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Normalizers of compact subgroups, the existence of commuting automorphisms, and applications to operator semistable measures

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Abstract. Let G be a Lie group with finitely many components and K a compact subgroup with identity component K_0. The normalizer, respectively, centralizer of K in G is denoted by $N(K,G)$, respectively, $Z(K,G)$. It is shown that $N(K,G)/K_0Z(K,G)$ is finite. This is applied to a problem in theoretical probability theory, namely, to the characterisation of operator semistable measures. The article explains these concepts and puts the present application into perspective.

1. A Structure Theorem on Lie Groups and Some Consequences

Motivated by applications to probability theory, notably to semistable laws on vector spaces and groups which we shall discuss in detail in Section 3 below, we present some results on the structure of Lie groups and locally compact groups which are of independent interest.

If K is a compact subgroup of a locally compact group L and C is a closed subgroup isomorphic to either $\mathbb{R}$ or $\mathbb{Z}$ such that $L = KC$, then the structure of L is known (see e.g. [14], pp.60–64; cf. also [11].)

Example 1.1. (The infinite dihedral group.) Let $\mathbb{S}^0$ be the multiplicative group of integers $\{1, -1\}$. On the product $\mathbb{Z}\times\mathbb{S}^0$ we consider the multiplication $(m_1, \varepsilon_1)(m_2, \varepsilon_2) = (m_1 + \varepsilon_1{\cdot}m_2, \varepsilon_1\varepsilon_2)$. Then $K := \{0\} \times \mathbb{S}^0$ is one maximal compact and not normal subgroup and $C := \mathbb{Z}\times\{1\}$ is an infinite cyclic normal subgroup. In fact, each of the subsets $K \stackrel{\text{def}}{=} \{(0, 1), (m, -1)\}$ is a maximal compact subgroup. Note that no nonidentity element commutes with $(0, -1)$. ■

We shall make frequent reference below to the special case that K is assumed to be normal; Example 1 shows how this may fail, and the structure theorems ([14], p.61) say that (at least in the case $C \cong \mathbb{R}$) it is not far from the

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worst that can happen. If K is normal and $C \cong \mathbb{R}$, the subgroup C can be chosen to commute elementwise with K, i.e., $L \cong K \times \mathbb{R}$ as we shall reprove in 1.9 below. In the case that K is open, i.e. $C \cong \mathbb{Z}$, the situation is more involved.

Let us use the following notation. If A is a subgroup of a group B, we denote by

$$N(A, B) = \{b \in B : bA = Ab\} \text{ the normalizer and by}$$
$$Z(A, B) = \{b \in B : ba = ab \text{ for all } a \in A\} \text{ the centralizer}$$

of A in B. Note that $AZ(A, B) = Z(A, B)A \subseteq N(A, B)$.

Example 1.2. Let K_1 be a compact group and α an automorphism of the topological group K_1. Form $L = K_1 \times \mathbb{Z}$ and define a multiplication

$$(k_1, m_1)(k_2, m_2) = (k_1 \alpha^{m_1}(k_2), m_1 + m_2).$$

Then $K := K_1 \times \{0\}$ is a compact normal subgroup of L and $C := \{1\} \times \mathbb{Z}$ is infinite cyclic and discrete. An element (k, m) is in the centralizer $Z(K, L)$ if and only if $\alpha^m = 1$ and k is in the center $Z(K_1, K_1)$. Thus

$$Z(K, L) = \begin{cases} Z(K_1, K_1) \times n\mathbb{Z} & \text{if } \alpha \text{ has finite order } n, \\ Z(K_1, K_1) \times \{0\} & \text{otherwise.} \end{cases}$$

In the former case, $KZ(K, L)$ has finite index n in L, and $KZ(K, L) = K_1 \times n\mathbb{Z}$ is a direct product. ■

Example 1.2.1. We write $K_1 \rtimes_\alpha \mathbb{Z}$ for the semidirect product L introduced in 1.2. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the torus and set $K_1 = \mathbb{T}^2$. Then by duality we have $\mathrm{Aut}(\mathbb{T}^2) \cong \mathrm{Aut}\,\mathbb{Z}^2 \cong \mathrm{GL}(2, \mathbb{Z})$. Since $\mathrm{GL}(2, \mathbb{Z})$ contains elements of infinite order such as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ we have an $\alpha \in \mathrm{Aut}\,\mathbb{T}^2$ of infinite order. Then $\mathbb{T}^2 \rtimes_\alpha \mathbb{Z}$ is a two dimensional metabelian Lie group with $Z(K, L) = K = \mathbb{T}^2 \times \{0\}$. In this Lie group, the maximal compact normal subgroup is far from splitting directly.■

In Exercise 1.11 below we shall observe that Example 1.2.1 is typical in the following sense: if the dimension of the center of K_1 is < 2 the phenomenon illustrated by Example 1.2.1 cannot occur.

Examples of the type of 1.2 arise as follows: Let K be a compact subgroup of a topological group G and let a be an element of the normalizer $N(K, G)$ of K in G such that $C := \langle a \rangle = \{a^n : n \in \mathbb{Z}\}$ is an infinite discrete subgroup. Then $L := KC \subseteq G$ is isomorphic to $K \rtimes_\alpha \mathbb{Z}$ with $\alpha \in \mathrm{Aut}(K)$ given by $\alpha(k) = aka^{-1}$. The order of α is n iff n is the smallest natural number such that $a^n \in Z(K, G)$.

We shall show that *whenever G is a Lie group with finitely many components, then for each $a \in N(K, G)$ there is an element k in the identity component of K and a natural number e such that $ka^e \in Z(K, G)$. In the case that $\langle a \rangle$ is infinite discrete, this entails that $K\langle a^e \rangle = K\langle ka^e \rangle$ is isomorphic to the direct product $K \times \mathbb{Z}$.*

This turns out to be a fairly quick consequence of a noteworthy structure theorem on Lie groups. If H is a topological group, then H_0 denotes its identity component. Our main theorem contributes to the finiteness results surrounding the concept of a Weyl group in a general vein.

Theorem A. *Let G be a Lie group with finitely many components and K a compact subgroup. Then $N(K,G)/K_0Z(K,G)$ is finite.*

The proof will be given in Section 2. Here we firstly point out sufficient conditions which secure the hypothesis that G has finitely many components and afterwards draw some consequences of Theorem A.

Lemma 1.3. *Let A be a subgroup of a topological group G and let K be a subgroup of A. Then*

$$N(K,A)/K_0Z(K,A) \cong N(K,A)K_0Z(K,G)/K_0Z(K,G) \subseteq N(K,G)/K_0Z(K,G).$$

In particular, if $N(K,G)/K_0Z(K,G)$ is finite, then $N(K,A)/K_0Z(K,A)$ is finite as well.

Proof. We note that $N(K,A) = N(K,G)\cap A$ and $Z(K,A) = Z(K,G)\cap A$ and thus, by the modular law, $K_0Z(K,A) = K_0Z(K,G)\cap A = K_0Z(K,G)\cap N(K,A)$. Thus

$$\begin{aligned} N(K,A)/K_0Z(K,A) &= N(K,A)/\big(N(K,A)\cap K_0Z(K,G)\big) \\ \cong N(K,A)K_0Z(K,G)/K_0Z(K,G) &\subseteq N(K,G)/K_0Z(K,G). \end{aligned}$$

The Lemma follows. ∎

Corollary 1.4. *If A is a closed subgroup of a real or complex affine algebraic group and K a compact subgroup of A, then $N(K,A)/K_0Z(K,A)$ is finite.*

Proof. Any real or complex affine algebraic group is contained up to isomorphism, as a closed subgroup, in some general linear group $G = \mathrm{GL}(n,\mathbb{R})$ for a suitable n. Thus A is a real Lie group contained in some group $\mathrm{GL}(n,\mathbb{R})$, $n \geq 1$, which is a Lie group with two components. Thus Theorem A applies and shows the finiteness of $N(K,G)/K_0Z(K,G)$. ∎

In particular, the 2-dimensional metabelian Lie group of Example 1.2.1 cannot be (isomorphic to) a subgroup of a real or complex affine algebraic group; in particular, it is not isomorphic to any matrix group.

Corollary 1.5. *Let H be a connected Lie group. If K is a compact subgroup of the automorphism group $\mathrm{Aut}\, H$, then $N(K,\mathrm{Aut}\, H)/K_0Z(K,\mathrm{Aut}\, H)$ is finite.*

Proof. Every automorphism $\varphi\colon H \to H$ determines uniquely a unique lifting to an automorphism $\widetilde{\varphi}\colon \widetilde{H} \to \widetilde{H}$ of the universal covering, and $\varphi \mapsto \widetilde{\varphi} : \mathrm{Aut}\, H \to \mathrm{Aut}\, \widetilde{H}$ is an isomorphism onto the closed subgroup of those automorphisms of $\widetilde{H}$ which preserves the kernel of the covering map $\widetilde{H} \to H$. On the other hand, $\mathrm{Aut}\, \widetilde{H}$ is a Lie group which is isomorphic to the linear Lie group $\mathrm{Aut}\, \mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of H and $\widetilde{H}$. But $\mathrm{Aut}\, \mathfrak{h}$ is an affine algebraic subgroup of $\mathrm{GL}(\mathfrak{h})$. Hence Corollary 1.4 proves the assertion. ∎

Corollary 1.6. *Let K be a compact subgroup of $\mathrm{GL}(V)$ for a finite dimensional real vector space V. Then $N\big(K,\mathrm{GL}(V)\big)/K_0Z\big(K,\mathrm{GL}(V)\big)$ is finite.*

Proof. This is an immediate consequence of Corollary 1.5 which we apply with $H = V$. ∎

Now we draw two conclusions from Theorem A; the first is Lie group theoretical, the second is arithmetical and group theoretical.

The Lie algebra of a Lie group G will be denoted by $\mathfrak{L}(G)$; we abbreviate $\mathfrak{L}(G)$ by $\mathfrak{g}$ and $\mathfrak{L}(K)$ by $\mathfrak{k}$.

Corollary B. *Let G be a Lie group with finitely many components and K a compact subgroup. Then $\mathfrak{L}\big(N(K,G)\big) = \mathfrak{k} + \mathfrak{L}\big(Z(K,G)\big)$.*

Proof. By Theorem A, the Lie group $N(K,G)/K_0Z(K,G)$ is finite, and thus $K_0Z(K,G)$ is open in $N(K,G)$. Consequently, $\mathfrak{L}\big(N(K,G)\big) = \mathfrak{L}\big(K_0Z(K,G)\big)$. But $\mathfrak{L}\big(K_0Z(K,G)\big) = \mathfrak{L}(K_0) + \mathfrak{L}\big(Z(K,G)\big)$. Since $\mathfrak{L}(K_0) = \mathfrak{L}(K) = \mathfrak{k}$, the assertion follows. ∎

We remark that in the case that K and G are connected, the Lie algebra $\mathfrak{L}\big(N(K,G)\big)$ is the normalizer of $\mathfrak{k}$ in $\mathfrak{g}$, and $\mathfrak{L}\big(Z(K,G)\big)$ is the centralizer of $\mathfrak{k}$ in $\mathfrak{g}$.

Corollary 1.7. *Let G be a Lie group with finitely many components and K a compact subgroup. Then the following conditions are equivalent.*

(1) *K is open in $N(K,G)$.*
(2) $\mathfrak{k} = \mathfrak{L}\big(N(K,G)\big)$.
(3) $\mathfrak{L}(Z(K,G)\big) \subseteq \mathfrak{k}$.
(4) $Z(K,G)_0 \subseteq K$.

Proof. (1) and (2) are equivalent since for any closed normal subgroup A in a Lie group B the relation $\mathfrak{L}(A) = \mathfrak{L}(B)$ is equivalent to $A_0 = B_0$, and that is equivalent to A being open in the Lie group B. By Corollary B we have $\mathfrak{L}\big(N(K,G)\big) = \mathfrak{L}(K) + \mathfrak{L}\big(Z(K,G)\big)$. Hence assertions (2) and (3) are equivalent. But (3) and (4) are both equivalent to $Z(K,G)_0 \subseteq K_0$. ∎

Now we draw a second conclusion. For this purpose recall that the *exponent* of a group F is

$$e(F) = \begin{cases} \min\{n \in \mathbb{N} : (\forall g \in G)\, g^n = 1\} & \text{if } \{|\langle g\rangle| : g \in F\} \text{ is bounded,} \\ \infty & \text{otherwise.} \end{cases}$$

If F is a finite group, then the exponent $e(F)$ is finite.

Lemma 1.8. *If K is a subgroup of a group G and if the exponent $e = e\big(N(K,G)/K_0Z(K,G)\big)$ is finite, then for all $a \in N(K,G)$ there is a $k \in K_0$ such that $(ka^e)x = x(ka^e)$ for all $x \in K$. In particular, $K\langle a^e\rangle = K\langle ka^e\rangle$ and the factors in the last product commute elementwise. Accordingly, $(x,n) \mapsto x(ka^e)^n: K \times Z \to K\langle a^e\rangle$ is an isomorphism of groups.*

Proof. By hypothesis and the definition of e we have $a^e \in K_0Z(K,G)$, i.e., there is a $k \in K_0$ such that $a^e = k^{-1}z$ for some $z \in Z(K,G)$. The claim follows. ∎

Now Lemma 1.8 and Theorem A imply

Corollary C. *Let G be a Lie group with finitely many components and K a compact subgroup. Let e be the exponent of $N(K,G)/K_0Z(K,G)$. Then for all $a \in N(K,G)$ there is a $k \in K_0$ such that $(ka^e)x = x(ka^e)$ for all $x \in K$.*

Moreover, $(x,n) \mapsto x(ka^e)^n: K\times\mathbb{Z} \to K\langle a^e\rangle$ is a morphism of topological groups which is an isomorphism if and only if $K \cap \langle a^e\rangle = \{1\}$. ∎

Corollary D. *Let G be a linear Lie group and K a compact normal subgroup such that G/K is either isomorphic to $\mathbb{R}$ or to $\mathbb{Z}$. Then in the first case*

(i) $G \cong K \times \mathbb{R}$.

In the second case,

(ii) *there is a natural number e such that for each $a \in G$ there is an element $k \in K_0$ such that $ka^e \in Z(K,G)$. Moreover, if $a \notin K$, then the open normal subgroup $K\langle a^e\rangle$ of G is isomorphic to $K\times\mathbb{Z}$. If $\langle Ka\rangle = G/K$, then $K\langle a^e\rangle$ has index e in G.*

Proof. Since K is normal in G, we have $N(K,G) = G$.

Case 1. Here K is not open in G. Hence by Corollary 1.7, $Z(K,G)_0 \not\subseteq K$, and since the only connected subgroups of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$, we have $G = KZ(K,G)_0$. Since $Z(K,G)_0/(Z(K,G)_0 \cap K) \cong \mathbb{R}$, there is a one parameter subgroup $E \cong \mathbb{R}$ contained in $Z(K,G)_0$. Then $K \cap E = \{1\}$ and $G = KE$. Therefore $(k,x) \mapsto kx: K\times E \to G$ is an isomorphism.

Case 2. Since G is a linear Lie group, Corollary 1.4 (or Corollary 1.6) shows that $G/K_0Z(K,G)$ is finite. Now Corollary C applies and shows that $(x,n) \mapsto x(ka^e)n: K\times\mathbb{Z} \to K\langle a^e\rangle$ is an isomorphism of topological groups. Clearly $G = K\langle a\rangle$ implies $|G/K\langle a^e\rangle| = |\mathbb{Z}/e\mathbb{Z}| = e$. ∎

The structure of an extension of a compact normal subgroup by a group isomorphic to $\mathbb{R}$ has long been settled. (See e.g [4]; or [14], Proposition 9.4; or [11], Lemma 1.25.) We can use the present results to present an independent proof.

Remark 1.9. Let K be a compact normal subgroup of a locally compact group G such that $G/K \cong \mathbb{R}$. Then $G \cong K \times \mathbb{R}$.

Proof. Claim 1. G/G_0 is compact. Again, since the only connected subgroups of $\mathbb{R}$ are $\{0\}$ and $\mathbb{R}$, we know that $KG_0/K = G/K$ and thus $G = KG_0$. Hence G/G_0 is compact as asserted.

Claim 2. K contains a compact normal subgroup N of G such that G/N is a Lie group. There are arbitrarily small compact normal subgroups N of G such that G/N is a Lie group. (See e.g. [23], p.175, Theorem 4.6.) As K is the unique maximal compact normal subgroup, $N \subseteq K$ follows.

Now G/N is a Lie group with finitely many components by Claims 1 and 2, while K/N is a compact normal subgroup and $(G/N)/(K/N) \cong G/N \cong \mathbb{R}$. Then by Corollary D we know $G/N \cong K/N \times \mathbb{R}$. Then $G \cong K \times \mathbb{R}$ follows at once. ∎

A straightforward generalisation of the Corollary D, $G/K \cong \mathbb{Z}$ along the lines of Remark 1.9 is not possible. Let $K_* = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ and let $\alpha: \mathbb{Z} \to \mathrm{Aut}(K_*)$ be defined by

$$\alpha(m)((x_n)_{n\in\mathbb{Z}}) = (x_{(n-m)})_{n\in\mathbb{Z}}.$$

Set $G = K_* \rtimes_\alpha \mathbb{Z}$, $K = K_* \times \{0\}$. Then $G/K \cong \mathbb{Z}$ and the unique smallest nondegenerate compact normal subgroup is K. Also $K_0Z(K,G) = K$. The essential conclusions of Corollary D fail. What remains is the following exercise.

Exercise 1.10. Let G be a locally compact group and K a compact normal subgroup such that $G/K \cong \mathbb{Z}$ and that G has arbitrarily small compact normal subgroups N such that G/N is a linear Lie group. Then there is a natural number e such that for each $a \in G \setminus K$ the open normal subgroup $K\langle a^e \rangle$ of G is isomorphic to $K \times \mathbb{Z}$. If $\langle Ka \rangle = G$, then $K\langle a^e \rangle$ has index e in G.

The following exercise rounds off the discussion illustrated by Example 1.2.1 which is shown to depend on the dimension of the center of K_1.

Exercise 1.11. Let K_1 be a compact connected Lie group, let Z be the identity component of the center, and let S be the (semisimple) commutator subgroup of K_1. The automorphism group of S is a finite extension of the inner automorphism group of S and is therefore compact. Now $\alpha \mapsto (\alpha|Z, \alpha|S)$: $\mathrm{Aut}(K_1) \to \mathrm{Aut}(Z) \times \mathrm{Aut}(S)$ is an embedding and thus $\mathrm{Aut}(K_1)$ is compact if $\mathrm{Aut}(Z)$ is compact, and this is the case if and only if $\dim Z < 2$. If $\mathrm{Aut}(K_1)$ is compact, then the natural semidirect product $P \stackrel{\mathrm{def}}{=} K_1 \rtimes \mathrm{Aut}(K_1)$ is a compact Lie group G with the normal subgroup $K \stackrel{\mathrm{def}}{=} K_1 \times \{1\}$. Trivially the number of components of G is finite. Hence, by Corollary C there is a natural number e such that for any $\alpha \in \mathrm{Aut}(K_1)$ there is an element $k \in K_1$ such that $(k_\alpha, \alpha^e) \stackrel{\mathrm{def}}{=} (k, \alpha)^e$ commutes elementwise with K.

Now let $\alpha \in \mathrm{Aut}(K_1)$ and recall the notation of 1.2.1. Then $\sigma\colon K_1 \rtimes_\alpha \mathbb{Z} \to P$, $\sigma(x, m) = (x, \alpha^m)$ is injective. Hence (k_α, e) commutes elementwise with $K_1 \times \{0\}$, and $K_1 \times e\mathbb{Z}$ is a direct product. ∎

2. The proof of Theorem A.

The proof proceeds through several steps.

Case 1. The case that K is connected and abelian. Let K be a connected compact abelian subgroup of a Lie group G. Since K is compact, each $\mathrm{ad}\, X$, $X \in \mathfrak{k}$ is semisimple because $\overline{e^{\mathbb{R}\cdot \mathrm{ad}\, X}} = \overline{\mathrm{Ad}(\exp \mathbb{R}\cdot X)}$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$. Hence there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{k}$. (See [1], Chap. 7, §2, n° 3, Proposition 10). Since the group $\exp \mathfrak{k} = K$ is compact and contained in the nilpotent Lie subgroup $H \stackrel{\mathrm{def}}{=} \exp \mathfrak{h}$, it is central in H. (Recall that any connected nilpotent Lie group has a unique maximal compact subgroup which is central.) Hence $[\mathfrak{k}, \mathfrak{h}] = \{0\} = [\mathfrak{k}, \mathfrak{k}]$. Now Theorem 4.9 on p. 131 of [13] applies and shows that the Weyl group $\mathcal{W}(\mathfrak{k}, \mathfrak{g})$ is finite. By Lemma 4.3 on p. 128 of [13] this means that $N(K, G_0)/Z(K, G_0)$ is finite. From Lemma 1.3, recalling $K = K_0 \subseteq Z(K, G)$, we note

$$N(K, G_0)/Z(K, G_0) \cong N(K, G_0)Z(K, G)/Z(K, G),$$

so this last group is finite. Further $N(K, G)/\big(N(K, G) \cap G_0) \cong N(K, G)G_0/G_0$. Now assume that G/G_0 is finite. Then the group $N(K, G)/N(K, G_0)$, being isomorphic to a subgroup of G/G_0, is finite. Then $N(K,G)/N(K,G_0)Z(K,G)$, as a quotient of $N(K,G)/N(K,G_0)$, is finite, too. Hence $N(K,G)/Z(K,G)$ is finite as asserted. Since $K = K_0 \subseteq Z(K,G)$, Theorem A is proved in this case.

Case 2. The case that K is connected. Any compact group is the product of the identity component $Z_0(K)$ of the center and the commutator group K' which is a semisimple Lie group; both of these subgroups are (fully) characteristic. We construct morphisms

$$\varphi\colon N(K,G)/KZ(K,G) \to N(Z_0(K),G)/Z(Z_0(K),G),$$
$$\psi\colon N(K,G)/KZ(K,G) \to \mathrm{Aut}(K')/\mathrm{Int}(K').$$

The range of φ is finite in view of the case that K is connected abelian, the range of ψ is finite by a result of Iwasawa's [15]. If we can show that $\ker\varphi\cap\ker\psi = \{1\}$, the assertion of Theorem A follows in this case.

Now $N(K,G) \subseteq N(Z_0(K),G)$ and $Z(K,G) \subseteq Z(Z_0(K),G)$. Moroever, $K' \subseteq Z(Z_0(K),G)$ and $Z_0(K) \subseteq Z(Z_0(K),G)$ whence

$$KZ(K,G) \subseteq Z_0(K)K'Z(Z_0(K),G) \subseteq Z(Z_0(K),G).$$

Hence φ is well defined by $\varphi\big(nKZ(K,G)\big) = nZ(Z_0(K),G)$ and has the kernel $Z(Z_0(K),G)/KZ(K,G)$.

Recall that K' is characteristic, hence normal in $N(K,G)$. Each inner automorphism I_n, $I_n(k) = nkn^{-1}$, $n \in N(K,G)$, $k \in K'$, gives us a morphism $n \mapsto I_n\colon N(K,G) \to \mathrm{Aut}(K')$. This morphism maps $KZ(K,G) = K'Z_0(K)Z(K,G) = K'Z(K,G)$ to the subgroup of inner automorphisms $\mathrm{Int}(K')$ and therefore defines ψ via $\psi\big(nZ(K,G)\big) = I_n{\cdot}\mathrm{Int}(K')$. The kernel of this morphism is $KZ(K',G)/KZ(K,G)$. Hence

$$\ker\varphi\cap\ker\psi = \big(Z(Z_0(K),G)\cap KZ(K',G)\big)/KZ(K,G).$$

Now

$$K' \subseteq Z(Z_0(K),G) \quad\text{and}\quad KZ(K',G) = K'Z(K',G).$$

Hence the numerator equals $K'\big(Z(Z_0(K),G)\cap Z(K',G)\big)$ by the modular law, and $Z(Z_0(K),G)\cap Z(K',G) = Z(K,G)$ because $K = Z_0(K)K'$. It follows that $\ker\varphi\cap\ker\psi$ is indeed singleton. Thus Theorem A is proved for connected $K = K_0$.

Case 3. The general case. Since K_0 is characteristic in K we have $N(K,G) \subseteq N(K_0,G)$. By the preceding $N(K_0,G)/\big(K_0Z(K_0,G)\big)$ is finite, and then

$$\big(N(K,G)Z(K_0,G)\big)/\big(K_0Z(K_0,G)\big) \cong N(K,G)/\big(N(K,G)\cap K_0Z(K_0,G)\big)$$

is finite. We claim that

$$\big(N(K,G)\cap K_0Z(K_0,G)\big)/K_0Z(K,G)$$

is finite. This will imply that $N(K,G)/K_0Z(K,G)$ is finite, as asserted in Theorem A.

We consider the morphism $\alpha\colon N(K,G) \to \operatorname{Aut}(K)$, $\alpha(n) = I_n|K$. Then $\ker\alpha = Z(K,G)$. Let $\operatorname{Int}_{K_0}(K) = \alpha(K_0)$. By a result of Iwasawa's [15], this group is the identity component $\operatorname{Aut}_0(K)$ of $\operatorname{Aut}(K)$. Thus α induces an injective morphism $\alpha_0\colon N(K,G)/K_0Z(K,G) \to \operatorname{Aut}(K)/\operatorname{Aut}_0(K)$. Let $\operatorname{Aut}_{K_0}(K)$ be the group of automorphisms of K fixing K_0 elementwise. Then $\alpha_0\bigl(N(K,G)\cap K_0Z(K_0,G)/K_0Z(K,G)\bigr)$ is contained in

$$\begin{aligned}\operatorname{Aut}_{K_0}(K)\operatorname{Aut}_0(K)/\operatorname{Aut}_0(K) &\cong \operatorname{Aut}_{K_0}(K)/\bigl(\operatorname{Aut}_{K_0}(K)\cap\operatorname{Aut}_0(K)\bigr)\\ &= \operatorname{Aut}_{K_0}(K)/\bigl(\operatorname{Aut}_{K_0}(K)\cap\operatorname{Int}_{K_0}(K)\bigr)\\ &= \operatorname{Aut}_{K_0}(K)/\operatorname{Int}_{Z(K_0)}(K).\end{aligned}$$

It thus suffices for our purposes to verify that $\operatorname{Aut}_{K_0}(K)/\operatorname{Int}_{Z(K_0)}(K)$ is finite. Now let $\operatorname{Aut}_{K_0,K/K_0}(K)$ be the subgroup of $\operatorname{Aut}_{K_0}(K)$ of all automorphisms $\sigma\in\operatorname{Aut}(K)$ with $\sigma(k)\in kK_0$. Then $\operatorname{Int}_{K_0}(K)\subseteq\operatorname{Aut}_{K_0,K/K_0}(K)\subseteq\operatorname{Aut}_{K_0}(K)$. The group $\operatorname{Aut}_{K_0,K/K_0}(K)/\operatorname{Int}_{Z(K_0)}(K)$ is finite according to Iwasawa [15]. It therefore remains to show that

$$\operatorname{Aut}_{K_0}(K)/\operatorname{Aut}_{K_0,K/K_0}(K)$$

is finite. Indeed, we notice that the homomorphism $\beta\colon\operatorname{Aut}_{K_0}(K)\to\operatorname{Aut}(K/K_0)$, $\beta(\sigma)(kK_0) = \sigma(k)K_0$ has the precise kernel $\operatorname{Aut}_{K_0,K/K_0}(K)$ and that the group $\operatorname{Aut}(K/K_0)$ is finite because K is a compact Lie group, whence K/K_0 is finite. Thus $\operatorname{Aut}_{K_0}(K)/\ker\beta$ is finite. This completes the proof of the theorem. ∎

3. Applications to probability theory

Theorem A, as we pointed out in the introduction, was motivated by an attempt to describe the structure of certain closed subgroups of $\mathrm{GL}(\mathbb{R}^d)$ which arise in the theory of infinitely divisible probability measures on $\mathbb{R}^d$. We describe the situation. Let $\mathbb{R}_+^\times$ denote the multiplicative group of positive real numbers. For a locally compact group G we let denote $\operatorname{Aut}(G)$ its group of automorphisms endowed with the standard refined compact open topology. If G is a connected Lie group, then $\operatorname{Aut}(G)$ is isomorphic to a subgroup of $\operatorname{Aut}(\mathfrak{g})\subseteq\mathrm{GL}(\mathfrak{g})$ (with the Lie algebra $\mathfrak{g}$ of G) and is therefore a linear Lie group. In particular, every subgroup of $\operatorname{Aut}(G)$ has a countable basis for its topology. For any group G we let $Z(G)$ denote the center of G. If G is a Lie group, $\mathfrak{g}$ is its Lie algebra, and $\mathfrak{z}(\mathfrak{g})$ is the Lie algebra of $Z(G)$.

Let $\mu_\bullet := (\mu_t)_{t\geq 0}$ be a continuous convolution semigroup. Then μ_0 is an idempotent, hence $\mu_0 = \omega_L$ a Haar measure on a compact subgroup L. To any probability measure μ define $\widetilde{\mu}$ by $\widetilde{\mu}(A) \stackrel{\text{def}}{=} \mu(A^{-1})$ for Borel sets A.

Definition 3.1. For a locally compact group G and a continuous convolution semigroup $\mu_\bullet := (\mu_t)_{t\geq 0}$ we set

$$\begin{aligned}\operatorname{Dec}(\mu_\bullet) &= \{A\in\operatorname{Aut}(G) : \bigl(\exists a\in\mathbb{R}_+^\times,\ X\in\mathfrak{z}(\mathfrak{g})\bigr)(\forall t\geq 0)\quad A\mu_t = \mu_{ta}*\varepsilon_{\exp t\cdot X}\}\\ \operatorname{Sym}(\mu_\bullet) &= \{A\in\operatorname{Aut}(G) : \bigl(\exists X\in\mathfrak{z}(\mathfrak{g})\bigr)(\forall t\geq 0)\quad A\mu_t = \mu_t*\varepsilon_{\exp t\cdot X}\}.\end{aligned}$$

The group $\mathrm{Dec}(\mu_\bullet)$ is called the *decomposability group*, $\mathrm{Sym}(\mu_\bullet)$ the *group of symmetries*. Indeed, it is easily shown that $\mathrm{Dec}(\mu_\bullet)$ and $\mathrm{Sym}(\mu_\bullet)$ are subgroups of $\mathrm{Aut}(G)$. ■

The projection $f_{\mu_\bullet}: \mathrm{Dec}(\mu_\bullet) \to (\mathbb{R}_+^\times, \cdot)$, $f_{\mu_\bullet}(A) = a$, is a continuous homomorphism with kernel $\mathrm{Sym}(\mu_\bullet)$ so that we have the following exact sequence before our eyes:

$$(\dagger) \qquad 0 \to \mathrm{Sym}(\mu_\bullet) \xrightarrow{\text{incl}} \mathrm{Dec}(\mu_\bullet) \xrightarrow{f_{\mu_\bullet}} \mathbb{R}_+^\times \underset{\cong}{\xrightarrow{\log}} \mathbb{R}.$$

Definition 3.2. A convolution semigroup $\mu_\bullet$ on a locally compact group G is called

(i) *semistable* if and only if $\mathrm{Dec}(\mu_\bullet) \neq \mathrm{Sym}(\mu_\bullet)$,
(ii) *stable* if and only if $\mathrm{im}(f_{\mu_\bullet}) = \mathbb{R}_+^\times$,
(iii) *properly semistable* if and only if $\mathrm{im}(f_{\mu_\bullet}) = (a^n : n \in \mathbb{Z})$ for some $a \in]0,1[$,
(iv) (A,a)*-semistable* if $A \in \mathrm{Dec}(\mu_\bullet)$ and $f_{\mu_\bullet}(A) = a$, and
(v) *strictly* (A,a)*-semistable* if $(\forall t \geq 0)\, A(\mu_t) = \mu_{at}$. ■

Assume for a moment that for a semistable law $\mu_\bullet$ on a connected Lie group G, we have the additional information that $\mathrm{Dec}(\mu_\bullet)$ is closed in $\mathrm{Aut}(G)$ and that $\mathrm{Sym}(\mu_\bullet)$ is compact. Then from (†) we know that the group $\mathrm{Dec}(\mu_\bullet)$ is the extension of the maximal compact normal subgroup $\mathrm{Sym}(\mu_\bullet)$ by a group isomorphic to $\mathbb{R}$ (if $\mu_\bullet$ is stable) or $\mathbb{Z}$ (if $\mu_\bullet$ is properly semistable). In these circumstances, Corollary D applies. This is the probability theoretical interest of the discourse in Section 1.

Now it is important for us to explain sufficient conditions for $\mathrm{Dec}(\mu_\bullet)$ to be closed in $\mathrm{Aut}(G)$ and for $\mathrm{Sym}(\mu_\bullet)$ to be compact. As usual ε_g denotes the probability measure concentrated on $\{g\}$. We shall say that $\mu_\bullet$ *is degenerate* if for each t there is a compact subgroup L_t of G and an element $x_t \in G$ such that $\mu = \omega_{L_t} * \varepsilon_{x_t}$, equivalently, if for each t the measure $\mu_t * \widetilde{\mu_t}$ is the normalized Haar measure of a compact subgroup of G.

Proposition 3.3. *Assume that G is a Lie group and that $\mu_\bullet$ is nondegenerate. Then* $\mathrm{Dec}(\mu_\bullet)$ *is a closed subgroup of* $\mathrm{Aut}(G)$. *Moreover, if G is connected, then* $\mathrm{Dec}(\mu_\bullet)$ *is a linear Lie group with a countable basis for its topology.*

Proof. The idempotent μ_0 is a Haar measure ω_L on a compact subgroup L. Assume $A_n \in \mathrm{Dec}(\mu_\bullet)$, $A = \lim_n A_n$ in $\mathrm{Aut}(G)$ and $A_n(\mu_t) = \mu_{ta_n} * \varepsilon_{\exp(tX_n)}$. We must show $A \in \mathrm{Dec}(\mu_\bullet)$. Now $A_n(\mu_t * \tilde{\mu}_t) = \mu_{ta_n} * \tilde{\mu}_{ta_n} \to A(\mu_t * \tilde{\mu}_t)$. Suppose firstly that $a_n \to 0$. Then $A(\mu_t * \tilde{\mu}_t) = \mu_0 * \tilde{\mu}_0 = \omega_L$, contradicting the hypothesis that $\mu_\bullet$ is nondegenerate. Suppose secondly $a_n \to \infty$. As we shall show in an appendix, $\mu_t * \tilde{\mu}_t$ converges vaguely for $t \to \infty$. Then $A(\mu_t * \tilde{\mu}_t) = \rho = \lim_{s\to\infty} \mu_s * \tilde{\mu}_s$ for all $t > 0$, again a contradiction. Hence the sequence a_n is bounded away from 0 and ∞. Assume $a_n \to a \in]0,\infty[$. As above we obtain now $A(\mu_t) = \lim_{n\to\infty} \mu_{a_n t} * \varepsilon_{\exp(tX_n)}$. Since $\mu_{a_n t}$ converges to μ_{at}, we obtain the boundedness of $\{\exp(tX_n)\}$ for all $t \geq 0$. Hence

$$(*) \qquad (\forall t \geq 0)(\exists z(t) \in Z(G)_0) \quad A(\mu_t) = \mu_{at} * \varepsilon_{z(t)}.$$

Lemma Z. *Assume that* $\mu_\bullet$ *is a convolution semigroup of a Lie group* G, *that* $A \in \operatorname{Aut}(G)$, *and that* $0 < a \in \mathbb{R}$. *If* Condition $(*)$ *is satisfied, then there is an* $X \in \mathfrak{z}(\mathfrak{g})$ *such that*

$$(**) \qquad (\forall t \geq 0) \quad \mu_{at} * \varepsilon_{z(t)} = \mu_{at} * \varepsilon_{\exp(t\cdot X)}.$$

In particular, $A \in \operatorname{Dec}(\mu_\bullet)$.

Once Lemma Z is proved it will be established that $\operatorname{Dec}(\mu_\bullet)$ is closed in $\operatorname{Aut}(G)$.

Proof of Lemma Z. This requires that we find an $X \in \mathfrak{z}(\mathfrak{g})$ such that $\mu_{at} * \varepsilon_{z(t)} = \mu_{at} * \varepsilon_{\exp(t\cdot X)}$. Define $I_r = \{x \in Z(G)_0 : \mu_r * \varepsilon_x = \mu_r\}$. Then I_r is a compact subgroup for each r. Since the function $r \mapsto I_r$ is increasing and continuous from the right, G is a Lie group, and the set of closed compact subgroups of a Lie group satisfies the descending chain condition, there is a $\delta > 0$ and a compact subgroup I such that $I_r = I$ for all $0 \leq r < \delta$. We compute $\mu_{a(s+t)} * \varepsilon_{z(s+t)} = A(\mu_{s+t}) = A(\mu_s) * A(\mu_t) = \mu_{as} * \varepsilon_{z(s)} * \mu_{at} * \varepsilon_{z(t)} = \mu_{as} * \mu_{at} * \varepsilon_{z(s)} * \varepsilon_{z(t)} = \mu_{a(s+t)} * \varepsilon_{z(s)z(t)}$ since $z(s)$ is central. Hence $z(s)z(t) \in z(s+t)I_{a(s+t)}$, and thus $z(s)z(t) \in z(s+t)I$ for $0 \leq s+t < \frac{\delta}{a}$. The local one-parameter subsemigroup $r \mapsto z(r)I : \left[0, \frac{\delta}{a}\right[\to Z_0(G)I/I \cong Z_0(G)/(Z_0(G) \cap T)$ gives rise to an element $X \in \mathfrak{z}(\mathfrak{g})$ such that $z(t)I = \exp(tX)I$ for $0 \leq t < \frac{\delta}{a}$. Then $A(\mu_t) = \mu_{at} * \varepsilon_{z(t)} = \mu_{at} * \varepsilon_{\exp(t\cdot X)}$ for $t \in \left[0, \frac{\delta}{a}\right[$. Since $t \mapsto A(\mu_t)$, $\mu_{at} * \varepsilon_{\exp(t\cdot X)} \colon [0, \infty[\to \mathcal{M}^1(G)$ are continuous one-parameter semigroups which agree on a neighborhood of 0, they agree. This proves statement $(**)$ of Lemma Z. From $(*)$ and $(**)$ and from the definition of $\operatorname{Dec}(\mu_\bullet)$ in 3.1 it follows that $A \in \operatorname{Dec}(\mu_\bullet)$. Thus Lemma Z is proved.

If G is connected, then $\operatorname{Aut}(G)$ is a Lie subgroup of $\operatorname{Aut}(\mathfrak{g}) \subseteq \operatorname{GL}(\mathfrak{g})$. In these circumstances, as a closed subgroup of $\operatorname{Aut}(G)$, the group $\operatorname{Dec}(\mu_\bullet)$ is a linear Lie group, and we conclude that it has a countable basis for its topology.■

In order to prove the compactness of $\operatorname{Sym}(\mu_\bullet)$ and closedness of $\operatorname{im} f_{\mu_\bullet}$ one needs a "Convergence of Types Theorem." We explain what this means. Namely, we say that *the Convergence of Types Theorem holds* if there is a subgroup $\mathcal{B} \subseteq \operatorname{Aut}(G)$ and an open subset $\mathcal{F} \subseteq \mathcal{M}^1(G)$ such that for $\mu, \nu \in \mathcal{F}$,

$$(\mu_n, A_n, x_n) \in \mathcal{M}^1(G) \times \mathcal{B} \times G, \quad n \in \mathbb{N},$$

such that

$$(\mu, \nu) = \lim_{n\to\infty} (\mu_n, A_n(\mu_n) * \varepsilon_{x_n}),$$

it follows that $\{A_n : n \in \mathbb{N}\}$ and $\{x_n : n \in \mathbb{N}\}$ are relatively compact in $\mathcal{B}$ and G, respectively. We shall also say that *a Convergence of Types Theorem holds for a convolution semigroup* $\mu_\bullet$ if a Convergence of Types Theorem holds in such a fashion that $\operatorname{Dec}(\mu_\bullet) \subseteq \mathcal{B}$ and $\mu_t \in \mathcal{F}$ for $t > 0$.

Proposition 3.4. *Let* G *be a connected Lie group and* $\mu_\bullet$ *a convolution semigroup. If a Convergence of Types Theorem holds for* $\mu_\bullet$, *then* $\operatorname{im} f_{\mu_\bullet}$ *is closed in* $\mathbb{R}_+^\times$, *and* $\operatorname{Sym}(\mu_\bullet)$ *is compact.*

Proof. (Cf. [7], [9], [24].) Assume that (a_n) is a sequence in $\operatorname{im} f_{\mu_\bullet}$ converging to $a \in \,]0, \infty[$. We must show that $a \in \operatorname{im} f_{\mu_\bullet}$. Pick $A_n \in \operatorname{Dec}(\mu_\bullet)$ with $f_{\mu_\bullet}(A_n) = a_n$ and find $X_n \in \mathfrak{z}(\mathfrak{g})$ such that

$$(1) \qquad \mu_{a_n t} = A_n(\mu_t) * \varepsilon_{\exp(-tX_n)}.$$

Then $\mu_{a_n t} \to \mu_{at}$, $t \geq 0$. By the Convergence of Types Theorem for $\mu_\bullet$ we obtain relative compactness of $\{A_n : n \geq 1\}$ and of $\{\exp -tX_n : n \geq 1\}$ for each $t > 0$. Hence there exist accumulation points $A \in \operatorname{Aut}(G)$ and $z(t) \in Z(G)_0$, such that

$$(*) \qquad (\forall t \geq 0)(\exists z(t) \in Z(G)_0) \quad A(\mu_t) = \mu_{at} * \varepsilon_{z(t)}.$$

By Lemma Z in the proof of 3.3 we find an element $X \in \mathfrak{z}(\mathfrak{g})$ such that

$$(**) \qquad (\forall t \geq 0) \quad A(\mu_t) = \mu_{at} * \varepsilon_{\exp(t \cdot X)}$$

and that $A \in \operatorname{Dec}(\mu_\bullet)$. This shows that $a \in \operatorname{im} f_{\mu_\bullet}$ and thus complete the proof that $\operatorname{im} f_{\mu_\bullet}$ is closed. In order to see the compactness of $\operatorname{Sym}(\mu_\bullet)$ we take $\{A_n : n \in \mathbb{N}\}$ in $\operatorname{Sym}(\mu_\bullet)$, and we have to show that this sequence has a cluster point. We find $X_n \in \mathfrak{z}(\mathfrak{g})$ such that

$$(2) \qquad \mu_t = A_n(\mu_t) * \varepsilon_{\exp(-t \cdot X_n)}.$$

This relation follows from (1) upon setting $a_n = 1$ for all n. Thus the argument in the first part of the proof shows that there is a cluster point A of $\{A_n : n \in \mathbb{N}\}$ in $\operatorname{Dec}(\mu_\bullet)$ and an $X \in \mathfrak{z}(\mathfrak{g})$ such that $A(\mu_t) = \mu_t * \varepsilon_{\exp(t \cdot X)}$. Hence $A \in \operatorname{Sym}(\mu_\bullet)$, and the compactness of $\operatorname{Sym}(\mu_\bullet)$ is established. ■

We shall discuss below situations in which a Convergence of Types Theorem is available for a given semigroup $\mu_\bullet$.

Corollary 3.5. *Assume the hypotheses of* Proposition 3.4 *and, in addition, that* $\mu_\bullet$ *is stable. Then* $\operatorname{Dec}(\mu_\bullet) \cong \operatorname{Sym}(\mu_\bullet) \times \mathbb{R}$.

Proof. The surjectivity of $f_{\mu_\bullet}$ implies that the following sequence is exact:

$$0 \to \operatorname{Sym}(\mu_\bullet) \xrightarrow{\text{incl}} \operatorname{Dec}(\mu_\bullet) \xrightarrow{\log \circ f_{\mu_\bullet}} \mathbb{R} \to 0.$$

Set $\Gamma \stackrel{\text{def}}{=} \operatorname{Dec}(\mu_\bullet)/\operatorname{Sym}(\mu_\bullet)$. Then $\log \circ f_{\mu_\bullet}$ induces canonically a bijective morphism $f\colon \Gamma \to \mathbb{R}$. Since $\operatorname{Sym}(\mu_\bullet)$ is compact and $\operatorname{im} f_{\mu_\bullet}$ is closed in $\mathbb{R}_+^\times$ by 3.4, by the Open Mapping Theorem for morphisms between locally compact groups, $f_{\mu_\bullet}$ is open onto its image, and thus is also a closed map because its kernel $\operatorname{Sym}(\mu_\bullet)$ is compact. Hence f is also closed and thus is an isomorphism. Now Remark 1.9 proves the corollary. ■

The conclusion of the previous Corollary may be obtained with slightly weaker hypotheses as follows. We continue to set $\Gamma \stackrel{\text{def}}{=} \operatorname{Dec}(\mu_\bullet)/\operatorname{Sym}(\mu_\bullet)$ and let $f\colon \Gamma \to \mathbb{R}$ be the morphism induced by $\log \circ f_{\mu_\bullet}$.

Remark 3.6. Let G be a locally compact group and $\mu_\bullet$ a stable convolution semigroup such that $\operatorname{Dec}(\mu_\bullet)$ is locally compact and $\operatorname{Sym}(\mu_\bullet)$ is compact. If $\operatorname{Dec}(\mu_\bullet)$ has a countable basis for its topology, which is the case if G is a connected Lie group and $\operatorname{Dec}(\mu_\bullet)$ is closed in $\operatorname{Aut}(G)$, then $\operatorname{Dec}(\mu_\bullet) \cong \operatorname{Sym}(\mu_\bullet) \times \mathbb{R}$.

Proof. Since $\mu_\bullet$ is stable, $f:\Gamma \to \mathbb{R}$ is a bijective morphism of topological groups. The connected subgroup $f(\Gamma_0)$ of $\mathbb{R}$ is either $\{0\}$ or $\mathbb{R}$ since $\mathbb{R}$ does not contain any proper connected nonsingleton subgroups. In the latter case, the bijectivity of f implies $\Gamma_0 = \Gamma$. Then, as a locally compact and connected group, Γ is σ-compact. Hence f is an isomorphism. If this fails, then $\Gamma_0 = \{1\}$, i.e., Γ is a totally disconnected locally compact abelian group. Hence it has an open compact subgroup C. Then $f(C) = \{0\}$ since $\mathbb{R}$ has no nondegenerate compact subgroups. Hence $C = \{1\}$ and thus Γ is discrete. Then $\mathrm{Dec}(\mu_\bullet)$ has uncountably many open components and thus cannot be contained in the group $\mathrm{Aut}(G)$ which has a countable basis for its topology. So this is impossible. The splitting is then concluded form 1.9 as in the proof of 3.5. ■

A particular consequence of Corollary 3.5 concerns information on convolution semigroups. Indeed, from 3.5 we obtain a one-parameter subgroup $t \mapsto \tau_t: \mathbb{R}_+^\times \to Z\big(\mathrm{Sym}(\mu_\bullet), \mathrm{Aut}(G)\big)$ such that $\mathrm{Dec}(\mu_\bullet) = \mathrm{Sym}(\mu_\bullet)\tau_{\mathbb{R}_+^\times}$ and that this product is direct.

Corollary 3.7. *Assume that G is a locally compact group and $\mu_\bullet$ a convolution semigroup such that $\mathrm{Dec}(\mu_\bullet) \cong \mathrm{Sym}(\mu_\bullet) \times \mathbb{R}$; by 3.6, this hypothesis is satisfied if G is a connected Lie group, if a Convergence of Types Theorem holds for $\mu_\bullet$, and if $\mu_\bullet$ is stable. Then there exists a continuous one-parameter group $t \mapsto \tau_t: \mathbb{R}_+^\times \to \mathrm{Dec}(\mu_\bullet)$, $\tau_{st} = \tau_s\tau_t$, such that $\mu_\bullet$ is (τ_t, t)-semistable for all $t > 0$. In other words, for all $s \geq 0$ and $t > 0$ we have*

$$\tau_t(\mu_s) = \mu_{ts} * \varepsilon_{\exp(tsX)}$$

for a suitable $X = X(t) \in \mathfrak{z}(\mathfrak{g})$. ■

Next we deal with the *properly semistable* variant of Proposition 3.4. For this purpose we introduce the following definition.

Definition 3.8. An element $A \in \mathrm{Dec}(\mu_\bullet)$ is called a *commuting normalization* if $AS = SA$ for all $S \in \mathrm{Sym}(\mu_\bullet)$. ■

We note the following. Assume that A is a commuting normalisation. Then we define $a \in \mathbb{R}_+^\times \setminus \{1\}$ by $f_{\mu_\bullet}(A) = a$. Now the semidirect product

$$\mathrm{Dec}_a(\mu_\bullet) \stackrel{\text{def}}{=} \{A' \in \mathrm{Dec}(\mu_\bullet) : f_{\mu_\bullet}(A') = a^n,\, n \in \mathbb{Z}\} = f_{\mu_\bullet}^{-1}\{a^n : n \in \mathbb{Z}\}$$

splits as a direct product over the compact subgroup $\mathrm{Sym}(\mu_\bullet)$ in the form

$$\mathrm{Dec}_a(\mu_\bullet) = \mathrm{Sym}(\mu_\bullet) \times \{A^n : n \in \mathbb{Z}\} \cong \mathrm{Sym}(\mu_\bullet) \times \mathbb{Z}.$$

Theorem 3.9. *Let G be a connected Lie group and $\mu_\bullet$ a properly semistable convolution semigroup of probability measures on G. If a Convergence of Types Theorem holds for $\mu_\bullet$, then the following conclusions hold,*

(I) $\mathrm{Dec}(\mu_\bullet) \cong \mathrm{Sym}(\mu_\bullet) \rtimes \mathbb{Z}$.

(II) *There exists a natural number $e \in \mathbb{N}$ such that for any $C \in \mathrm{Dec}(\mu_\bullet)$ there is an $S \in \mathrm{Sym}(\mu_\bullet)_0$ such that $A \stackrel{\text{def}}{=} C^eS$ is a commuting normalization.*

(III) *Let* $c = f_{\mu_\bullet}(C)$. *Then*

$$f_{\mu_\bullet}^{-1}\big(\{c^{en} : n \in \mathbb{Z}\}\big) = \mathrm{Sym}(\mu_\bullet)\{A^n : n \in \mathbb{N}\},$$

and this product is direct and isomorphic to $\mathrm{Sym}(\mu_\bullet) \times \mathbb{Z}$.

(IV) *Let* $C \in \mathrm{Dec}(\mu_\bullet)$ *be such that* $f_{\mu_\bullet}(C)$ *is a generator of* $\mathrm{im}(f_{\mu_\bullet})$. *Then there is an element* $S \in \mathrm{Sym}(\mu_\bullet)_0$ *such that the subgroup* $\mathrm{Sym}(\mu_\bullet)\langle C^e S\rangle$ *of* $\mathrm{Dec}(\mu_\bullet)$ *is a direct product and* $\mathrm{Dec}(\mu_\bullet)/\mathrm{Sym}(\mu_\bullet)\langle C^e S\rangle$ *is a cyclic group of order* e.

Proof. (I) It follows from 3.4 that $\mathrm{im}(f_{\mu_\bullet})$ is a closed subgroup of $\mathbb{R}_+^\times$; it is proper since $\mu_\bullet$ is properly semistable and nondegenerate since $\mu_\bullet$ is semistable. Therefore it is infinite cyclic discrete. Thus $\mathrm{im}(f_{\mu_\bullet})$ is free in the category of topological groups and (I) follows.

(II) By (I) there is an element $T \in \mathrm{Dec}(\mu_\bullet)$ such that $\mathrm{Dec}(\mu_\bullet) = \mathrm{Sym}(\mu_\bullet)\langle T\rangle$ with $\langle T\rangle = \{T^n : n \in \mathbb{Z}\}$, and T normalizes $\mathrm{Sym}(\mu_\bullet)$. By 3.4 we may consider $\mathrm{Dec}(\mu_\bullet)$ as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ and thus as a closed subgroup of $\mathrm{GL}(\mathfrak{g})$. Therefore the assumptions of Corollary D are fulfilled, and the assertion follows from Corollary D(ii).

(III) Moreover, since $f_{\mu_\bullet}(\langle T\rangle) = \mathrm{im}\, f_{\mu_\bullet}$ is infinite cyclic in $\mathbb{R}_+^\times$, it follows that $\mathrm{Dec}(\mu_\bullet)/\mathrm{Sym}(\mu_\bullet)$ is infinite cyclic discrete. Thus the assertion follows from Corollary D.

(IV) If we take $C = T$ with T as in the proof of (II), then (II) applies and yields the assertions straightforwardly. ∎

This theorem was an original motivation for the theory presented in Sections 1 and 2. In the remainder of the paper we illustrate situations in which we have a Convergence of Types Theorem so that 3.4 and its corollaries, notably 3.9 become applicable. Firstly, all simply connected nilpotent Lie groups admit a Convergence of Types Theorem with $\mathcal{B} = \mathrm{Aut}(G)$, generalizing the well-known Convergence of Types Theorem for vector groups $\mathbb{R}^d$. A precise formulation will be given below. However, there are other types of Lie groups admitting a Convergence of Types Theorem, e.g., certain connected Lie groups without central tori of positive dimension or certain algebraic groups where the automorphisms are supposed to be algebraic [3]. Yet we shall restrict our considerations to simply connected nilpotent Lie groups. This is justified since the existence of a semistable continuous convolution semigroup on a connected Lie group causes these Lie groups to come into focus as was shown in [10]; we will overview this fact now.

Reduction to simply connected nilpotent Lie groups

We begin on a purely topological group theory level by considering a topological group G and an automorphism $A \in \mathrm{Aut}(G)$ of topological groups. Set

$$C(A) \stackrel{\mathrm{def}}{=} \{x \in G : \lim_{n\to\infty} A^n(x) = 1\}.$$

Then $C(A)$ is an A-invariant subgroup of G. Let L be a compact subgroup of G with $A(L) \subseteq L$. We take $x \in C(A)$ and $g \in L$. The space $S \stackrel{\mathrm{def}}{=}$

$\{Ax, A^2x, \ldots\} \cup \{1\}$ is compact. For each $n \in \mathbb{N}$ we write $g_n = A^n g A^{-n}$ and get $A^n(gxg^{-1}) = g_n(A^n x)g_n^{-1}$. Assume that $n(j)$ is a subnet of the sequence of natural numbers such that $h = \lim_j A^{n(j)}gxg^{-1}$ in the compact space LSL^{-1}. Since L is compact, there is a subnet $n(j_k)$, such that $c = \lim_k g_{n(j_k)}$ exists. Thus we obtain $h = c1c^{-1} = 1$. Hence $\lim_{n\to\infty} A^n(gxg^{-1}) = 1$ and $gxg^{-1} \in C(A)$. Therefore *every A-invariant compact subgroup L of G normalizes $C(A)$.*

For a finite dimensional real vector space $\mathfrak{g}$ and an endomorphism α we write $\mathfrak{g}_0(\alpha) = \{X \in \mathfrak{g} : \lim_{n\to\infty} \alpha^n(X) = 0\}$. Now assume that G is a Lie group with Lie algebra $\mathfrak{g} = \mathfrak{L}(G)$ and that $A \in \mathrm{Aut}(G)$ is an automorphism. Denote by $\mathfrak{L}(A) \in \mathrm{Aut}(\mathfrak{g})$ the automorphism induced on the Lie algebra. As is shown in [10], $C(A)$ is an analytic subgroup with Lie algebra $\mathfrak{L}\big(C(A)\big) = \mathfrak{g}_0\big(\mathfrak{L}(A)\big)$.

For a compact subgroup L of G we shall also consider the homogeneous space $G/L = \{xL : x \in G\}$ and define the subgroup

$$C_L(A) \stackrel{\text{def}}{=} \{x : \lim_{k\to\infty} A^k(x)L = L \text{ in } G/L\},$$

the *L-contractible part of G*. Clearly $C(A) \subseteq C_L(A)$. If $A(L) = L$, then L normalizes $C(A)$ and A induces an autodiffeomorphism A_L of the homogeneous space G/L in such a way that $x \in C_L(A)$ iff $\lim_{k\to\infty} A_L^k(xL) = L$ in G/L. In particular, $L \subseteq C_L(A)$ and thus $C(A)L \subseteq C_L(A)$. In Theorem 2.4 of [10] equality is shown. The normalizer of $C(A) \cap L$ contains $C(A)L = C_L(A)$, and thus $D \stackrel{\text{def}}{=} \overline{C(A) \cap L}$ is a compact normal subgroup of $\overline{C_L(A)}$.

The case $G = \overline{C_L(A)}$. This suggests that the hypothesis $G = \overline{C_L(A)}$ which we assume now is of particular interest. Then D is a compact normal subgroup of G and we can form the Lie groups

$$\begin{aligned} &\mathcal{G} \stackrel{\text{def}}{=} G/D, \\ &\mathcal{L} \stackrel{\text{def}}{=} L/D, \text{ and the autmorphism} \\ &\mathcal{A} \in \mathrm{Aut}(\mathcal{G}) \text{ induced by } A \in \mathrm{Aut}(G). \end{aligned}$$

The analytic subgroup $C(A)D/D$ of $\mathcal{G}$ becomes a Lie group when it is endowed with its intrinsic Lie group structure; this Lie group we shall call $\mathcal{N}$, and the inclusion morphism yields an injective morphism of Lie groups $j_{\mathcal{N}}: \mathcal{N} \to \mathcal{G}$ with bijective image $C(A)D/D$. The automorphism $\mathcal{A} \in \mathrm{Aut}(\mathcal{G})$ induces on $\mathcal{N}$ an automorphism $\mathcal{A}_{\mathcal{N}}$ such that $C(\mathcal{A}_{\mathcal{N}}) = \mathcal{N}$. It is shown in [10] that $\mathcal{N}$ *is a simply connected nilpotent Lie group.* The inner automorphisms implemented by elements of $\mathcal{L}$ on $C(A)D/D$ implement automorphisms of $\mathcal{N}$ and thus provide us with a morphism of Lie groups $\beta: \mathcal{L} \to \mathrm{Aut}(\mathcal{N})$. This allows us to form the Lie group $\Gamma \stackrel{\text{def}}{=} \mathcal{N} \rtimes_\beta \mathcal{L}$. Since $\mathcal{L}$ as a compact Lie group has finitely many components, the Lie group Γ has finitely many components. We let $\mathcal{C}$ denote the subgroup $C_L(A)/D$ of $\mathcal{G}$. The definition of D implies $C(A)D/D \cap L/D = \{D\}$. By Theorem 2.4 of [10] we have $C_L(A) = C(A)L$. Hence the morphism of topological groups $p: \Gamma \to \mathcal{C} \subseteq \mathcal{G}$, $p(n,x) = j_{\mathcal{N}}(n)x$ is bijective. The subgroup $p(\Gamma_0) \subseteq \mathcal{C}$ of $\mathcal{G}$ is analytic and has finite index in $\mathcal{C}$. The group $\mathcal{C}$ is σ-compact (i.e. is a countable union of compact subspaces) since it is a continuous image of the σ-compact space Γ; hence $\mathcal{C}$ is Borel measurable, and the inverse map $p^{-1}: \mathcal{C} \to \varGamma$ is Borel measurable. In passing we remark from the fact that D and $C_L(A)/D$ have finitely many arc components, that $C_L(A)$ has finitely many arc components. The arc component $C_L(A)_0$ of the identity is an analytic subgroup of G which is dense in G_0. In particular, $C_L(A)$ is Borel measurable. Thus G/G_0 is finite and $(G_0)'$, the commutator subgroup of G_0, is contained in $C_L(A)_0$.

The desintegration of convolution semigroups. Now assume that $\mu_\bullet$ is a convolution semigroup of probability measures on G. Then μ_0 is an idempotent probability measure and therefore is a Haar measure ω_L of a compact subgroup L of G. In addition we assume that $\mu_\bullet$ is semistable with $A \in \mathrm{Dec}(\mu_\bullet)$, $0 < a = f_{\mu_\bullet}(A) < 1$. Then $L \stackrel{\mathrm{def}}{=} \operatorname{supp} \mu_0$ is an A-invariant compact subgroup. It is no loss of generality if *we restrict our attention to the smallest closed A-invariant subgroup supporting $\mu_\bullet$.* The smallest closed subgroup supporting $\mu_\bullet$ is $H \stackrel{\mathrm{def}}{=} \overline{\langle \bigcup_{t>0} \operatorname{supp}(\mu_t) \rangle}$. If

$$H = \overline{\langle \bigcup_{t>0} \operatorname{supp}(\mu_t * \tilde{\mu}_t) \rangle}, \tag{+}$$

then H is A-invariant and thus agrees with G by our assumption. With the methods of [10] it follows that for any $t > 0$ $\mu_t * \tilde{\mu}_t$ is concentrated on the Borel measurable subgroup $C_L(A)$. Thus $C_L(A)$ is dense in G if (+) holds. In [10] it is shown that, if $\mu_\bullet$ is *strictly* (A, a)-semistable (see 3.2(v)), then, more strongly,

$$\mu_t\big(C_L(A)\big) = 1 \text{ for all } t \geq 0. \tag{++}$$

Thus again $C_L(A)$ is dense in G in this case. We observe that the condition

$$1 \in \operatorname{supp}(\mu_t) \text{ for all } t > 0$$

implies both (+) and (++); it therefore implies in particular, that $C_L(A)$ is a dense subgroup of G.

The bimeasurable homomorphism p defines an injective weakly continuous homomorphism $\mathcal{M}^1(\varGamma) \to \mathcal{M}^1(\mathcal{G})$ of convolution semigroups. Indeed, p establishes a bijection between the set of measures $\nu \times \omega_{\mathcal{L}}$, where ν is an $\mathcal{L}$-invariant measure on the subgroup $\mathcal{N}$ of $\varGamma$, and the set of measures $\lambda = \omega_{\mathcal{L}} * \lambda * \omega_{\mathcal{L}}$ in $\mathcal{M}^1(\mathcal{G})$ with the property $\lambda(\mathcal{C}) = 1$. This applies, in particular, to semistable continuous convolution semigroups on $\varGamma$ and on $\mathcal{G}$, respectively. Let $\pi: G \to \mathcal{G}$ denote the quotient morphism. For a measure ν on G we write $\pi(\nu)$ for its image on $\mathcal{G}$.

The significance of these structural results for the semigroup $\mu_\bullet$ is summarized in the following proposition ([10]). We denote the automorphism induced by $\mathcal{A}$ on $\mathcal{N}$ by $\mathcal{A}_{\mathcal{N}} \in \mathrm{Aut}(\mathcal{N})$.

Let $\mu_\bullet$ be an (A, a)-semistable convolution semigroup on a connected Lie group G in which we consider a compact subgroup L. For the sake of the formulation of the following proposition let us say that $\mu_\bullet$ *satisfies hypothesis* (S) if

(i) $\mu_0 = \omega_L$

(ii) $\mu_\bullet$ is strictly semistable, or $1 \in \operatorname{supp}(\mu_t)$ for all $t \geq 0$.

The property (S) guarantees that $C_A(L)$ is dense and the results we discussed apply. We have seen that for a $\mu_\bullet$ satisfying (S), $\mu_\bullet$ and $\pi(\mu_\bullet)$ are concentrated on $C_L(A)$ and on $C_{\mathcal{L}}(\mathcal{A})$, respectively. The notation introduced in the preceding discussion is maintained.

Proposition 3.10. *Let G be a connected Lie group G. Then the following conclusions hold.*

(i) *For each convolution semigroup $\mu_\bullet$ on G satisfying hypothesis* (S), *there is a continuous, $\mathcal{L}$-invariant, and $(\mathcal{A}_{\mathcal{N}}, a)$-semistable convolution semigroup $\lambda_\bullet = (\lambda_t)_{t\geq 0}$ on $\mathcal{N}$ such that $\lambda_0 = \varepsilon_1$ and*

$$(\#) \qquad \pi(\mu_\bullet) = p(\lambda_\bullet \times \omega_{\mathcal{L}}) = j_{\mathcal{N}}(\lambda_\bullet) * \omega_{\mathcal{L}} = j_{\mathcal{N}}(\lambda_\bullet) * \pi(\omega_L)$$

(ii) *Conversely, given an $\mathcal{L}$-invariant $(\mathcal{A}_{\mathcal{N}}, a)$-semistable convolution semigroup $\lambda_\bullet = (\lambda_t)_{t\geq 0}$ on $\mathcal{N}$, then $t \mapsto \lambda_t \times \omega_{\mathcal{L}}$ lifts uniquely to an (A, a)-semistable convolution semigroup $\mu_\bullet$ on G in the sense that $(\#)$ holds.*

Proof. After the preceding comments, only the last assertion needs proof. Note that $D \subseteq L$ and thus π sets up a bijection between L-biinvariant measures on G and $\mathcal{L}$-biinvariant measures on $\mathcal{G}$. ■

The following consequence of Proposition 3.10 collects relevant conclusions.

Corollary 3.11. *For a given connected Lie group G, an $A \in \operatorname{Aut}(G)$, the compact subgroup L of G, and the corresponding simply connected nilpotent Lie group $\mathcal{N}$ defined by A and L, there is a bijective correspondence between strictly (A, a)-semistable convolution semigroups $\mu_\bullet$ on G with $\mu_0 = \omega_L$ on the one hand, and strictly $(A_{\mathcal{N}}, a)$-semistable, and hence $\mathcal{L}$-invariant convolution semigroups $\lambda_\bullet$ with $\lambda_0 = \varepsilon_1$ on the simply connected nilpotent Lie group $\mathcal{N}$ on the other.* ■

Therefore, in order to investigate (strictly) semistable convolution semigroups on Lie groups G it is largely sufficient to investigate semistable convolution semigroups on simply connected nilpotent Lie groups N. We aim to apply Theorem 3.9 to this situation.

Convergence of Types Theorem on simply connected nilpotent Lie groups

Let G be a simply connected nilpotent Lie group. The subsequent remarks follow [9]. We define π be the homomorphism resulting from the composition

$$G \xrightarrow{\text{quot}} G/[G,G] \xrightarrow{\cong} \mathfrak{g}/[\mathfrak{g},\mathfrak{g}].$$

Recall that in the classical situation a probability measure on a vector space $\mathbb{R}^d$ is called *full* if $\operatorname{supp}(\mu)$ is not concentrated on a coset of a proper linear subspace. A probability convolution semigroup $\mu_\bullet$ on G is said to be *full* if $\pi(\mu_\bullet)$ is a full probability convolution semigroup on the vector space $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Let $\mathcal{F}$ denote the set of full measures and $\mathcal{B} \stackrel{\text{def}}{=} \operatorname{Aut}(G)$. In this case an analog of the Convergence of Types Theorem for Vector Spaces ([9], Theorem 2.3, [7]) holds. In particular, for full convolution semigroups $\mu_\bullet$ on G, the morphism $f_{\mu_\bullet}$ is a proper map, and the group $\operatorname{Sym}(\mu_\bullet)$ is compact. (The above description implies directly that μ_t, $t > 0$ is full iff μ_1 is full.)

An application of Theorem 3.9 to this situation yields

Theorem 3.12. *Let G be a simply connected nilpotent Lie group and $\mu_\bullet$ a full convolution semigroup of probability measures on G. Assume that $\mu_\bullet$ is properly semistable. Then there exists a natural number e with the following property. For each $C \in \operatorname{Dec}(\mu)$ there is an $S \in \operatorname{Sym}(\mu)_0$ such that $A \stackrel{\text{def}}{=} C^e S$ is a commuting normalization. Accordingly, the subgroup $\operatorname{Sym}(\mu_\bullet)\langle C^e S\rangle$ of $\operatorname{Dec}(\mu_\bullet)$ is a direct product and $\operatorname{Dec}(\mu_\bullet)/\operatorname{Sym}(\mu_\bullet)\langle C^e S\rangle$ is a cyclic group of order e.* ∎

We set $c = f_{\mu_\bullet}(C)$ and note that $\mu_\bullet$ is (A, c^e)-semistable in these circumstances.

The case $G = \mathbb{R}^d$

We assume now that μ is a full probability on $\mathbb{R}^d$, a special case of what we discussed in Theorem 3.12. Nevertheless, from the point of view of applications we prefer to reformulate Theorem 3.11 in the context of semistable laws on finite dimensional vector spaces. In this context, (semi-)stable probabilities μ are traditionally called *operator (semi-)stable* (see [16], [17], [18], [25]).

On $\mathbb{R}^d$ infinitely divisible probabilities μ are uniquely embeddable into continuous convolution semigroups $\mu_\bullet$ such that $\mu = \mu_1$. Therefore, semistability is a property of a probability $\mu = \mu_1$, and the notations $f_\mu \stackrel{\text{def}}{=} f_{\mu_\bullet}$, $\operatorname{Sym}(\mu) \stackrel{\text{def}}{=} \operatorname{Sym}(\mu_\bullet)$, and $\operatorname{Dec}(\mu) \stackrel{\text{def}}{=} \operatorname{Dec}(\mu_\bullet)$ are justified.

For simply connected nilpotent Lie groups only a weaker result is available: Let $\mu_\bullet$ and $\mu_\bullet'$ be convolution semigroups on G which are strictly semistable and assume $\mu_1 = \mu_1'$. Then $\mu_t = \mu_t'$ for all $t \geq 0$ (see [8]). Hence strict semistability is a property of a probability $\mu = \mu_1$, and then again the notations $f_\mu \stackrel{\text{def}}{=} f_{\mu_\bullet}$, $\operatorname{Sym}(\mu) \stackrel{\text{def}}{=} \operatorname{Sym}(\mu_\bullet)$, and $\operatorname{Dec}(\mu) \stackrel{\text{def}}{=} \operatorname{Dec}(\mu_\bullet)$ are justified.

Several remarks are in order. Firstly,

Remark 3.13. Semistable, respectively, stable laws are frequently defined as limit laws of operator-normalized sums of i.i.d. random variables. In the case of full measures these definitions coincide. ∎

Secondly, for a probability μ on $\mathbb{R}^d$ the *decomposability semigroup* is defined as

$$\mathrm{D}(\mu) := \{A \in \operatorname{End}(\mathbb{R}^d) : \big(\exists \nu \in \mathcal{M}^1(\mathbb{R}^d)\big) \quad \mu = A(\mu) * \nu\}.$$

If $A \in \operatorname{Dec}(\mu)$ and $f_\mu(A) < 1$, then $\mu_t = A(\mu_t) * \mu_{t-at}$ for any $t > 0$. Thus we have

Remark 3.14. The subset $\mathrm{Dec}(\mu)_+ := \{A \in \mathrm{Dec}(\mu) : a = f_{\mu_\bullet}(A) \in]0,1]\}$ is a subsemigroup of the decomposability semigroup $\mathrm{D}(\mu)$. ■

The semigroup $\mathrm{D}(\mu)$ is an important tool for investigations of selfdecomposable (or Lévy's) probabilities. See e.g., [17] or [12].

On $\mathbb{R}^d$ one-parameter groups $\tau: \mathbb{R}_+^\times \to \mathrm{Aut}(\mathbb{R}^d) = \mathrm{GL}(\mathbb{R}^d)$ with $\tau_{st} = \tau_s\tau_t$ are all of the form $\tau_t = t^E \stackrel{\mathrm{def}}{=} \exp\big((\log t)\cdot E\big)$, $E \in M(\mathbb{R}, d)$. Now Corollary 3.7 applies to the present situation and yields instantly

Remark 3.15. For full operator stable laws there exist *commuting exponents*, i.e., there exists a one-parameter group $(t^{E_c})_{t>0}$ with *exponent* $E_c \in \mathrm{GL}(\mathbb{R}^d)$ such that $t^{E_c} \in \mathrm{Dec}(\mu)$ with $f_\mu(t^{E_c}) = t$ for all $t > 0$ and such that t^{E_c} commutes with all $S \in \mathrm{Sym}(\mu)$. ■

For a direct proof see e.g. [17]. For the corresponding result for groups see e.g. [5] or [6]. Recall that according to Definition 3.8 an automorphism $A \in \mathrm{Dec}(\mu)$ is called *commuting normalization* if $AS = SA$ for all $S \in \mathrm{Sym}(\mu)$. In particular, for full stable laws the existence of commuting exponents provides commuting normalizations t^{E_c} for all $t > 0$.

In contrast with the situation of stable laws, for properly semistable laws on $\mathbb{R}^d$, up to now, only partial results were known, namely, under the hypothesis that the symmetry group $\mathrm{Sym}(\mu)$ is finite [19] or that $d \leq 3$ [20], Theorem 7. As a corollary of 3.12 we obtain now:

Corollary 3.16. *Let μ be a full semistable law on $\mathbb{R}^d$. Then there is a natural number e depending only on* $\mathrm{Sym}(\mu)$ *such that for each $C \in \mathrm{Dec}(\mu)$ there exists an $S \in \mathrm{Sym}(\mu)_0$ for which $A \stackrel{\mathrm{def}}{=} C^e S$ is a commuting normalization.* ■

It can easily be shown that, in contrast with the stable case, the exponent e in Theorem 3.12, will in general not be 1. Thus in general $\mathrm{Dec}(\mu)$ itself does not split as a direct product. We saw in Example 1.2.1 that a semidirect extension of a compact group K by $\mathbb{Z}$ is in general not a direct product even if K is a Lie group; however, not every compact subgroup of $\mathrm{GL}(\mathbb{R}^d)$ is representable as $\mathrm{Sym}(\mu)$. For a discussion see [21]. Thus we shall now exhibit examples $K \subseteq \mathrm{GL}(\mathbb{R}^d)$ representable as the symmetry group $K = \mathrm{Sym}(\mu)$ of a full semistable law μ with a semidirect nonsplitting extension:

Example 3.17. Consider $\mathbb{R}^2$ and let $K \stackrel{\mathrm{def}}{=} \{\pm I, \pm D\}$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $A := \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$ normalizes K, but the centralizer $Z(K, \mathrm{GL}(\mathbb{R}^d))$ of K is the group of diagonal matrices. Therefore $AC \notin Z(K, \mathrm{GL}(\mathbb{R}^d))$ for all $C \in K$, but of course $A^2 = \alpha\beta I$ is central. Hence we have $e = 2$. ■

Remark 3.18. If $0 < \alpha < \beta$ and $\alpha\beta < a < 1$ then there exist (A, a)-semistable laws μ on $\mathbb{R}^d$ with $\mathrm{Sym}(\mu) = K$.

Proof. We see this easily by constructing a suitable K-invariant Lévy measure η on the union of orbits $\bigcup_{X \in K}\{XA^k e_1 : k \in \mathbb{Z}\}$, e_i denoting the i-th unit vector, such that $A(\eta) = a\eta$. Note that $\mathrm{Spec}(A) = \{\pm(\alpha\beta)^{1/2}\}$. Hence such Lévy measures η exist, cf. [16]. ■

Theorem 3.12 illuminates the fine structure of the decomposability group $\mathrm{Dec}(\mu)$ and has various applications. As an example we briefly sketch one particular result which we now derive easily from Theorem 3.12. (For corresponding results in the situation of stable laws cf. [22], Theorem 2.1 and Corollary 2.1.)

Proposition 3.19. *Let μ be a full semistable law. Assume $A \in \mathrm{Dec}(\mu)$ and $f_\mu(A) = a$. Then the absolute values of the eigenvalues of A, thus the spectral radii $\rho(A)$ and $\rho(A^{-1})$, and therefore the rate of decay of $\|A^n\|$ depend only on $a = f_\mu(A)$ and not on the particular choice of A.*

Proof. (1) For $T \in \mathrm{GL}(\mathbb{R}^d)$ let $T = S_T U_T$ be the (multiplicative) Jordan decomposition, S_T and U_T denoting the semisimple and unipotent part of T, respectively. Assume $P, Q \in \mathrm{Dec}(\mu)$ such that Q is a commuting normalization and that $f_\mu(P) = f_\mu(Q)$. Then, since $PQ^{-1} \in \mathrm{Sym}(\mu) = \ker f_\mu$, we obtain that Q and PQ^{-1} commute, hence $PQ = QP$ and thus S_P, S_Q, U_P, and U_Q commute. Therefore $PQ^{-1} = (S_P S_Q^{-1})(U_P U_Q^{-1})$ is the Jordan decomposition. From the fact that $\mathrm{Sym}(\mu)$ is compact we conclude $U_P U_Q^{-1} = I$, hence $U_P = U_Q$. Moreover, $S_P S_Q^{-1} = S_Q^{-1} S_P \in \mathrm{Sym}(\mu)$ is semisimple and therefore $\mathrm{Spec}(P) = \mathrm{Spec}(Q)$ up to multiplication with unimodular constants.

(2) Let now $A \in \mathrm{Dec}(\mu)$. According to Theorem 3.12, let $e \in \mathbb{N}$ and $C \in \mathrm{Sym}(\mu)$ such that $Q \overset{\text{def}}{=} A^e C \in Z(\mathrm{Sym}(\mu), \mathrm{GL}(\mathbb{R}^d))$. We apply Part (1) to $P \overset{\text{def}}{=} A^e$ and find that the absolute values of the eigenvalues of A^e and of Q are equal, $r_1, \ldots, r_d$, say. Therefore, by the Spectral Mapping Theorem, the absolute values of the eigenvalues of A are $r_i^{1/e}$, $i = 1, \ldots, d$. The assertion follows. ■

Appendix: Vague Convergence.

Let $\mu_\bullet$ a continuous convolution semigroup on a second countable locally compact group G. Then the vague limit $\lim_{t\to\infty} \mu_t * \tilde{\mu}_t$ exists.

Proof. (1) According to a theorem by Csiszàr [2] for any sequence ν_n in $\mathcal{M}^1(G)$, there exists a sequence x_n in G such that for $\lambda_n \overset{\text{def}}{=} \nu_1 * \cdots * \nu_n$ either $\lambda_n * \varepsilon_{x_n}$ converges weakly or the concentration function $\sup_{x\in G} \lambda_n(Kx)$ converges to 0 for all compact $K \subseteq G$. In either case we obtain weak, respectively, vague convergence of $\lambda_n * \widetilde{\lambda}_n$.

(2) We apply this to continuous convolution semigroups $\mu_\bullet$: For any increasing sequence $t_n \nearrow \infty$ put $\nu_n \overset{\text{def}}{=} \mu_{t_n - t_{n-1}}$; hence $\lambda_n = \nu_1 * \cdots * \nu_n = \mu_{t_n}$, and we obtain that the vague limit $\lim_{n\to\infty} \mu_{t_n} * \widetilde{\mu}_{t_n} \overset{\text{def}}{=} \rho$ exists.

Assume ρ and σ to be the limits for sequences s_n and $t_n \nearrow \infty$. Without losing generality we may assume that $t_n \le s_n < t_{n+1}$. Combining these sequences we obtain a new sequence r_n, and again the limit exists. We conclude that $\rho = \sigma$. This proves our assertion. ■

References

[1] Bourbaki, N., "Groupes et algèbres de Lie, Chapitres 7 et 8," Hermann, Paris, 1975.

[2] Csiszàr, I., *On infinite products of random elements and infinite convolutions of probability distributions on locally compact groups*, Z. Wahrscheinlichkeitstheorie verw. Geb. **5** (1966), 279–295.

[3] Dani, S. G., *Invariance groups and convergence of types of measures on Lie groups*, Math. Proc. Cambridge Phil. Soc. **112** (1992), 91-108.

[4] Freudenthal, H., *La structure des groupes à deux bouts et des groupes triplements transitifs*, Indag. Math. **13** (1951), 288–294.

[5] Hazod, W., *On the decomposability group of a convolution semigroups*, In: Probabilities on Vector Spaces, Lecture Notes in Math. (Springer) **1391** (1988), 99–111.

[6] —, *Stable probability measures on groups and on vector spaces. A survey*, in: Probability Measures on Groups VIII, Lecture Notes in Math. (Springer) **1375** (1989), 304–352.

[7] —, *Remarks on the convergence of types theorem for finite dimensional vector spaces*, Publ. Math. Debrecen **50** (1997), 197–219.

[8] —, *On some convolution semi- and hemigroups appearing as limit distributions of normalized products of group valued random variables*, Preprint Univ. Dortmund, 1997.

[9] Hazod, W., and S. Nobel, *Convergence of types theorem for simply connected nilpotent groups*, in: Probability Measures on Groups IX, Lecture Notes in Math. (Springer) **1375** (1989), 99–106.

[10] Hazod, W., and E. Siebert, *Automorphisms on a Lie group contracting modulo a compact subgroup and applications to semistable convolution semigroups*, J. Theor. Probability **1** (1988), 211–226.

[11] Hofmann, K. H., *Locally compact semigroups in which a subgroup with compact complement is dense*, Trans. Amer. Math. Soc. **106** (1991) 19–63.

[12] Hofmann, K. H., and Z. J. Jurek, *Some analytical semigroups occuring in probability theory*, J. of Theor. Probability **9** (1996), 745–763.

[13] Hofmann, K. H., J. D. Lawson, and W. A. F. Ruppert, *Weyl groups are finite—and other finiteness properties* Math. Nachrichten **179** (1996), 119-143.

[14] Hofmann, K. H., and P. S. Mostert, "Splitting in Topological Groups," Memoir Amer. Math. Soc. **43**, 1963, 82pp.

[15] Iwasawa, K., *On some types of topological groups*, Annals of Math. **50**(3) (1949), 507–558.

[16] Jajte, R., *Semistable measures on* $\mathbb{R}^N$, Studia Math. **61** (1977), 29–39.

[17] Jurek, Z. J., and J. D. Mason, "Operator-Limit Distributions in Probability Theory," J. Wiley, New York etc., 1993.

[18] Luczak, A., *Elliptical symmetry and characterization of operator stable and operator semistable measures*, Ann. Prob. **12** (1984), 1217–1223.

[19] —, *On some symmetry properties of infinite divisible and operator semi-stable measures*, in: Stochastic Methods in Experimental Design, World Scientific 1990, 304–311.

[20] —, *Remarks on operator semistable measures on finite dimensional vector spaces*, to be published.

[21] Meerschaert, M., and J. A. Veeh, *Symmetry-groups in d-space*, Statistics and Probability Letters **22** (1995), 1–6.

[22] —, *The structure of the exponents and symmetries of an operator stable measure*, J. Theor. Probability **6** (1993), 713–726.

[23] Montgomery, D., and L. Zippin, "Topological Transformation Groups," Interscience Publishers, New York, 1955.

[24] Nobel, S., *Limit theorems for probability measures on simply connected nilpotent Lie groups*, J. Theor. Probability **4** (1991), 261–284.

[25] Siebert, E., *Supplements to operator stable and operator semistable laws on Euclidian vector spaces*, J. Mult. Analysis **19** (1986), 329–341.

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