebras of type $\mathfrak{a}_l$, we need to consider the following four possibilities.

1) α_1 and $\alpha_l \in \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$, then taking $\sigma = \tau$ we have the Lemma.

2) Neither α_1 nor α_l belong to $\tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ then, keeping in mind that in formula (3) equality holds, we have that $\pm(\mu - \alpha_1)$ and

$$\pm(\mu - \alpha_1) \in \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right).$$

Since the reflection σ_μ corresponding to the maximal root μ satisfies

$$\sigma_\mu(\alpha_j) = \begin{cases} \alpha_j & j \neq 1, l \\ \alpha_j - \mu & j = 1, l \end{cases}$$

by taking $\sigma = \sigma_\mu \tau$, we have the Lemma.

3) $\alpha_1 \in \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ and $\alpha_l \notin \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$. If we call $\pi' = \{-\mu, \alpha_1, ..., \alpha_{l-1}\}$, this is also a system of simple roots for $\Delta(\mathfrak{g}, \mathfrak{h})$ and therefore there exists a θ in the Weyl group of $\mathfrak{a}_l$ such that $\theta(\pi') = \pi$. Since π' is contained in $\tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ then by taking $\sigma = \theta\tau$ we have the Lemma.

4) It is clear that the situation $\alpha_l \in \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ and $\alpha_1 \notin \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ is totally analogous to (3) and so we also have the Lemma in this case. This concludes the proof of the Lemma for the algebras of type $\mathfrak{a}_l$.

Let $\mathfrak{g}$ be an algebra of type $\mathfrak{b}_l$. We proceed with the proof by induction on l. The lemma is immediate for $l = 1, 2$ ($\mathfrak{b}_1 \approx \mathfrak{a}_1$). Set $l \geq 3$. By Lemma 3.4 we may assume that $\mu \in \widetilde{\Delta}$. Due to Lemma 3.2, the assertion that we want to prove is equivalent to find an element σ in the Weyl group such that the system of simple roots π is contained in $\sigma\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$.

With Δ_0 and Δ_1 indicated in Lemma 3.3 for the algebra $\mathfrak{b}_l$, we have $\#(\Delta_1) = 2(2l-3)$ and $(\widetilde{\Delta} \cap \Delta_0) - \{\alpha_1\}$ is a presymmetric set for an algebra of type $\mathfrak{b}_{l-2}$ ($\alpha_3, ..., \alpha_l$ are its simple roots). By Theorem 1.1, its cardinality is

$$\#\left((\widetilde{\Delta} \cap \Delta_0) - \{\alpha_1\}\right) \leq \frac{1}{2}(l-2)(l-1) \tag{4}$$

and

$$\#\left(\widetilde{\Delta} \cap \Delta_1\right) \leq 2l - 3 \tag{5}$$

because $\mu \in \widetilde{\Delta}$.

Now since

$$\#\left(\widetilde{\Delta}\right) = \#\left(\widetilde{\Delta} \cap \Delta_0\right) + \#\left(\widetilde{\Delta} \cap \Delta_1\right) + 1 = \frac{1}{2} l(l+1)$$

we get that formulas (4) and (5) are in fact equalities and $\alpha_1 \in \widetilde{\Delta}$. Then $(\widetilde{\Delta} \cap \Delta_0) - \{\alpha_1\}$ is a presymmetric set of maximal cardinality for an algebra of type $\mathfrak{b}_{l-2}$. Thus, by inductive hypothesis, there exists τ_1 in the Weyl group of $\mathfrak{b}_{l-2}$ such that

$$\tau_1\left[\left((\widetilde{\Delta} \cap \Delta_0) - \{\alpha_1\}\right) \cup -\left((\widetilde{\Delta} \cap \Delta_0) - \{\alpha_1\}\right)\right] \supset \{\alpha_3, ..., \alpha_l\}.$$

Let τ be the natural extension of τ_1 to the Weyl group of $\mathfrak{b}_l$. Since τ is product of reflections σ_{α_j} with $3 \leq j \leq l$, it is clear that $\tau(\alpha_1) = \alpha_1$ and therefore

$$\{\alpha_1, \alpha_3, ..., \alpha_l\} \subset \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right).$$

If $\alpha_2 \in \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ then taking $\sigma = \tau$ we have the Lemma.

If $\alpha_2 \notin \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$ then, keeping in mind that in formula (5) equality holds, we have that $\pm(\mu - \alpha_2) \in \tau\left(\widetilde{\Delta} \cup (-\widetilde{\Delta})\right)$. Since the reflection σ_μ corresponding to the maximal root μ satisfies

$$\sigma_\mu(\alpha_j) = \begin{cases} \alpha_j & j \neq 2 \\ \alpha_2 - \mu & j = 2, \end{cases}$$

by taking $\sigma = \sigma_\mu \tau$ we conclude the proof for the algebras of type $\mathfrak{b}_l$. ■

Proof of Theorem 1.2. In the paper [6, Remark 4.1] we proved that if $\mathfrak{p}$ is the tangent space of the inner symmetric space G_u/K at $[K]$ then there exists a root $\gamma^* \in \pi$ such that $\mathfrak{p}$ is of the form

$$\mathfrak{p} = \textstyle\sum_{\gamma\in\Delta^*} \mathfrak{m}_\gamma, \quad \textit{where} \quad \Delta^* = \{\gamma \in \Delta^+ : k_\gamma(\gamma^*) = 1\}.$$

Furthermore by Proposition 2.1 we know that $RP\,(\mathfrak{p}) \subset X\,[M]$. Now from Theorem 2.2 and Lemma 3.5 it follows that there exists an element τ in the Weyl group of $\Delta\,(\mathfrak{g}, \mathfrak{h})$ such that $\tau\,(\Delta^* \cup (-\Delta^*)) \cap \Delta^+ = \Gamma$ (set of positive roots of odd height). Then, if $\mathfrak{p}_\Gamma = \sum_{\gamma\in\Gamma} \mathfrak{m}_\gamma$ there exists h in the normalizer $N_{G_u}(T)$ of the maximal torus T in the group G_u such that $Ad\,(h)\,\mathfrak{p}_\Gamma = \mathfrak{p}$.

By Lemma 3.5, for our subspace $\widetilde{\mathfrak{p}}$ there exits a g in $N_{G_u}(T)$ such that $Ad\,(g)\,\widetilde{\mathfrak{p}} = \mathfrak{p}_\Gamma$. Thus $Ad\,(hg)\,\widetilde{\mathfrak{p}} = \mathfrak{p}$ and since $Ad\,(hg)$ is orthogonal with respect to the Killing form in $\mathfrak{g}_u$, it takes $\widetilde{\mathfrak{k}} = \widetilde{\mathfrak{p}}^\perp$ onto $\mathfrak{k} = \mathfrak{p}^\perp$. Since the set of fixed points of the torus T acting on G_u/K is the orbit of the point $[K]$ by the normalizer $N_{G_u}(T)$ the Theorem follows. ■

4. Some applications.

We present here some consequences and comments concerning the results of the previous sections. First of all we introduce some notation and terminology which is convenient for our purposes.

Let us recall a definition, as given for instance in [12, p. 70]. If $Y \subset RP^n$ is an algebraic variety and $G\,(k,n)$ denotes the Grassmannian of real projective subspaces RP^k contained in the projective space RP^n [12, p. 63], we may consider the set of k-planes contained in Y

$$F_k\,(Y) = \{\Lambda \in G\,(k,n) : \Lambda \subset Y\}.$$

This is an algebraic variety ([12, p. 70]) which is a subvariety of the Grassmannian. For each k this variety is a very interesting geometric object naturally associated to the original variety Y which it is usually called the k-th *Fano variety associated to the variety* Y.

We want to take a look here to a particular Fano variety associated to the variety $X\,[M]$ for our flag manifold $M = G_u/T$ and present an application of our results.

Let us take the Fano variety $F_{d-1}\,(X\,[M]) \subset G\,(d-1, n-1)$ where $d = d\,(G_u)$ and $n = \dim\,(M)$. The points of $F_{d-1}\,(X\,[M])$ are the real projective subspaces of dimension $d-1$ contained in the variety $X\,[M]$. It is clear that the projective subspaces described in Theorems 2.2 and 2.3 are points in $F_{d-1}\,(X\,[M])$ and so this variety is not empty. In order to write down our result we need to introduce some notation.

Recall that for each compact connected simple Lie group G_u we have the corresponding flag manifold of complete flags $M = G_u/T$ and also associated to the group G_u we have an inner symmetric space G_u/K of maximal dimension $d\,(G_u)$

(see Table I). There is one of them for each group G_u and so the flag manifold M determines uniquely the inner symmetric space and hence the subgroup K (without loosing generality we may assume $T \subset K$).

Associated to the subgroup K we have its normalizer $N_{G_u}(K)$ in the group G_u. It is well known that the Lie algebra of $N_{G_u}(K)$ coincides with that of K. This means that the quotient group $\Sigma = N_{G_u}(K)/K$ is discrete and hence finite.

There is a right free action of the group Σ on the symmetric space G_u/K (compare [2, p. 24]) given by

$$(gK).(hK) = ghK, \quad for\ hK \in \Sigma \ .$$

In this situation we may consider the quotient of the symmetric space by the free action, which is then a manifold, and denote it by $\Sigma \setminus (G_u/K)$.

As usual we denote by $\chi(L)$ the Euler-Poincaré characteristic of the topological space L. Theorems 1.1 and 1.2 in this paper and results from [6] allow us to present the following theorem concerning the Euler-Poincaré characteristic of the variety $F_{d-1}(X[M])$.

Theorem 4.1. *The Euler-Poincaré characteristic of the $(d-1)-th$ Fano variety associated to the variety $X[M]$ coincides with that of the locally symmetric manifold $\Sigma \setminus (G_u/K)$; i.e.*

$$\chi(F_{d-1}(X[M])) = \chi(\Sigma \setminus (G_u/K)).$$

Proof. Set $o = eT \in M$. The linear action (via $Ad_{G_u}(T)$) of T on $\mathfrak{m} = T_o(M)$ induces an action of T on the Grassmannian

$$G(d, T_o(M)) = G(d-1, RP(T_o(M))) = G(d-1, n-1).$$

Since the variety $X[M]$ is defined in terms of the invariant tensors α an D it is itself invariant by the action of the torus T (see [5, pp. 227 and 231]). This clearly implies that there is a natural action of the torus T on the Fano variety $F_{d-1}(X[M])$.

It is a classical fact (see for instance [1, p. 163, 10.9] or [5, p. 234]) that the Euler-Poincaré characteristic of a compact topological space supporting an action of a torus, is equal to that of its fixed point set. For our action of T on $F_{d-1}(X[M])$ the fixed point set clearly consists of those subspaces of the form $RP(\mathfrak{q})$ where $\mathfrak{q}$ is a subspace of $T_o(M)$ invariant by the torus action on this tangent space and such that $RP(\mathfrak{q}) \subset X[M]$. Theorem 2.2 identifies these subspaces precisely, and so we see that the Euler-Poincaré characteristic of $F_{d-1}(X[M])$ is just the number of these subspaces.

On the other hand Theorem 1.2 indicates that all these subspaces are tangent to the symmetric space G_u/K at the fixed points of the natural action of the torus $T \subset K$ on G_u/K.

Since that action of the torus has only isolated fixed points it is also clear, and well known, (by the same classical result mentioned above) that the Euler-Poincaré characteristic of G_u/K is precisely the cardinality of this finite set Θ of fixed points i.e. $\chi(G_u/K) = |\Theta|$.

Since the action of Σ on G_u/K is free the space G_u/K is a $|\Sigma|$-sheeted covering of the manifold $\Sigma \setminus (G_u/K)$ and then

$$\chi(\Sigma \setminus (G_u/K)) = \frac{\chi(G_u/K)}{|\Sigma|} = \frac{|\Theta|}{|\Sigma|}.$$

The group Σ acts simply transitively on the set

$$\Omega = \{s \in G_u/K : (G_u)_s = K\}.$$

In fact, if $s = uK \in \Omega$ then $uKu^{-1} = K$ which means $u \in N_{G_u}(K)$ and so the action is transitive (we already mentioned that the action of Σ is free). Of course the same thing is true for each of the sets $h\Omega = \left\{h(s) \in G_u/K : (G_u)_{h(s)} = hKh^{-1}\right\}$ for each $h \in N_{G_u}(T)$.

Now we observe that the union of all these sets is Θ

$$\Theta = \bigcup_{h \in N_{G_u}(T)} h\Omega$$

(since the sets $h\Omega$ are in fact orbits of Σ, they either coincide or they are disjoint), notice that $h \in N_K(T)$ implies $h\Omega = \Omega$;

For each $s \in \Omega$ the isotropy subalgebra is precisely $\mathfrak{k}$ and so the tangent spaces $T_s(G_u/K)$ (the subspace $\mathfrak{p}$ orthogonal to $\mathfrak{k}$ by the Killing form in $\mathfrak{g}_u$) coincide as subspaces of $\mathfrak{m}$. Thus they give a single $RP(\mathfrak{p})$ contained in $X[M]$ and therefore one point in $F_{d-1}(X[M])$ fixed by the torus T. Naturally the same thing is true for each of the sets $h\Omega$. This proves that there are $\frac{|\Theta|}{|\Sigma|}$ points in the fixed point set $F(F_{d-1}(X[M]), T)$ and concludes de proof of the theorem. ∎

The group Σ is a well known object associated to the inner symmetric space G_u/K. It "counts", in some sense, how many points in that space have the same isotropy subgroup or, as we have preferred to do above, how many points have "the same" tangent space (as long as we identify the tangent space at a point with the orthogonal complement of the isotropy subalgebra in $\mathfrak{g}_u$ with respect to the Killing form).

The reader may find a list of the corresponding groups Σ for the classical irreducible inner symmetric spaces for which it is not trivial in [2, p. 24]. The authors indicate there that the group is trivial for $G_2/SO(4)$.

We have been able to see that the group Σ is also trivial for the symmetric spaces $F_4/Sp(3)Sp(1)$ and $E_6/SU(6)Sp(1)$. This was obtained from the last theorem, by computing specifically all the presymmetric sets of roots for the algebras $\mathfrak{f}_4$ and $\mathfrak{e}_6$ and noticing that the number of those sets coincides with the Euler-Poincaré characteristic of the symmetric spaces.

There are many problems that we leave open because their consideration seems beyond the scope of our methods.

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