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A Classification of Multiplicity Free Representations

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Abstract. Let G be a connected reductive linear algebraic group over $\mathbb{C}$ and let (ρ, V) be a regular representation of G. There is a locally finite representation $(\hat{\rho}, \mathbb{C}[V])$ on the affine algebra $\mathbb{C}[V]$ of V defined by $\hat{\rho}(g)f(v) = f(g^{-1}v)$ for $f \in \mathbb{C}[V]$. Since G is reductive, $(\hat{\rho}, \mathbb{C}[V])$ decomposes as a direct sum of irreducible regular representations of G. The representation (ρ, V) is said to be multiplicity free if each irreducible representation of G occurs at most once in $(\hat{\rho}, \mathbb{C}[V])$. Kac has classified all irreducible multiplicity free representations. In this paper, we classify arbitrary regular multiplicity free representations, and for each new multiplicity free representation we determine the monoid of highest weights occurring in its affine algebra.

1. Facts about Multiplicity Free Representations

Throughout this paper G will denote a connected reductive linear algebraic group over the complex numbers $\mathbb{C}$. B will denote a Borel subgroup of G, and we will write $B = HN$ to denote the decomposition of B into a maximal torus H and a unipotent radical N. We will suppose (ρ, V) is a regular representation of G and let $(\hat{\rho}, \mathbb{C}[V])$ denote the representation of G on the affine algebra $\mathbb{C}[V]$ of V defined by $(\hat{\rho}(g)(f))(v) = f(\rho(g^{-1})v)$. Since G is reductive and $(\hat{\rho}, \mathbb{C}[V])$ is locally regular, the affine algebra decomposes as a direct sum of irreducible G modules. (ρ, V) also induces the structure of an irreducible affine G–variety on V, and the following definition makes sense for any affine G–variety V:

Definition 1.1. The representation (ρ, V) is said to be multiplicity free provided that the decomposition of $(\hat{\rho}, \mathbb{C}[V])$ into a direct sum of irreducible G modules contains no irreducible module more than once.

We let Λ (or Λ_G) denote the set of highest weights of G. Let $S[V] = \bigoplus_{\chi \in \Lambda} S[V]_\chi$ be the decomposition of the symmetric algebra $S[V] = P[V^*]$ into its isotypic components and define $\Lambda(V) = \{\chi \in \Lambda | S[V]_\chi \neq 0\} \subseteq \mathfrak{h}^*$. Likewise, let $P[V] = S[V^*] = \bigoplus_{\chi \in \Lambda} S[V^*]_\chi$ be the decomposition of $P[V]$ into its isotypic

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components and define $\Lambda(V^*) = \{\chi \in \Lambda | S[V^*]_\chi \neq 0\} \subseteq \mathfrak{h}^*$. Note that $\Lambda(V^*) = -w_0\Lambda(V)$, where w_0 is the longest element of the Weyl group.

We first review some basic facts concerning multiplicity free representations and multiplicity free affine actions in general. Let $\mathbb{C}[V]^N$ denote the set of N–invariant polynomials in $\mathbb{C}[V]$ and $\mathbb{C}(V)^B$ denote the B–invariant rational functions on V. Recall that the highest weight vectors in $\mathbb{C}[V]$ are just the elements of $\mathbb{C}[V]^N$ which are eigenvectors for the action of H. Also, recall that an affine G–variety Z is said to be spherical if B has a dense orbit in Z. The following theorem gives several characterizations of multiplicity free represenations.

Theorem 1.2. *The following are equivalent:*

(i) (ρ, V) *is multiplicity free.*

(ii) V *is a spherical* G*–variety.*

(iii) $\mathbb{C}(V)^B = \mathbb{C}$.

(iv) B *has only finitely many orbits in* V.

(v) $\mathbb{C}[V]^N = \mathbb{C}[g_1, \dots, g_n]$ *where the* g_i *are algebraically independent and the weights of the* g_i *are* $\mathbb{Q}$*–linearly independent.*

Proof. $(i) \iff (ii) \iff (iii)$ is proved for irreducible G–varieties in [9, p. 199]. $(iv) \implies (ii)$ is clear from basic properties of orbits of algebraic groups. $(ii) \implies (iv)$ was proved independently by Brion [4] and Vinberg [15]. (This result has also been obtained by Knop in [8, Corollary 2.6] and can be deduced from a result of Matsuki [13].) $(v) \implies (i)$ is clear, since the highest weight vectors in $P[V]$ are exactly the monomials in the g_i. For $(iii) \implies (v)$, note that [2, Lemma 6] shows that $P[V]^N$ is a polynomial algebra. So we can write $P[V]^N = \mathbb{C}[g_1, \dots, g_n]$, where the g_i are algebraically independent and, again, the highest weight vectors are the monomials in the g_i. The space of highest weight vectors corresponding to each weight will be one–dimensional exactly when the weights of the g_i are $\mathbb{Q}$–linearly independent. ■

The next corollary, first proved by Kac in [7], follows immediately from Theorem 1.2 *(ii)* by noting that $\dim B = \frac{1}{2}(\dim G + \operatorname{rank} G)$.

Corollary 1.3. *Suppose* (ρ, V) *is a multiplicity free representation of a group* G. *Then*

$$\dim V \leq \frac{1}{2}(\dim G + \operatorname{rank} G).$$

By Theorem 1.2 *(v)*, if (ρ, V) is multiplicity free, $\Lambda(V^*)$ is a free abelian monoid with generators the weights of $g_1, \dots, g_n$. We shall denote the corresponding generators of the monoid $\Lambda(V^*)$ [resp., $\Lambda(V)$] by $\Lambda^+(V^*)$ [resp., $\Lambda^+(V)$]. It is clear that (ρ, V) is multiplicity free if and only if (ρ^*, V^*) is multiplicity free. For computational reasons, it is often more convenient to deal with $\Lambda(V)$ rather than $\Lambda(V^*)$, so we will consider $\Lambda(V)$ in what follows.

The irreducible multiplicity free representations were classified by Kac [7]:

Theorem 1.4. *A complete list of multiplicity free irreducible linear actions of connected reductive linear algebraic groups is [up to equivalence] as follows:*

(1) SL_n, Sp_n, $SO_n \otimes \mathbb{C}^*$, S^2GL_n, $\Lambda^2 SL_n$ *for* n *odd,* $\Lambda^2 GL_n$ *for* n *even,* $SL_m \otimes SL_n$ *for* $m \neq n$, $GL_n \otimes SL_n$, $GL_2 \otimes Sp_n$, $GL_3 \otimes Sp_n$, $GL_4 \otimes Sp_4$, $SL_n \otimes Sp_4$ *for* $n > 4$, $Spin_7 \otimes \mathbb{C}^*$, $Spin_9 \otimes \mathbb{C}^*$, $Spin_{10}$, $G_2 \otimes \mathbb{C}^*$, $E_6 \otimes \mathbb{C}^*$.

(2) $G \otimes \mathbb{C}^*$ *for the semisimple groups* G *from list (1).*

Here, the representation denoted by a group G is the representation corresponding to the first fundamental weight of the group. $G_1 \otimes G_2$ denotes the action of $G_1 \times G_2$ on the tensor product of the representations corresponding to the respective first fundamental weights. S^2GL_n, $\Lambda^2 GL_n$, etc., follow the same pattern.

Remark 1.5. Each representation in the theorem actually stands for an equivalence class of representations: Given two triples (G, ρ, V) and (G', ρ', V'), we say that $(G, \rho, V) \sim (G', \rho', V')$ if and only if there exists an isomorphism $\psi : V \mapsto V'$ such that the induced map $GL(\psi) : GL(V) \mapsto GL(V')$ has $GL(\psi)(\rho(G)) = \rho'(G')$. In addition to (ρ, V) the equivalence class will contain the dual representation (ρ^*, V^*) and $(\rho \circ i, V)$, where $i : \tilde{G} \to G$ is a surjective homomorphism with a finite kernel. Thus, any representation obtained by an automorphism of the Dynkin diagram is in the same equivalence class. Since any reductive group can be covered by a group $S \times (\mathbb{C}^*)^r$, where S is a semisimple group, we can assume G is of this form when necessary.

Howe and Umeda [6] discuss the structure of $\mathbb{C}[V]$ for each of the representations in Theorem 1.4. Table 1 summarizes some of their results. Note that we denote the i^{th} fundamental weight of the first simple group (reading from left to right in the graph) by ω_i (with $\omega_n = 0$ for SL_n), the i^{th} fundamental weight of the second simple group by ω_i', and so forth. The fundamental weight of the $\mathbb{C}^*$ acting on the first irreducible representation will be denoted by ϵ. In subsequent sections, the fundamental weight of the $\mathbb{C}^*$ acting on the second irreducible representation will be denoted by ϵ', and so forth.

2. A Summary of Our Results

Since Kac' classification of multiplicity free representations, there has been additional work on this topic. In particular, Brion [3] has extended Theorem 1.4 to a classification of all multiplicity free representations of simple groups. The goal of this paper is the extension of Theorem 1.4 to a classification of multiplicity free representations for arbitrary reductive groups. (This result was also obtained independently by Benson and Ratcliff in [1].) We shall also determine $\Lambda(V)$ for each of these multiplicity free representations.

The next lemma follows by noting that if Y is a closed G–invariant subvariety of an affine G–variety Z, then the restriction map $\mathbb{C}[Z] \longrightarrow \mathbb{C}[Y]$ is a surjective G–homomorphism and hence takes isotypic components onto isotypic components.

Representation	$\Lambda^+(V)$
$SL_n \otimes \mathbb{C}^*$	$\omega_1 + \epsilon$
$SL_n \otimes SL_m \otimes \mathbb{C}^*$	$\omega_1 + \omega_1' + \epsilon, \omega_2 + \omega_2' + 2\epsilon, \ldots, \omega_r + \omega_r' + r\epsilon$ where $r = \min(m, n)$ and $\omega_r = 0$ (resp., $\omega_r' = 0$) if $n = r$ (resp., $m = r$)
$S^2\mathbb{C}^n \otimes \mathbb{C}^*$	$2\omega_1 + \epsilon, \ldots, 2\omega_{n-1} + (n-1)\epsilon, n\epsilon$
$\Lambda^2\mathbb{C}^n \otimes \mathbb{C}^*$	$\omega_2 + \epsilon, \omega_4 + 2\epsilon, \ldots, \omega_{2\lfloor \frac{n}{2} \rfloor} + \lfloor \frac{n}{2} \rfloor \epsilon$, where $\omega_n = 0$
$Sp_n \otimes \mathbb{C}^*$	$\omega_1 + \epsilon$
$Sp_n \otimes SL_2 \otimes \mathbb{C}^*$	$\omega_1 + \omega_1' + \epsilon, \omega_2 + 2\epsilon, 2\epsilon$
$Sp_n \otimes SL_3 \otimes \mathbb{C}^*$	$\omega_1 + \omega_1' + \epsilon, \omega_2 + \omega_2' + 2\epsilon, \omega_2' + 2\epsilon, \omega_1 + 3\epsilon, \omega_2 + \omega_1' + 4\epsilon, \omega_3 + 3\epsilon$ where the last weight is zero when $n = 4$
$Sp_4 \otimes SL_n \otimes \mathbb{C}^*$	$\omega_1 + \omega_1' + \epsilon, \omega_2 + \omega_2' + 2\epsilon, \omega_1 + \omega_3' + 3\epsilon, \omega_2' + 2\epsilon, \omega_2 + \omega_1' + \omega_3' + 4\epsilon, \omega_4' + 4\epsilon$ where the last weight is zero when $n = 3$
$SO_n \otimes \mathbb{C}^*$	$\omega_1 + \epsilon, 2\epsilon$
$Spin_{10} \otimes \mathbb{C}^*$	$\omega_1 + 2\epsilon, \omega_5 + \epsilon$

Table 1: $\Lambda^+(V)$ for some irreducible representations.

Lemma 2.1. *The restriction of a multiplicity free affine G–action to any closed G–invariant subvariety is multiplicity free.*

Let (ρ, V) be a multiplicity free representation of G and suppose $V = V_1 \oplus \cdots \oplus V_k$ as a direct sum of irreducible modules. Lemma 2.1 implies that $(\rho|_{V_i}, V_i)$ is an irreducible multiplicity free representation of G.

Definition 2.2. Let (ρ, V) be a representation of a group G. We say that (ρ, V) is indecomposable if (ρ, V) is not equivalent (cf., Remark 1.5) to $(\rho_1, V_1) \oplus (\rho_2, V_2)$, where (ρ_1, V_1) and (ρ_2, V_2) are multiplicity free representations of G_1 and G_2, respectively, and $G = G_1 \times G_2$.

Note that by restricting the torus in the representation $(\mathbb{C}^* \otimes SL_n, \mathbb{C}^n) \oplus \cdots \oplus (\mathbb{C}^* \otimes SL_n, \mathbb{C}^n)$ to the diagonal subgroup $\{(z, \ldots, z) | z \in \mathbb{C}^*\} \subseteq (\mathbb{C}^*)^n$, one trivially obtains an indecomposable representation. The following definition eliminates examples like this.

Definition 2.3. A representation (ρ, V) of G is said to be saturated if $V = V_1 \oplus \cdots \oplus V_k$ as a sum of irreducible G–modules and $\rho(G) \supseteq (\mathbb{C}^*)^k$.

Theorem 1.4 shows that each irreducible multiplicity free representation has at most two simple factors acting on it. Thus, the following definition makes sense.

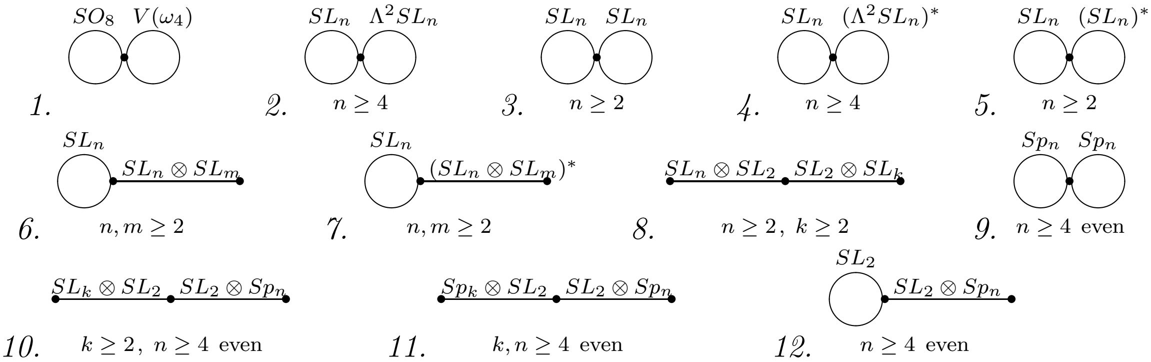
Definition 2.4. Let (ρ, V) be a representation of a group G and let $V = V_1 \oplus \cdots \oplus V_k$ be a decomposition of V into a direct sum of irreducible modules. Then the representation diagram of (ρ, V) is a graph with one edge for each V_i and one vertex for each simple factor of G and such that a vertex is an endpoint of an edge if the corresponding factor of G acts nontrivially on the irreducible submodule. (By definition, an irreducible submodule on which only one simple factor acts has both ends connected to the vertex corresponding to that simple factor.)

The representation diagrams of the representations in Theorem 1.4 have two forms: or . Note that a representation is indecomposable if and only if its representation diagram is a connected graph. With this notation in hand we can state our first result.

Theorem 2.5. *The following is the list [up to equivalence] of all saturated indecomposable multiplicity free representations.*

(i) A representation from Theorem 1.4 of a group with exactly one $\mathbb{C}^*$ factor.

(ii) One of the following representations:



The edges in these graphs are labeled according to which representation in Theorem 1.4 they correspond. Note that each representation is unique only up to the equivalence given in Remark 1.5. Our second result characterizes which subgroups of the torus will yield multiplicity free representations upon restriction.

Theorem 2.6. *Let (ρ, V) be a representation of a group G. Suppose that $V = V_1 \oplus \cdots \oplus V_s$ is a decomposition of V into a direct sum of irreducible submodules. Suppose also that $(\mathbb{C}^*)^k \subseteq \rho(G)$. Let $S = \langle \Lambda(V) \rangle \cap X$, where X is the character group of $(\mathbb{C}^*)^s$, and suppose H is an algebraic subgroup of $(\mathbb{C}^*)^s$ which has character group X/T. Then the representation obtained by restricting the $\mathbb{C}^*$ factors of G to H is multiplicity free if and only if the quotient map $S \to X/T$ is an injection (equivalently, if and only if $S \cap T = 0$).*

Here, $\langle \Lambda(V) \rangle$ denotes the group generated by $\Lambda(V)$. To ease our computations, we will determine $S_{\mathbb{Q}}$, a basis for the $\mathbb{Q}$–linear span of S.

These two theorems together yield a method for determining all multiplicity free representations: By adding $\mathbb{C}^*$ factors, if necessary, and applying a homomorphism with a finite kernel, we can assume every multiplicity free representation is obtained by restriction from a saturated representation. If (ρ, V) is a saturated multiplicity free representation of a group G, then by Lemma 2.1 $G = G_1 \times \cdots \times G_k$ and $V = W_1 \oplus \cdots \oplus W_k$, where $(\rho|_{W_i}, W_i)$ is an indecomposable multiplicity free representation of G_i and G_i acts trivially on W_j for $j \neq i$. Conversely, let (ρ_i, W_i) be a multiplicity free representation of G_i. If B_i is a Borel subgroup of G_i and $B_i w_i \subseteq W_i$ is an open orbit, then $B_1 w_1 \times \cdots \times B_k w_k$ is an open orbit of the Borel subgroup $B_1 \times \cdots \times B_k$ of $G = G_1 \times \cdots \times G_k$ in V. Hence, by Theorem 1.2 *(ii)*, (ρ, V) is multiplicity free and it follows that every saturated representation is a sum of indecomposable saturated representations. Consequently, every multiplicity free representation is obtained by restriction from a sum of indecomposable saturated representations. The conditions on $\rho(G)\cap(\mathbb{C}^*)^k$ such that a representation (ρ, V) is multiplicity free is precisely the condition given in Theorem 2.6.

In short, all multiplicity free representations can be obtained as follows:

1. Choose finitely many saturated indecomposable representations (G_i, V_i) and let $V = \bigoplus_i V_i$ and $\bar{G} = \prod G_i$.
2. Choose a subspace T of $X_Q = \bigoplus_i (X_i)_Q$. This corresponds to a connected normal subgroup G of $\bar{G}$ such that $\bar{G}/G$ is a torus.
3. Then (G, V) is multiplicity free if and only if the condition in Theorem 2.6 is met with respect to $S_Q = \bigoplus_i (S_i)_Q$.

In Section 3 we present more general facts about multiplicity free representations and develop an initial list of possible saturated indecomposable multiplicity free representations which can be written as a sum of two irreducible summands (Lemma 3.5). The section ends with a characterization of such representations in terms of the decomposition of tensor products within their affine algebras (Lemma 3.7). In Section 4 we use this additional characterization to determine which representations in Lemma 3.5 are not multiplicity free. In Section 5, we show the remaining representations in Lemma 3.5 are multiplicity free, and describe how to determine their affine algebras. In Section 6, we demonstrate that there are no indecomposable multiplicity free representations which can be written as the sum of more than two irreducible summands. In Section 7, we give the proof of Theorem 2.6. A summary of our results is given in Table 2

3. An Initial List of Possible Saturated Representations

As noted previously, the irreducible multiplicity free representations in Theorem 1.4 are only representatives of a class of representations which include, in particular, the dual of the representation. It is not clear *a priori* that $V \oplus W$ being a multiplicity free representation of G implies that $V \oplus W^*$ must be a multiplicity free representation of G. We next show that this is in fact true. This substantially reduces the number of cases that must be considered in Lemma 3.5.

For a representation (ρ, V) of an algebraic group H, let $\#V/H$ denote the number of orbits of H in V and $H_v = \{h \in H | hv = v\}$ denote the isotropy subgroup of $v \in V$ in H. Now suppose (ρ, V) and (σ, W) are two representations of H.

Lemma 3.1. *$\#(V \oplus W)/H < \infty$ if and only if $\#V/H < \infty$ and $\#W/H_v < \infty$ for all $v \in V$.*

Proof. Clearly, $\#(V \oplus W)/H < \infty$ implies $\#V/H < \infty$. Also, (v, w) and (v, w') are in the same H orbit if and only if w and w' are in the same H_v orbit. This implies $\#W/H_v < \infty$. Conversely, suppose $v_1, \ldots, v_s$ are representatives of the finitely many H orbits in V and $w_{i1}, \ldots, w_{ir_i} \in W$ are representatives of the H_{v_i} orbits in W. Then $\{(v_i, w_j) | 1 \le i \le s, 1 \le j \le r_i\}$ are the representatives of the H orbits in $V \oplus W$. ■

Rep.	$\Lambda^+(V)$	$S_{\mathbb{Q}}$
1.	$\omega_1+\epsilon$, $\omega_3+\epsilon+\epsilon'$, $\omega_4+\epsilon'$, 2ϵ, $2\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$
2.	$\omega_{2k}+k\epsilon', \omega_{2k-1}+\epsilon+(k-1)\epsilon'$ $k=1,\ldots,\lfloor n/2\rfloor$	$\langle\epsilon\rangle_{\mathbb{Q}}$ n even $\langle\epsilon+(n-1)\epsilon'\rangle_{\mathbb{Q}}$ n odd
3.	$\omega_1+\epsilon$, $\omega_1+\epsilon'$, $\omega_2+\epsilon+\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $n=2$ $\langle\epsilon-\epsilon'\rangle_{\mathbb{Q}}$ $n>2$
4.	n even: $\omega_{2k-1}+\epsilon+(\lfloor n/2\rfloor-(k-1))\epsilon'$, $\omega_{2k-2}+(\lfloor n/2\rfloor-(k-1))\epsilon'$, $(k=1,\ldots,\lfloor n/2\rfloor)$, $\omega_1+\epsilon$ n odd: $\omega_{2k-1}+(\lfloor n/2\rfloor-(k-1))\epsilon'$, $\omega_{2k}+\epsilon+(\lfloor n/2\rfloor-(k-1))\epsilon'$, $(k=1,\ldots,\lfloor n/2\rfloor)$, $\omega_1+\epsilon$	$\langle\epsilon\rangle_{\mathbb{Q}}$ n even $\langle\epsilon-(n-1)\epsilon'\rangle_{\mathbb{Q}}$ n odd
5.	$\omega_1+\epsilon$, $\omega_{n-1}+\epsilon'$, $\epsilon+\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $n=2$ $\langle\epsilon+\epsilon'\rangle_{\mathbb{Q}}$ $n>2$
6.	$\omega_k+\omega'_k+k\epsilon'$, $k=1,\ldots,\min(m,n)$, $\omega_k+\omega'_{k-1}+\epsilon+(k-1)\epsilon'$, $k=1,\ldots,\min(m,n)$, $\omega_{m+1}+\omega'_m+\epsilon+m\epsilon'$ for $n>m$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $n=m$ $\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $n=m+1$ $\langle\epsilon-\epsilon'\rangle_{\mathbb{Q}}$ $n>m+1$ $\langle\epsilon\rangle_{\mathbb{Q}}$ $m>n$
7.	$\omega_{n-k}+\omega'_{m-k}+k\epsilon'$, $k=1,\ldots,\min(m,n)$, $\omega_{n-k+1}+\omega'_{m-k}+\epsilon+k\epsilon'$, $k=1,\ldots,\min(m,n)-1$, $\omega_1+\epsilon$, $\omega_{n-m+1}+\epsilon+m\epsilon'$ for $n>m$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $n=m$ $\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $n=m+1$ $\langle\epsilon+\epsilon'\rangle_{\mathbb{Q}}$ $n>m+1$ $\langle\epsilon\rangle_{\mathbb{Q}}$ $m>n$
8.	$\omega_1+\omega'_1+\epsilon$, $\omega_2+2\epsilon$, $\omega'_1+\omega''_1+\epsilon'$, $\omega''_2+2\epsilon'$, $\omega_1+\omega''_1+\epsilon+\epsilon'$	0 $n,k>2$ $\langle\epsilon\rangle_{\mathbb{Q}}$ $n=2,k>2$ $\langle\epsilon'\rangle_{\mathbb{Q}}$ $k=2,n>2$ $\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $k=n=2$
9.	$\omega_1+\epsilon$, $\omega_1+\epsilon'$, $\omega_2+\epsilon+\epsilon'$, $\epsilon+\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$
10.	$\omega_1+\omega'_1+\epsilon$, $\omega_2+2\epsilon$, $\omega'_1+\omega''_1+\epsilon'$, $\omega_1+\omega''_1+\epsilon+\epsilon'$, $\omega''_2+2\epsilon'$, $2\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$ $k=2$ $\langle\epsilon'\rangle_{\mathbb{Q}}$ $k>2$
11.	$\omega_1+\omega'_1+\epsilon$, $\omega_2+2\epsilon$, 2ϵ, $\omega'_1+\omega''_1+\epsilon'$, $\omega''_2+2\epsilon'$, $2\epsilon'$, $\omega_1+\omega''_1+\epsilon+\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$
12.	$\omega_1+\epsilon$, $\omega_1+\omega'_1+\epsilon'$, $\omega'_2+2\epsilon'$, $2\epsilon'$, $\omega'_1+\epsilon+\epsilon'$	$\langle\epsilon,\epsilon'\rangle_{\mathbb{Q}}$

Table 2: $\Lambda^+(V)$ and $S_{\mathbb{Q}}$ for the representations in Theorem 2.5. By convention, $\omega_0=0$ and $\omega_n=0$ for SL_n.

Proposition 3.2. *$\#(V \oplus W)/H < \infty$ if and only if $\#(V \oplus W^*)/H < \infty$.*

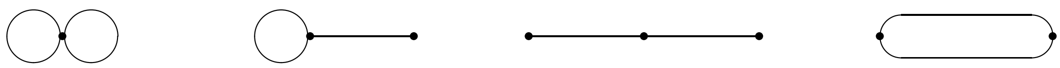
Proof. By Lemma 3.1, $\#(V \oplus W)/H < \infty$ if and only if $\#V/H < \infty$ and $\#W/H_v < \infty$ for all $v \in V$. By a result of Pyasetskii [14], if H is an algebraic group and V is a representation of H, then $\#V/H$ if finite if and only if $\#V^*/H$ is finite. In particular, $\#W/H_v < \infty$ for all $v \in V$ if and only if $\#W^*/H_v < \infty$ for all $v \in V$. But then the Lemma implies that $\#(V \oplus W^*)/H < \infty$. ∎

The following corollary then follows from Theorem 1.2.

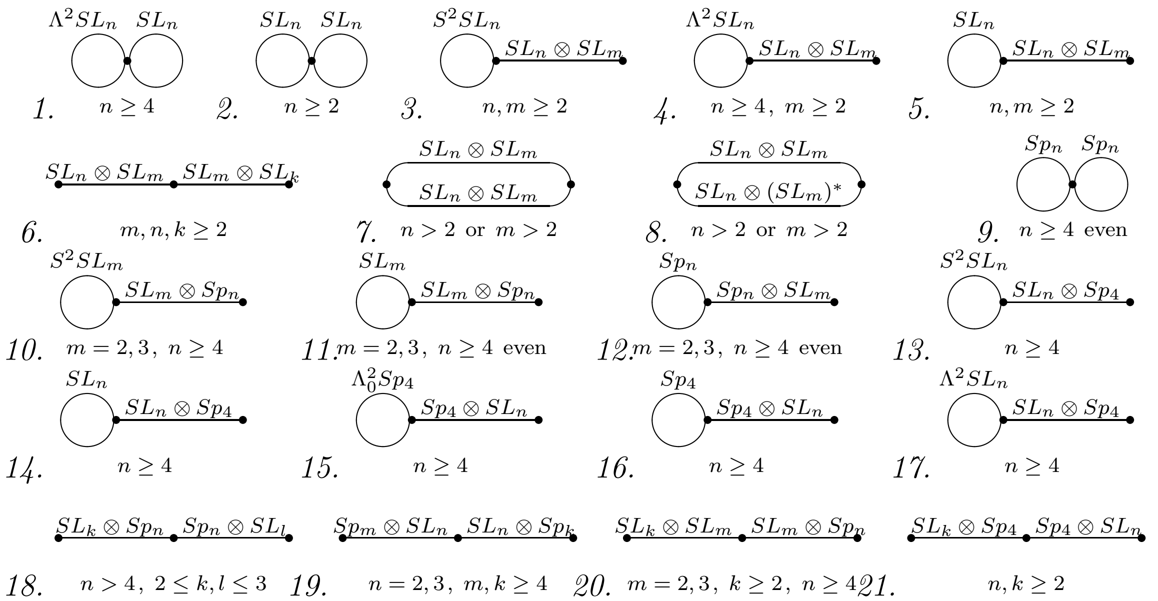
Corollary 3.3. *Suppose (ρ, V) and (σ, W) are two representations of G. The representation $(\rho \oplus \sigma, V \oplus W)$ is multiplicity free if and only if $(\rho \oplus \sigma^*, V \oplus W^*)$ is multiplicity free.*

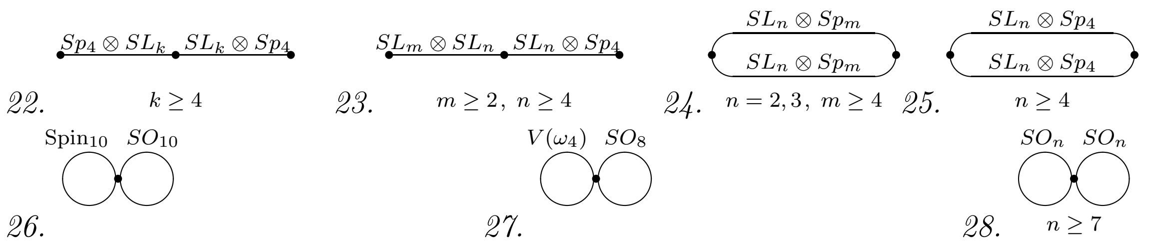
By induction, it follows that a representation which decomposes as a direct sum of several irreducible modules will be multiplicity free if and only if the representation obtained by replacing any number of the irreducible summands with their duals is multiplicity free. We now make use of these results to determine an initial list of possible saturated indecomposable multiplicity free representations which can be written as a sum of two irreducible summands.

Remark 3.4. Given an indecomposable representation with two irreducible summands, its representation diagram is a connected graph with exactly two edges. The graph cannot have more than three vertices, since then it would be disconnected, so it must be one of the following:



Lemma 3.5. *Let (ρ, V) be a saturated indecomposable representation with two nontrivial irreducible submodules. Then, up to replacing one of the summands by its dual, it is necessary that (ρ, V) be one of the following:*





Remark 3.6. Let (ρ, V) be a representation of G. Suppose that $\iota : H \to G$ is an isomorphism and that (σ, W) is a representation of H. By Remark 1.5 we must also consider the representation $(\rho \circ \iota \oplus \sigma, V \oplus W)$ as a representation of H which may be multiplicity free. Thus, we have had to consider representations (such as 15 and 27) obtained by the isomorphisms of the low–dimensional Lie algebras. (Here $\Lambda_0^2\mathbb{C}^4$ is the irreducible, five–dimensional representation of Sp_4 of weight ω_2.) The other representations obtained by low–level isomorphisms in this manner yield representations which are subsumed by other cases.

Proof. The possible representation diagrams of an indecomposable multiplicity free representation with two nontrivial irreducible submodules are given in Remark 3.4. Theorem 1.4 gives all of the irreducible multiplicity free representations, up to the conditions discussed in Remark 1.5. One obtains a list of possible saturated indecomposable multiplicity free representations by filling in the diagrams allowed by the remark with all possible combinations of representations from Theorem 1.4, noting that by Corollary 3.3 a representation in Lemma 3.5 will be multiplicity free if and only if the representation obtained by replacing one of the summands by its dual is multiplicity free. From this list, all representations which do not satisfy the condition in Corollary 1.3 are eliminated to arrive at the result. ■

We now determine which of these representations are multiplicity free. Suppose (ρ, V) and (σ, W) are two multiplicity free representations of G and consider $(\rho \oplus \sigma, V \oplus W)$. Write G as a product $G_1 \times G_2 \times G_3$ of reductive groups, where G_1 acts trivially on V and G_3 acts trivially on W. The decomposition of $S[V]$ into a direct sum of irreducible $G = G_1 \times G_2 \times G_3$ modules is $S[V] = \bigoplus_{\alpha,\beta} V(\alpha) \otimes V(\beta)$ where $\alpha \in \Lambda_{G_1}$, $\beta \in \Lambda_{G_2}$, and the sum is taken over all pairs of highest weights α and β of G_1 and G_2, respectively, such that $V(\alpha) \otimes V(\beta) \subset S[V]$. Similarly, $S[W] = \bigoplus_{\beta',\gamma} V(\beta') \otimes V(\gamma)$ where $\beta' \in \Lambda_{G_2}$, $\gamma \in \Lambda_{G_3}$, and the sum is taken over all pairs of highest weights β' and γ of G_2 and G_3, respectively, such that $V(\beta') \otimes V(\gamma) \subset S[W]$. (ρ, V) (resp., (σ, W)) is multiplicity free when each irreducible module $V(\alpha) \otimes V(\beta)$ (resp., $V(\beta') \otimes V(\gamma)$) appears exactly once. The next lemma is obvious from the fact that $S[V \oplus W] = S[V] \otimes S[W] = \bigoplus_{\alpha,\beta,\beta',\gamma} V(\alpha) \otimes V(\beta) \otimes V(\beta') \otimes V(\gamma)$.

Lemma 3.7. *Let (ρ, V) and (σ, W) be multiplicity free representations of a group G and assume the notation of the previous paragraph. Then $(\rho \oplus \sigma, V \oplus W)$ is multiplicity free if and only if*

(1) for every β and β' such that $V(\alpha) \otimes V(\beta)$ is an irreducible module in $S[V]$ and $V(\beta') \otimes V(\gamma)$ is an irreducible module in $S[W]$ the decomposition of

$V(\beta)\otimes V(\beta')$ into a direct sum of irreducible modules contains no module more than once; and

(2) if $V(\alpha)\otimes V(\beta_1)$ and $V(\alpha)\otimes V(\beta_2)$ occur in $S[V]$ and $V(\beta_1')\otimes V(\gamma)$ and $V(\beta_2')\otimes V(\gamma)$ occur in $S[W]$ (with $\beta_1\neq\beta_2$ or $\beta_1'\neq\beta_2'$), then $V(\beta_1)\otimes V(\beta_1')$ and $V(\beta_2)\otimes V(\beta_2')$ do not share a summand.

The generators of the set of highest weight vectors in $S[V]$ for the relevant irreducible modules in Theorem 1.4 are given in Table 1. From this set of generators, one can then write down the module structure of $S[V]$ for each of these representations. Tensor products of representations of SL_n can be decomposed using the Littlewood–Richardson Rule and the Clebsch–Gordon Formula. (See [5] for a summary of this method and [12] for a proof.) The computations in these cases are straightforward and will be omitted. Note that to show a representation is *not* multiplicity free it will suffice to find a single example of a tensor product which does not satisfy condition (1) or (2) of Lemma 3.7.

4. Non–Multiplicity Free Representations in Lemma 3.5

Lemma 4.1. *The following representations enumerated in Lemma 3.5 are not multiplicity free: 3 with $n\geq 3$, 4 with $n\geq 5$, 6 with $m\geq 3$, 7, and 8.*

Proof. These representations are not multiplicity free because they fail to satisfy condition (1) of Lemma 3.7.

The symmetric algebra in 6 can be written as:

$$\bigoplus_{a_i,b_j\geq 0} V((a_1+\cdots+ra_r)\epsilon)\otimes V(a_1\omega_1+\cdots+a_r\omega_r)\otimes V(a_1\omega_1'+\cdots+a_r\omega_r')$$
$$\otimes V(b_1\omega_1'+\cdots+b_s\omega_s')\otimes V(b_1\omega_1''+\cdots+b_s\omega_s'')\otimes V((b_1+\cdots+sb_s)\epsilon') \quad (1)$$

where $r=\min(m,n)$ and $s=\min(n,k)$. Set $a_1=a_2=b_1=b_2=1$ and the rest of the coefficients to zero. (This is possible by the assumption that $m\geq 3$.) This gives the module $V(\omega_1'+\omega_2')\otimes V(\omega_1'+\omega_2')$ and the module $V(\omega_1'+\omega_2'+\omega_3')$ has multiplicity two in this tensor product.

The symmetric algebra in 4 can be written as:

$$\bigoplus_{a_i,b_j\geq 0} V\left(\left(a_2+\cdots+\left\lfloor\frac{n}{2}\right\rfloor a_{2\lfloor\frac{n}{2}\rfloor}\right)\epsilon\right)\otimes V\left(a_2\omega_2+\cdots+a_{2\lfloor\frac{n}{2}\rfloor}\omega_{2\lfloor\frac{n}{2}\rfloor}\right)$$
$$\otimes V(b_1\omega_1+\cdots+b_r\omega_r)\otimes V(b_1\omega_1'+\cdots+b_r\omega_r')\otimes V((b_1+\cdots+rb_r)\epsilon') \quad (2)$$

When $a_2=a_4=1$ and $b_1=b_2=1$ we have $V(\omega_2+\omega_4)\otimes V(\omega_1+\omega_2)$, and $V(\omega_1+\omega_3+\omega_5)$ occurs with multiplicity two in the decomposition of this tensor product. (The assumption that $n\geq 5$ is necessary for this to be true.)

The symmetric algebra in 3 can be written as:

$$\bigoplus_{a_i,b_j\geq 0} V((a_1+\cdots+na_n)\epsilon)\otimes V(2a_1\omega_1+\cdots+2a_{n-1}\omega_{n-1})$$
$$\otimes V(b_1\omega_1+\cdots+b_r\omega_r)\otimes V(b_1\omega_1'+\cdots+b_r\omega_r')\otimes V((b_1+\cdots+rb_r)\epsilon') \quad (3)$$

When $a_1 = a_2 = b_1 = b_2 = 1$, we have $V(2\omega_1 + 2\omega_2) \otimes V(\omega_1 + \omega_2)$, and $V(2\omega_1 + 2\omega_2 + \omega_3)$ occurs with multiplicity two in the decomposition of this tensor product. (Note that it is necessary for $n \geq 3$ for this to occur.)

The symmetric algebra in 7 can be written as:

$$\bigoplus_{a_i, b_j \geq 0} V((a_1 + \cdots + ra_r)\epsilon) \otimes V(a_1\omega_1 + \cdots + a_r\omega_r) \otimes V(a_1\omega_1' + \cdots + a_r\omega_r') \\ \otimes V(b_1\omega_1 + \cdots + b_r\omega_r) \otimes V(b_1\omega_1' + \cdots + b_r\omega_r') \otimes V((b_1 + \cdots + rb_r)\epsilon') \quad (4)$$

The lemma is true for $m = n = 2$ by dimension considerations so without loss of generality we assume $m \geq 3$. When $a_1 = a_2 = b_1 = b_2 = 1$, we have $V(\omega_1 + \omega_2) \otimes V(\omega_1 + \omega_2)$. Now argue as in 3.

For the representation in 8, note that the symmetric algebra in this case is obtained by replacing the second SL_m modules by its dual (and ϵ' by $-\epsilon'$) in the previous case. We may assume $m > 2$ or $n > 2$, since otherwise SL_2 is self–dual and we are then in the previous case. Then we may assume that $n > 2$ by taking a dual of one of the summands of the representation, if necessary. Now argue as in 7. ■

Lemma 4.2. *Representation 3 with $n = 2$ and 4 with $n = 4$ in Lemma 3.5 are not multiplicity free.*

Proof. These representations are not multiplicity free because they fail to satisfy satisfy condition (2) of Lemma 3.7.

The symmetric algebra in 3 can be written as:

$$\bigoplus_{a_1, a_2, b_1, b_2 \geq 0} V((a_1 + 2a_2)\epsilon) \otimes V(2a_1\omega_1) \otimes V(b_1\omega_1) \\ \otimes V(b_1\omega_1' + b_2\omega_2') \otimes V((b_1 + 2b_2)\epsilon') \quad (5)$$

(Note $\omega_2' = 0$ if $m = 2$.) The Clebsch–Gordan Formula implies that $V(2\omega_1) \otimes V(3\omega_1)$ contains $V(5\omega_1)$, so when $a_1 = a_2 = 1$, $b_1 = 3$ and $b_2 = 0$, we will obtain the irreducible summand $V(3\epsilon) \otimes V(5\omega_1) \otimes V(3\omega_1') \otimes V(3\epsilon')$. The Clebsch–Gordan formula also implies that $V(6\omega_1) \otimes V(3\omega_1)$ contains $V(5\omega_1)$. So when $a_1 = b_1 = 3$, and $a_2 = b_2 = 0$, we obtain this summand again.

The symmetric algebra in 4 can be written as:

$$\bigoplus_{a_2, a_4, b_j \geq 0} V((a_2 + 2a_4)\epsilon) \otimes V(a_2\omega_2) \otimes V(b_1\omega_1 + \cdots + b_r\omega_r) \\ \otimes V(b_1\omega_1' + \cdots + b_r\omega_r') \otimes V((b_1 + \cdots + rb_r)\epsilon') \quad (6)$$

where $r = \min(4, m)$. When $a_4 = b_1 = b_2 = 1$, $V(4\epsilon) \otimes V(\omega_1 + \omega_2) \otimes V(\omega_1' + \omega_2') \otimes V(3\epsilon')$ is an irreducible module in this term (where $\omega_2' = 0$ when $m = 2$). When $a_2 = 2$ and $b_1 = b_2 = 1$, this irreducible module also appears. ■

Corollary 4.3. *Representations 10, 13, 17, 18, 21, 22, 23, 24, and 25 of Lemma 3.5 are not multiplicity free. In addition representation 19 with $n = 3$ and representation 20 with $m = 3$ are also not multiplicity free.*

Proof. Suppose one of the representations in the lemma is multiplicity free. If each Sp_n is replaced by an SL_n, the resulting representation is multiplicity free. (Compare the orbits of Borel subgroups.) This contradicts the results of the previous Lemma. ∎

For representations of Sp_n and $Spin_n$ we make use of a method for decomposing tensor products developed by Littelmann. This technique for decomposing tensor products and a discussion of generalized Young tableaux can be found in [10].

Lemma 4.4. *The tensor product $V(2\omega_1+\omega_2)\otimes V(2\omega_1)$ of irreducible Sp_n modules ($n\geq 4$ even) contains the module $V(2\omega_1+\omega_2)$ with multiplicity two.*

Proof. We must find all those tableaux T of shape $p(2\omega_1)=(4,0,\ldots,0)$ which are $(2\omega_1+\omega_2)$–dominant and such that $(2\omega_1+\omega_2)+\nu(T)=2\omega_1+\omega_2$. (Equivalently, $\nu(T)=0$.) The shape of $p(2\omega_1)$ is $\boxed{}\boxed{}\boxed{}\boxed{}$.

Write $n=2m$. The condition $\nu(T)=0$ implies that the number j $(1\leq j\leq m)$ appears in T the same number of times as the number $2m-j+1$ appears in T. (This follows directly from the definition of $\nu(T)$.) But then $(2\omega_1+\omega_2)$–dominance implies that $1\leq j\leq 2$, since otherwise we can use the fact that T is weakly increasing in the columns to construct a dominant weight with the coefficient of ω_j a negative number. We are left with T being either $\boxed{1}\boxed{1}\boxed{2m}\boxed{2m}$ or $\boxed{2}\boxed{2}\boxed{2m-1}\boxed{2m-1}$ and it is trivial to check that both of these are Sp_n–standard and $(2\omega_1+\omega_2)$–dominant. ∎

Lemma 4.5. *Representations 12 and 16 from Lemma 3.5 are not multiplicity free.*

Proof. None of these representations is multiplicity free because each fails to satisfy condition (1) of Lemma 3.7.

The symmetric algebra in 12 can be written in two ways, depending on whether $m=2$ or $m=3$. For the first case, the symmetric algebra is:

$$\bigoplus_{k,r,s,t\geq 0} V((s+2t+2r)\epsilon')\otimes V(s\omega_1')\otimes V(s\omega_1+t\omega_2)\otimes V(k\omega_1)\otimes V(k\epsilon).$$

When $s=k=2$ and $t=1$, the resulting term contains the tensor product $V(2\omega_1+\omega_2)\otimes V(2\omega_1)$. Now use Lemma 4.4.

For the second case in representation 12 as well as representation 16, it suffices to find a term in the symmetric algebra of both representations which contains the tensor product in Lemma 4.4. Moreover, since the representation of Sp_n on $\mathbb{C}^n$ is a factor in all three cases, it suffices to find a term in the symmetric algebra of $Sp_4\otimes SL_n$ and $Sp_n\otimes SL_3$ containing the Sp_n module $V(2\omega_1+\omega_2)$. But this is clear from the module structure in each case. ∎

Lemma 4.6. *The Sp_4 module $V(2\omega_2)$ is a submodule of the tensor product $V(\omega_1+\omega_2)\otimes V(2\omega_2)$ of Sp_4 modules.*

Proof. The shape of the partition $p(2\omega_2)$ is $\begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; & \; & \; & \; \\ \hline \end{array}$. The tableau $T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 4 & 4 \\ \hline \end{array}$ is Sp_4–standard and $(\omega_1 + \omega_2)$–dominant. Since $\nu(T) = 0$, it follows that $V(\omega_1 + \omega_2)$ is a summand in the decomposition of the tensor product into irreducible submodules. ■

Lemma 4.7. *Representations 11 with $m = 3$, 13, and 15 of Lemma 3.5 are not multiplicity free.*

Proof. The representations are not multiplicity free because they do not satisfy condition (2) of Lemma 3.7.

The symmetric algebra for representation 11 with $m = 3$ can be written as:

$$\bigoplus_{k,t_i \geq 0} V(k\epsilon) \otimes V(k\omega_1) \otimes V((t_1 + t_6)\omega_1 + (t_2 + t_4)\omega_2)$$
$$\otimes V((t_1 + t_5)\omega_1' + (t_2 + t_6)\omega_2' + t_3\omega_3') \otimes V((t_1 + 2t_2 + 3t_3 + 2t_4 + 3t_5 + 4t_6)\epsilon') \tag{7}$$

For $k = t_1 = t_4 = t_5 = 1$, we have the term

$$V(\epsilon) \otimes V(\omega_1) \otimes V(\omega_1 + \omega_2) \otimes V(2\omega_1') \otimes V(6\epsilon') \tag{8}$$

The irreducible module $V(\epsilon) \otimes V(2\omega_1 + \omega_2) \otimes V(2\omega_1') \otimes V(6\epsilon')$ occurs in the decomposition of this module into a direct sum of irreducibles. For $k = 1$ and $t_1 = t_4 = 2$, we have the term

$$V(\epsilon) \otimes V(\omega_1) \otimes V(2\omega_1 + 2\omega_2) \otimes V(2\omega_1') \otimes V(6\epsilon') \tag{9}$$

Since $\omega_3 = 0$, the irreducible module is a submodule of this term as well. This proves that condition (2) of Lemma 3.7 does not hold.

The symmetric algebra in representation 13 can be written as:

$$\bigoplus_{k,t_i \geq 0} V(k\epsilon) \otimes V(k\omega_1) \otimes V((t_1 + t_5)\omega_1 + (t_2 + t_4)\omega_2 + (t_3 + t_5)\omega_3 + t_6\omega_4)$$
$$\otimes V((t_1 + t_3)\omega_1' + (t_2 + t_5)\omega_2') \otimes V((t_1 + 2t_2 + 3t_3 + 2t_4 + 4t_5 + 4t_6)\epsilon) \tag{10}$$

For $k = t_1 = t_2 = t_3 = 1$, the irreducible module

$$V(\epsilon) \otimes V(2\omega_1 + \omega_2 + \omega_3) \otimes V(2\omega_1' + \omega_2') \otimes V(6\epsilon') \tag{11}$$

is a submodule of the resulting term. When $t_1 = 2$ and $k = t_5 = 1$, the irreducible module (11) is a submodule of this term as well. This proves that condition (2) of Lemma 3.7 does not hold.

For representation 15, note that since the representation $\Lambda_0^2\mathbb{C}^4$ is obtained via the isomorphism $\mathfrak{so}_5 \cong \mathfrak{sp}_4$, the generators of the monoid $\Lambda(V)$ are 2ϵ and $\omega_2+\epsilon$ (since the isomorphism $\mathfrak{so}_5 \cong \mathfrak{sp}_4$ interchanges the indices of the fundamental weights). The symmetric algebra of this representation can thus be written as:

$$\bigoplus_{r,s,.t_i \geq 0} V((r + 2s)\epsilon) \otimes V(r\omega_2) \otimes V((t_1 + t_5)\omega_1 + (t_2 + t_5)\omega_2)$$
$$\otimes V((t_1+t_5)\omega_1'+(t_2+t_4)\omega_2'+(t_3+t_5)\omega_3'+t_6\omega_4') \otimes V((t_1+2t_2+3t_3+4t_5+4t_6)\epsilon') \tag{12}$$

Now set $t_1 = t_2 = 1$ and $t_i = 0$ for $i > 2$. For $r = 0$ and $s = 1$, we have the term $V(2\epsilon) \otimes V(\omega_1 + \omega_2) \otimes V(\omega_1' + \omega_2') \otimes V(3\epsilon')$ When $r = 2$ and $s = 0$, Lemma 4.6 implies this tensor product contains the irreducible module as well. ∎

Lemma 4.8. *The tensor products $V(\omega_1 + 5\omega_5) \otimes V(\omega_1)$ and $V(\omega_1 + 5\omega_5) \otimes V(3\omega_1)$ of $Spin_{10}$ modules both contain $V(2\omega_1 + 5\omega_5)$.*

Proof. The shape of the partition $p(\omega_1)$ is $\square\square$ and it is easy to see that the tableau $T = \boxed{1}\boxed{1}$ is $Spin_{10}$–standard and $(\omega_1 + 5\omega_5)$–dominant. Since $(\omega_1 + 5\omega_5) + \nu(T) = 2\omega_1 + 5\omega_5$, the result follows in this case. Similarly, the shape of the partition $p(3\omega_1)$ is $\square\square\square\square\square\square$ and it is easy to see that the tableau $T = \boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{10}\boxed{10}$ is $Spin_{10}$–standard and $(\omega_1 + 5\omega_5)$–dominant. Since $(\omega_1 + 5\omega_5) + \nu(T) = 2\omega_1 + 5\omega_5$, the result follows in this case. ∎

Lemma 4.9. *Representations 26 and 28 of Lemma 3.5 are not multiplicity free.*

Proof. These representations fail to be multiplicity free because they do not satisfy condition (2) of Lemma 3.7.

The symmetric algebra in representation 28 can be written as:

$$\bigoplus_{k,m,r,s \geq 0} V((k + 2m)\epsilon) \otimes V(k\omega_1) \otimes V(r\omega_1) \otimes V((r + 2s)\epsilon') \tag{13}$$

For $k = 2$ and $m = r = s = 4$, we have the term $V(10\epsilon) \otimes V(2\omega_1) \otimes V(4\omega_1) \otimes V(12\epsilon')$. When $k = 4$, $m = 3$, $r = 2$, and $s = 5$, this term also results.

The symmetric algebra in representation 26 can be written as:

$$\bigoplus_{k,m,r,s \geq 0} V((k + 2m)\epsilon) \otimes V(k\omega_1) \otimes V(r\omega_1 + s\omega_5) \otimes V((s + 2r)\epsilon') \tag{14}$$

For $k = r = 1$, $m = 2$, and $s = 5$, we have the term $V(5\epsilon) \otimes V(\omega_1) \otimes V(\omega_1 + 5\omega_5) \otimes V(7\epsilon')$ For $k = 3$, $m = r = 1$, and $s = 5$, we have the term $V(5\epsilon) \otimes V(3\omega_1) \otimes V(\omega_1 + 5\omega_5) \otimes V(7\epsilon')$ We can conclude from Lemma 4.8 that the irreducible term $V(5\epsilon) \otimes V(2\omega_1 + 5\omega_5) \otimes V(7\epsilon')$ appears in both of these summands. ∎

5. Multiplicity Free Representations in Lemma 3.5

We will next show that the remaining representations in Lemma 3.5 are multiplicity free and describe how to determine $\Lambda^+(V)$ for each representations.

Lemma 5.1. *Representation 6 in Lemma 3.5 is multiplicity free and $\Lambda^+(V) = \{\omega_1 + \epsilon, \omega_1 + \epsilon', \omega_2 + \epsilon + \epsilon'\}$.*

Proof. The symmetric algebra of the representation can be written as:

$$\bigoplus_{k,m\geq 0} V(k\epsilon)\otimes V(k\omega_1)\otimes V(m\omega_1)\otimes V(m\epsilon') \tag{15}$$

For fixed $k,m\geq 0$, the Littlewood–Richardson Rule shows

$$V(k\omega_1)\otimes V(m\omega_1)=\bigoplus_{j=0}^{\min(k,m)} V((k+m-2j)\omega_1+j\omega_2) \tag{16}$$

(Note $\omega_2=0$ when $n=2$.) This shows condition (1) of Lemma 3.7 holds.

Now consider a term in the direct sum (15). For $k\neq k'$, $V(k\epsilon)\neq V(k'\epsilon)$. Similarly, for $m\neq n$, $V(m\epsilon')\neq V(n\epsilon')$. So condition (2) of Lemma 3.7 holds trivially.

To determine $\Lambda^+(V)$, note that Equations (15) and (16) imply

$$S[V]=\bigoplus_{\substack{k,m\geq 0\\ j=0,\dots,\min(k,m)}} V(k\epsilon)\otimes V((k+m-2j)\omega_1+j\omega_2)\otimes V(m\epsilon')$$

Setting $k=1$ in shows that $\omega_1+\epsilon$ is a highest weight in $S[V]$. Similarly, setting $m=1$ and (resp., $k=m=j=1$) shows that $\omega_1+\epsilon'$ (resp., $\omega_2+\epsilon+\epsilon'$) is a highest weight in $S[V]$. Since

$$\begin{aligned}(k-j)(\omega_1+\epsilon)+j(\omega_2+\epsilon+\epsilon')+(m-j)(\omega_1+\epsilon')=&\\ =k\epsilon+(k+m-2j)\omega_1+j\omega_2+m\epsilon'&\end{aligned}$$

every highest weight in $S[V]$ is an $\mathbb{N}$–linear combination of these weights. ∎

The calculation to determine $S[V]$ and $\Lambda(V)$ is similar in the remaining cases, so it will be omitted. A table summarizing these results for all of the indecomposable multiplicity free representations we determine can be found in Section 2.

Lemma 5.2. *Representation 5 of Lemma 3.5 is multiplicity free.*

Proof. The symmetric algebra of the representation can be written as:

$$\begin{aligned}\bigoplus_{k,a_i\geq 0} V(k\epsilon)\otimes V(k\omega_1)\otimes V(a_1\omega_1+\cdots+a_r\omega_r)&\\ \otimes V(a_1\omega_1'+\cdots+a_r\omega_r')\otimes V((a_1+\cdots+ra_r)\epsilon')&\end{aligned} \tag{17}$$

where $r=\min(m,n)$. There are two cases to consider.

First, suppose $r<n$. Then $\omega_r\neq 0$. For fixed k and a_i,

$$\begin{aligned}&V(k\omega_1)\otimes V(a_1\omega_1+\cdots+a_r\omega_r)\\ &\quad=\bigoplus_{\substack{0\leq b_j\leq a_{j-1}\\ \sum_{j=1}^{r+1} b_j=k}} V((a_1+b_1-b_2)\omega_1+\cdots+(a_r+b_r-b_{r+1})\omega_r+b_{r+1}\omega_{r+1})\end{aligned} \tag{18}$$

Note that we may not necessarily assume $\omega_{r+1} \neq 0$. To show that condition (1) of Lemma 3.7 holds, we view the b_i as a vector $b = (b_1, \dots, b_{r+1}) \in \mathbb{C}^{r+1}$ and suppose there is another vector $b' = (b'_1, \dots, b'_{r+1}) \in \mathbb{C}$ such that

$$V((a_1 + b_1 - b_2)\omega_1 + \cdots + (a_r + b_r - b_{r+1})\omega_r + b_{r+1}\omega_{r+1}) \\ = V((a_1 + b'_1 - b'_2)\omega_1 + \cdots + (a_r + b'_r - b'_{r+1})\omega_r + b'_{r+1}\omega_{r+1})$$

and $\sum_{j=1}^{r+1} b_j = \sum_{j=1}^{r+1} b'_j = k$. The uniqueness of the highest weight corresponding to an irreducible module and the independence of the weights $\omega_1, \dots, \omega_r$ imply that b_i and b'_i satisfy $a_i + b_i - b_{i+1} = a_i + b'_i - b'_{i+1}$ $(i = 1, \dots, r)$ and $\sum_{j=1}^{r+1} b_j = \sum_{j=1}^{r+1} b'_j$. This system is equivalent to an equation $A_{r+1}b = A_{r+1}b'$, where A_{r+1} is an $(r+1) \times (r+1)$ invertible matrix. So $b = b'$ and condition (1) of Lemma 3.7 holds. To show that condition (2) of Lemma 3.7 holds, consider a term in the direct sum (17). For $k \neq k'$, $V(k\epsilon) \neq V(k'\epsilon)$. For $(a_1, \dots, a_{r-1}) \neq (a'_1, \dots, a'_{r-1})$, $V(a_1\omega'_1 + \cdots + a_{r-1}\omega'_{r-1}) \neq V(a'_1\omega'_1 + \cdots + a'_{r-1}\omega'_{r-1})$ (The condition $r < n$ implies $r = m$ and so $\omega'_r = 0$.) Finally, given that $a_1, \dots, a_{r-1}$ are uniquely determined, $a_r \neq a'_r$ implies $V((a_1 + \cdots + (r-1)a_{r-1} + ra_r)\epsilon') \neq V((a_1 + \cdots + (r-1)a_{r-1} + ra'_r)\epsilon')$. So condition (2) of Lemma 3.7 holds trivially.

Now suppose $r = n$ (so that $\omega_r = 0$). Then

$$V(k\omega_1) \otimes V(a_1\omega_1 + \cdots + a_{r-1}\omega_{r-1}) \\ = \bigoplus_{\substack{0 \le b_j \le a_{j-1} \\ \sum_{j=1}^{r} b_j = k}} V((a_1 + b_1 - b_2)\omega_1 + \cdots + (a_{r-1} + b_{r-1} - b_r)\omega_{r-1}) \quad (19)$$

If there are two isomorphic modules in the summation (19), this means we have $b = (b_1, \dots, b_r), b' = (b'_1, \dots, b'_r) \in \mathbb{C}^r$ satisfying $A_r b = A_r b'$. So, as before, $b = b'$ and condition (1) of Lemma 3.7 is satisfied. The argument for condition (2) of Lemma 3.7 proceeds as in the preceding case. ∎

Lemma 5.3. *Representation 6 of Lemma 3.5 is multiplicity free when $m = 2$.*

Proof. The symmetric algebra of the representation can be written as:

$$\bigoplus_{p,q,r,s \ge 0} V((p + 2s)\epsilon) \otimes V(p\omega_1 + s\omega_2) \otimes V(p\omega'_1) \otimes V(q\omega'_1) \\ \otimes V(q\omega''_1 + r\omega''_2) \otimes V((q + 2r)\epsilon') \quad (20)$$

(Note we may not necessarily assume that ω_2 or ω''_2 is nonzero.) Condition (1) of Lemma 3.7 is immediate from the Clebsch–Gordan formula applied to $V(p\omega'_1) \otimes V(q\omega'_1)$. For $p \neq p'$, $V(p\omega_1 + s\omega_2) \neq V(p'\omega_1 + s\omega_2)$. But for fixed p and $s \neq s'$, we have $V((p + 2s)\epsilon) \neq V((p + 2s')\epsilon)$. A similar argument works for r and s. This implies condition (2) holds. ∎

Lemma 5.4. *Representation 1 of Lemma 3.5 is multiplicity free.*

Proof. The symmetric algebra of the representation can be written as:

$$\bigoplus_{k,a_{2i}\geq 0} V(k\epsilon)\otimes V(k\omega_1)\otimes V\left(a_2\omega_2+\cdots+a_{2\lfloor\frac{n}{2}\rfloor}\omega_{2\lfloor\frac{n}{2}\rfloor}\right) \otimes V\left(\left(a_2+\cdots+\left\lfloor\frac{n}{2}\right\rfloor a_{2\lfloor\frac{n}{2}\rfloor}\right)\epsilon'\right) \quad (21)$$

If n is even, then

$$V(k\omega_1)\otimes V(a_2\omega_2+\cdots+a_{n-2}\omega_{n-2}) = \bigoplus_{\substack{0\leq b_i\leq a_{i-1}\\ \sum b_i=k}} V(b_1\omega_1+(a_2-b_3)\omega_2+b_3\omega_3+\cdots+(a_{n-2}-b_{n-1})\omega_{n-2}+b_{n-1}\omega_{n-1}) \quad (22)$$

To show that condition (1) of Lemma 3.7 holds, fix k and a_i and suppose we have two sequences $(b_1,b_3,\ldots,b_{n-1})$ and $(b'_1,b'_3,\ldots,b'_{n-1})$ such that $0\leq b_j,b'_j\leq a_{j-1}$, $\sum b_j=\sum b'_j=k$, and

$$\begin{aligned} &V(b_1\omega_1+(a_2-b_3)\omega_2+b_3\omega_3+\cdots+(a_{n-2}-b_{n-1})\omega_{n-1}+b_{n-1}\omega_{n-1}) \\ &\quad = V(b'_1\omega_1+(a_2-b'_3)\omega_2+b'_3\omega_3+\cdots+(a_{n-2}-b'_{n-1})\omega_{n-2}+b'_{n-1}\omega_{n-1}) \end{aligned} \quad (23)$$

The independence of $\omega_1,\ldots,\omega_{n-1}$ imply that $b_i=b'_i$ and this is sufficient to show condition (1) of Lemma 3.7.

Condition (2) of Lemma 3.7 holds trivially: For $k\neq k'$, $V(k\epsilon)\neq V(k\epsilon')$. Now suppose there exists some $a'_2,\ a'_4,\ \ldots,\ a'_{n-2}$ and $b'_1,\ b'_3,\ \ldots,\ b'_{n-1}$ such that

$$\begin{aligned} &V(b_1\omega_1+(a_2-b_3)\omega_2+b_3\omega_3+\cdots+(a_{n-2}-b_{n-1})\omega_{n-2}+b_{n-1}\omega_{n-1}) \\ &\quad = V(b'_1\omega_1+(a'_2-b'_3)\omega_2+b'_3\omega_3+\cdots+(a'_{n-2}-b'_{n-1})\omega_{n-2}+b'_{n-1}\omega_{n-1}) \end{aligned} \quad (24)$$

Then the independence of the fundamental weights implies

$$\begin{aligned} &b_1=b'_1,\quad b_3=b'_3,\quad \ldots\quad ,b_{n-1}=b'_{n-1}, \\ &\qquad a_2-b_3=a'_2-b'_3,\quad \ldots\quad ,a_{n-2}-b_{n-1}=a'_{n-2}-b'_{n-1} \end{aligned} \quad (25)$$

and hence $a_2=a'_2,\ldots,a_{n-2}=a'_{n-2}$. But then $a_n\neq a'_n$ implies

$$\begin{aligned} &V\left(\left(a_2+\cdots+((n-1)/2)a_{n-1}+(n/2)a_n\right)\epsilon'\right)\neq \\ &\qquad V\left(\left(a_2+a_4\cdots+((n-1)/2)a_{n-1}+(n/2)a'_n\right)\epsilon'\right). \end{aligned} \quad (26)$$

If n is odd,

$$V(k\omega_1)\otimes V(a_2\omega_2+\cdots+a_{n-1}\omega_{n-1}) = \bigoplus_{\substack{0\leq b_i\leq a_{i-1}\\ \sum b_i=k}} V(b_1\omega_1+(a_2-b_3)\omega_2+b_3\omega_3+\cdots+(a_{n-1}-b_n)\omega_{n-1}) \quad (27)$$

and the argument proceeds exactly as in the preceding case. ■

Lemma 5.5. *Representations 11 with $m=2$, 19 with $n=2$, and 20 with $m=2$ in Lemma 3.5 are multiplicity free.*

Proof. In all cases, it is immediate from the Clebsch–Gordan formula that condition (1) of Lemma 3.7 holds. The symmetric algebra in representation 20 ($m=2$) can be written (as a sum of irreducible modules) as

$$\bigoplus_{\substack{m,p,r,s,t\geq 0\\ j=0,\ldots,\min(m,r)}} V((m+2p)\epsilon)\otimes V(m\omega_1+p\omega_2)\otimes V((m+r-2j)\omega_1') \otimes V(r\omega_1''+s\omega_2'')\otimes V((r+2s+2t)\epsilon')$$

The symmetric algebra in representation 19 ($n=2$) can be written as:

$$\bigoplus_{\substack{m,p,q,r,s,t\geq 0\\ j=0,\ldots,\min(m,r)}} V((m+2p+2q)\epsilon)\otimes V(m\omega_1+p\omega_2)\otimes V((m+r-2j)\omega_1') \otimes V(r\omega_1''+s\omega_2'')\otimes V((r+2s+2t)\epsilon')$$

The symmetric algebra in representation 11 ($m=2$) can be written as:

$$\bigoplus_{\substack{k,r,s,t\geq 0\\ j=0,\ldots,\min(k,r)}} V(k\epsilon)\otimes V((k+r-2j)\omega_1)\otimes V(r\omega_1'+s\omega_2')\otimes V((r+2s+2t)\epsilon')$$

For condition (2) of Lemma 3.7 we consider only the first case, as the other cases are similar. If $m\neq m'$ [resp., $p\neq p'$], $V(m\omega_1'+p\omega_2')\neq V(m'\omega_1'+p\omega_2')$ [resp., $V(m\omega_1'+p\omega_2')\neq V(m\omega_1'+p'\omega_2')$]. A similar argument works to show that r and s must remain fixed. But then for $t\neq t'$, $V((r+2s+2t)\epsilon')\neq V((r+2s+2t')\epsilon')$. ∎

Lemma 5.6. *The tensor product $V(j\omega_1)\otimes V(k\omega_1)$ of modules of Sp_n can be written as a direct sum of irreducible Sp_n modules as*

$$V(j\omega_1)\otimes V(k\omega_1) = \bigoplus_{\substack{0\leq r\leq \min(k,j)\\ 0\leq s\leq \min(j-r,k-r)}} V((j+k-2r-2s)\omega_1+s\omega_2) \tag{28}$$

Proof. We can write $n=2m$. We must find the $k\omega_1$–dominant Sp_n–standard Young tableau of shape $p(j\omega_1)$. The shape $p(j\omega_1)$ is $\square\square\ \cdots\ \square\square$ ($2j$ boxes). Being $k\omega_1$–dominant implies that the numbers $3,4,\ldots,2m-1$ cannot occur in the tableau. Let $2r$ denote the number of boxes in the tableau that contain the number $2m$. Clearly, $0\leq r\leq j$. By $k\omega_1$–dominance, the weight $2k\omega_1+\nu_{2r}(T)$ must be in the dominant Weyl Chamber of Sp_{2m}. By the condition of being weakly increasing in the rows, $c_{T(2r)}(2m)=2r$ and $c_{T(2r)}(j)=0$ for $j\neq 2m$. It follows that $2k\omega_1+\nu_{2r}(T)=(2k-2r)\epsilon_1$, which is in the dominant Weyl chamber of Sp_{2m} if and only if $2k-2r\geq 0$. It follows that $0\leq r\leq \min(k,j)$.

Now let $2s$ denote the number of boxes in the tableau that contain the number 2. Clearly, $0 \leq s \leq j-r$. By $k\omega_1$–dominance, the weight $2k\omega_1+\nu_{2j+2r}(T)$ must be in the dominant Weyl chamber of Sp_{2m}. But $2k\omega_1 + \nu_{2j+2r}(T) = (2k-2r)\epsilon_1 + 2s\epsilon_2 = (2k-2r-2s)\omega_1 + 2s\omega_2$. So we must have $0 \leq s \leq \min(k-r, j-r)$. The remaining boxes in the tableau must be filled with ones. Since any tableau satisfying these inequalities must be a Sp_n–standard Young tableaux of shape $p(j\omega_1)$, it follows that the decomposition of the tensor product into a direct sum of irreducibles is given by Equation 28. ∎

In [11], Littelmann determines all pairs of fundamental weights (ω, ω') such that $V(k\omega)\otimes V(l\omega')$ decomposes without multiplicities. For these tensor products, he also determines how they decompose as a direct sum of irreducibles. Lemma 5.6 can also be deduced from this result.

Lemma 5.7. *Representation 9 in Lemma 3.5 is multiplicity free.*

Proof. The symmetric algebra of the representation can be written as:

$$\bigoplus_{k,j\geq 0} V(k\epsilon) \otimes V(k\omega_1) \otimes V(j\omega_1) \otimes V(j\epsilon') \tag{29}$$

Condition (2) of Lemma 3.7 is clear: $k \neq k'$ implies $V(k\epsilon) \neq V(k'\epsilon)$ and $j \neq j'$ implies $V(j\epsilon') \neq V(j'\epsilon')$.

Condition (1)is clear from Lemma 5.6. ∎

Lemma 5.8. *The tensor product $V(m\omega_1) \otimes V(k\omega_4)$ of irreducible modules of the simple group $Spin_8$ can be written as a direct sum of irreducible modules as*

$$V(m\omega_1) \otimes V(k\omega_4) = \bigoplus_{0\leq j\leq \min(m,k)} V((m-j)\omega_1 + j\omega_3 + (k-j)\omega_4) \tag{30}$$

Proof. We must determine all $Spin_8$–standard tableaux that are of shape $p(k,k,k,k)$ and $m\omega_1$–dominant. (See the appendix of [10].) Note that a tableau of shape $p(k,k,k,k)$ consists of four left–justified rows of k boxes each—i.e., there are k columns of four boxes each. It is not difficult to show that the only possible $Spin_8$ standard tableau that are of shape $p(k,k,k,k)$ and $m\omega_1$–dominant are

$$\overbrace{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array}\cdots\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline 4\\ \hline\end{array}\underbrace{\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 5\\ \hline 8\\ \hline\end{array}\cdots\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline 5\\ \hline 8\\ \hline\end{array}}_{j}}^{k}$$

where $0 \leq j \leq \min(m,k)$. If T is a tableau of this form, then each column of the form $\boxed{1}\boxed{2}\boxed{3}\boxed{4}$ contributes $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4$ to $\nu(T)$, while each column of the form $\boxed{2}\boxed{3}\boxed{5}\boxed{8}$ contributes $-\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4$ to $\nu(T)$. If there are $0 \leq j \leq \min(k,m)$

columns of this latter type (and thus $k-j$ columns of the former type), it follows that

$$\begin{aligned} m\omega_1 + \nu(T) &= m\epsilon_1 + \frac{1}{2}\left[(k-2j)\epsilon_1 + \epsilon_2 + k\epsilon_3 + (k-2j)\epsilon_4\right] \\ &= (m-j)\omega_1 + j\omega_3 + (k-j)\omega_4 \end{aligned}$$

which implies the result. ∎

Lemma 5.9. *Representation 27 of Lemma 3.5 is multiplicity free.*

Proof. The representation $V(\omega_4)$ of $Spin_8$ is obtained by composing the representation $V(\omega_1)$ by an isomorphism obtained from triality. So we have

$$S[V(\omega_1) \oplus V(\omega_4)] = \bigoplus_{p,q,s,t \geq 0} V((p+2q)\epsilon) \otimes V(p\omega_1) \otimes V(s\omega_4) \otimes V((s+2t)\epsilon)$$

Hence it is clear that condition (1) of Lemma 3.7 holds from Lemma 5.8. For condition (2) of the lemma, we must show that distinct values of the parameters lead to non–isomorphic modules. Suppose first that $q \neq q'$. (The argument for t will be the same.) Then $V((p+2q)\epsilon) \neq V((p'+2q')\epsilon)$ unless $p \neq p'$. So we are in the case where $p \neq p'$. (The argument for $s \neq s'$ will also be the same.) If we have

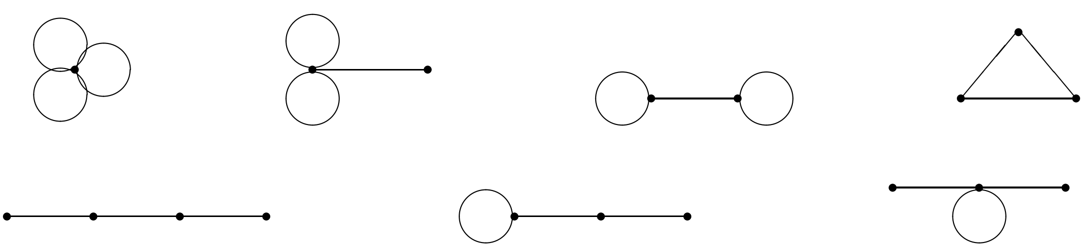
$$\begin{aligned} &V((p+2q)\epsilon) \otimes V((p-j)\omega_1 + j\omega_3 + (s-j)\omega_4) \otimes V((s+2t)\epsilon') \\ &\qquad = V((p'+2q')\epsilon) \otimes V((p'-j')\omega_1 + j'\omega_3 + (s'-j')\omega_4) \otimes V((s'+2t')\epsilon') \end{aligned}$$

then $p \neq p'$ and $p-j = p'-j'$ imply $j \neq j'$. But then the coefficients of ω_3 in the modules are distinct and they cannot be isomorphic. ∎

This shows that the saturated indecomposable multiplicity free representations which can be written as a sum of two irreducible summands are precisely those given in the statement of Theorem 2.5. We will show there are no saturated indecomposable multiplicity free representation with more irreducible summands.

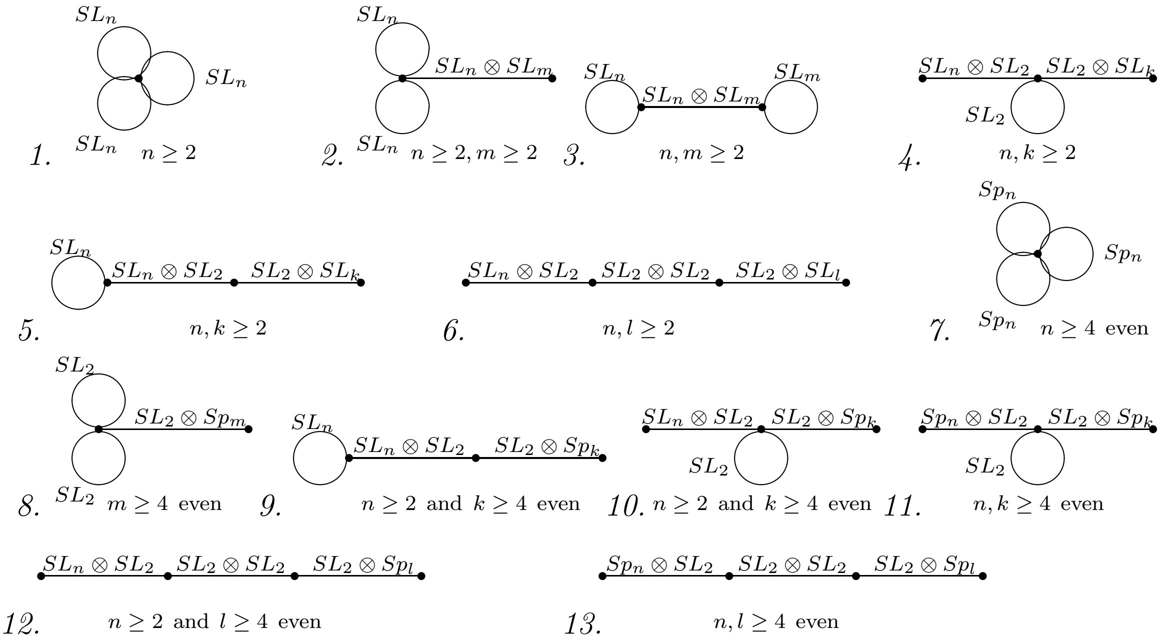
6. Indecomposable Representations: Three Irreducible Summands

Suppose (ρ, V) is a saturated indecomposable multiplicity free representation which can be written as a direct sum of three irreducible submodules. Then Remark 2.1 implies that the restriction of the representation to any two of these submodules is also multiplicity free. Hence, after removing an edge from the representation diagram, the resulting representation diagram should be a diagram for a representation given in Theorem 2.5. The only connected graphs with exactly three edges which satisfy these conditions are:



The next lemma is proved exactly as Lemma 3.5, subject to the additional constraint that the restriction to any two of the edges must be an element of Theorem 2.5 (b).

Lemma 6.1. *If (ρ, V) is a saturated indecomposable multiplicity free representation of a group which can be written as a direct sum of three irreducible summands, then it is necessary (but not sufficient) that—up to replacing one of the summands by its dual—the representation be one of the following:*



Lemma 6.2. *Consider a representation (ρ, V) of a group with an SL_n simple factor, where $V = V_1 \oplus \cdots \oplus V_k$ as a sum of irreducible submodules. If the SL_n module $V(\omega_1)$ appears in a term of $S[V_i]$ for at least three distinct V_i, then (ρ, V) is not multiplicity free.*

Proof. If this is the case, then $V(\omega_1) \otimes V(\omega_1) \otimes V(\omega_1)$ appears in a term of the symmetric algebra $S[V]$ and the module $V(\omega_1 + \omega_2)$ appears as a submodule of this twice. (When $n = 2$, $\omega_2 = 0$ and this is the module $V(\omega_1)$.) So the representation does not satisfy condition (1) of Lemma 3.7. ∎

Corollary 6.3. *Representations 1, 2, and 4 of Lemma 6.1 are not multiplicity free.*

Proof. For each of these representations there is an SL_n factor operating on all three of the irreducible factors and the irreducible SL_n module $V(\omega_1)$ appears in a term of the decomposition of each of the symmetric algebras. So the previous Lemma shows that in each case the representation is not multiplicity free. ∎

Lemma 6.4. *Representations 3, 5, and 6 of Lemma 6.1 are not multiplicity free.*

Proof. We will show that each of the representations does not satisfy condition (2) of Lemma 3.7.

The symmetric algebra in 3 can be written as:

$$\bigoplus_{k,l,a_i\geq 0} V(k\epsilon)\otimes V((a_1+\cdots+ra_r)\epsilon')\otimes V(l\epsilon'') \otimes V(k\omega_1)\otimes V(a_1\omega_1+\cdots+a_r\omega_r)\otimes V(l\omega_1')\otimes V(a_1\omega_1'+\cdots+a_r\omega_r') \quad (31)$$

(Here $r=\min(m,n)$ and we have $\omega_r=0$ [resp., $\omega_r'=0$] if $r=n$ [resp., $r=m$].) For $k=l=1$ and $a_1=2$, the irreducible module

$$\hat{V}=V(\epsilon)\otimes V(2\epsilon')\otimes V(\epsilon'')\otimes V(\omega_1+\omega_2)\otimes V(\omega_1'+\omega_2')$$

appears as a summand in the decomposition of this tensor product. (Note that $\omega_2=0$ [resp., $\omega_2'=0$] if $n=2$ [resp., $m=2$].) Likewise, if we have $k=l=a_2=1$, $\hat{V}$ is an element of the tensor product decomposition of this module as well. (Note, $V(\omega_2)=0$ [resp., $V(\omega_2')=0$] if $n=2$ [resp., $m=2$].)

The symmetric algebra in 5 can be written as:

$$\bigoplus_{a_i,b_j,k\geq 0} V(k\epsilon)\otimes V((a_1+2a_2)\epsilon')\otimes V((b_1+2b_2)\epsilon'') \otimes V(k\omega_1)\otimes V(a_1\omega_1+a_2\omega_2)\otimes V(a_1\omega_1')\otimes V(b_1\omega_1')\otimes V(b_1\omega_1''+b_2\omega_2'') \quad (32)$$

For $a_1=2$ and $b_1=k=1$, the irreducible module $\hat{V}=V(2\epsilon')\otimes V(\epsilon)\otimes V(\epsilon'')\otimes V(\omega_1+\omega_2)\otimes V(\omega_1')\otimes V(\omega_1'')$ appears as a summand in the decomposition of this module into a direct sum of irreducible modules. (As above, $\omega_2=0$ when $n=2$.) For $a_2=k=b_1=1$ $\hat{V}$ appears as a summand of the decomposition of this module into a sum of irreducible modules as well.

The symmetric algebra in 6 can be written as:

$$\bigoplus_{a_i,b_j,c_k\geq 0} V((a_1+2a_2)\epsilon)\otimes V((b_1+2b_2)\epsilon')\otimes V((c_1+2c_2)\epsilon'')\otimes V(a_1\omega_1+a_2\omega_2) \otimes V(a_1\omega_1')\otimes V(b_1\omega_1')\otimes V(b_1\omega_1'')\otimes V(c_1\omega_1'')\otimes V(c_1\omega_1'''+c_2\omega_2''') \quad (33)$$

For $a_1=c_1=1$ and $b_1=2$, the irreducible module $\hat{V}=V(\epsilon)\otimes V(2\epsilon')\otimes V(\epsilon'')\otimes V(\omega_1)\otimes V(\omega_1')\otimes V(\omega_1'')\otimes V(\omega_1''')$ appears as a term in the decomposition of this module into a direct sum of irreducible modules. For $a_1=c_1=b_2=1$ and $b_1=a_2=c_2=0$, we have $\hat{V}$ again. ∎

The proof of the following Corollary is the same as the proof of Corollary 4.3.

Corollary 6.5. *None of representations 7, 8, 9, 10, 11, 12, and 13 is multiplicity free.*

This completes the proof of Theorem 2.5.

7. Non–Saturated Multiplicity Free Representations

Suppose that $V = W_1 \oplus \cdots \oplus W_r$ is a decomposition of W into indecomposable multiplicity free submodules. (Note that $r \leq s$ and each W_i is either an irreducible module or a sum of two irreducible modules.) Then $S[V] = S[W_1] \otimes \cdots \otimes S[W_r]$ Since there are distinct semisimple and $\mathbb{C}^*$ factors acting on each W_i, the monoid $\Lambda(V)$ of highest weights in $S[V]$ is the direct sum of the monoids of highest weights occurring in each of the $S[W_i]$. Let $X \cong \mathbb{Z}^s$ be the character group of $(\mathbb{C}^*)^s$ and let $S = \langle \Lambda(V) \rangle \cap X$. Recall that if $H \subseteq (\mathbb{C}^*)^s$ is an algebraic subgroup, then the character group of H is X/T where $T = \{\chi \in X | \chi(H) = 1\}$. We will now prove Theorem 2.6.

Proof. We show that the restriction of the representation to H is not multiplicity free if and only if $S \cap T \neq \{0\}$.

We use the characterization of multiplicity free representations given in Theorem 1.2 *(v)*. Since restricting to H does not change the highest weight vectors, it suffices to determine when the condition that the elements of $\Lambda^+(V)$ be linearly independent holds.

For $i = 1, \dots, r$, let $\{\chi_i^j | j = 1, \dots, s_j\} \subseteq \Lambda(W_j)$ be the generators of $\Lambda(W_j)$. If the restriction to H is not multiplicity free, then there is a nontrivial linear dependency relation $\sum a_{ij}\chi_i^j = 0$ among the generators of the monoid of highest weights. Each generator χ_i^j is a linear combination of fundamental highest weights and the fundamental highest weights are linearly independent. So, by grouping like terms, the equation $\sum a_{ij}\chi_i^j = 0$ is equivalent to a system of homogeneous linear equations given by the coefficients of the distinct fundamental weights. In particular, the coefficients of the fundamental weights of the semisimple factors are zero. Thus, if we consider the weight $\sum a_{ij}\chi_i^j$ as a weight of the original group G (with the same semisimple factors and the maximal number of $\mathbb{C}^*$ factors), it must be that $\nu \in X$. But $\nu(H) = 1$. Thus, $\nu \in T$ and we have $\nu \in S \cap T$.

Conversely, suppose $\nu \in S \cap T$, $\nu \neq 0$. Since $\nu \in T$, the restriction of ν to H is zero. Since $\nu \in S = \langle \Lambda(V) \rangle \cap X$, by writing ν as a $\mathbb{Q}$–linear combination of elements of $\Lambda(V)$, clearing the denominators of the coefficients of this linear combination, and bringing the negative coefficients to the other side of the equation ($\nu(H) = 0$), it is clear that we can find two distinct weights in $\Lambda(V)$ which yield the same weight upon restriction to H. This means that when restricted to H the representation is not multiplicity free. ∎

We now discuss indecomposable multiplicity free representations in this context. A weight in $\langle \Lambda(V) \rangle$ will be in S exactly when it can be written as a linear combination of the generators of $\Lambda(V)$ in such a way that all of coefficients of the fundamental weights of the simple factors vanish. Thus, to determine S we consider weights in $\Lambda^+(V)$ as weights of the semisimple part of G and determine which linear combinations of these vanish. A basis for this space will be a basis of S. (In reality, we will work in $\langle \Lambda(V) \rangle_{\mathbb{Q}}$ and $X \cong \mathbb{Q}^2$, so we will determine $S_{\mathbb{Q}}$.)

There are three cases that can occur: (1) If $\dim S_{\mathbb{Q}} = 2$, $S_{\mathbb{Q}} = X_{\mathbb{Q}}$ and so $S_{\mathbb{Q}} \cap T_{\mathbb{Q}} \neq 0$ unless $T_{\mathbb{Q}} = 0$. Thus, we must have $H = (\mathbb{C}^*)^2$. (2) If $\dim S_{\mathbb{Q}} = 0$, then $S_{\mathbb{Q}} \cap T_{\mathbb{Q}} = 0$ for all T and so H could be any subgroup of $(\mathbb{C}^*)^2$. (In particular, H could be the trivial subgroup of $(\mathbb{C}^*)^2$.) (3) if $\dim S_{\mathbb{Q}} = 1$, then H could be any subgroup of $(\mathbb{C}^*)^2$ such that $T_{\mathbb{Q}}$ and $S_{\mathbb{Q}}$ have trivial intersection. (In particular, H must be either one–dimensional or two–dimensional.)

As an example, we will perform the computations for representation 2 of Theorem 2.5. A complete summary of the results for the other cases is given in Section 2. When restricted to the semisimple part of G (i.e., SL_n), the elements of $\Lambda^+(V)$ for the representation are ω_1, ω_1, and ω_2. Now consider the equation $a\omega_1 + b\omega_1 + c\omega_2 = 0$ For $n > 2$, we must have $c = 0$ and $b = -a$. Thus, $S_{\mathbb{Q}}$ is spanned by $(\omega_1 + \epsilon) - (\omega_1 + \epsilon') = \epsilon - \epsilon'$. When $n = 2$, we may have $c \neq 0$ (since $\omega_2 = 0$). Then $S_{\mathbb{Q}} = \langle \epsilon - \epsilon', \epsilon + \epsilon' \rangle_{\mathbb{Q}} = \langle \epsilon, \epsilon' \rangle_{\mathbb{Q}}$

Note that this is equivalent to saying that when only one $\mathbb{C}^*$ factor acts by $t \mapsto t^p I_1 \oplus t^q I_2$, the representation will be multiplicity free if and only if $n > 2$ and $p \neq q$. Moreover, when there are no $\mathbb{C}^*$ factors, the representation will not be multiplicity free.

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