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Wavelet Transforms and Symmetric Tube Domains

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Abstract. We extend wavelet analysis to the symmetric tube domains and their Shilov boundaries. Our approach is based on the theory of Jordan algebra.

One-dimensional wavelet analysis can be explained in terms of square-integrable representation of the affine group (cf. [4], [6]). It is an intermediate between the function theory on the upper half-plane of one complex variable and the harmonic analysis on the real line (cf. [7], [9]). In this paper we extend wavelet analysis to the symmetric tube domains and their Shilov boundaries, the higher dimensional analogues of the upper half-plane and the real line. We assume that V is a simple Euclidean Jordan algebra, Ω is the associated symmetric cone and T_Ω is the symmetric tube domain over Ω. In §1, we recall some notations and facts about Jordan algebras and symmetric cones, especially the Iwasawa subgroup P of the holomorphic automorphism group of T_Ω. P has a natural unitary representation π on $L^2(V)$. In §2, we decompose $L^2(V)$ into the direct sum of the irreducible invariant closed subspaces under π. In §3, we give an explicit characterization of the admissibility condition in terms of Fourier transform and Jordan algebra. We also give a family of admissible wavelets, which is a complete orthonormal system in a sense. Finally in §4, we use wavelet transforms to decompose the weighted L^2-space on the tube domain T_Ω into a direct sum of subspaces such that the first component is exactly the weighted Bergman space.

A good reference on Jordan algebras, symmetric cones and tube domains is the book [3] by J. Faraut and A. Korányi. Various authors developed the theory of continuous wavelet in view of square-integrable group representations, for example, in [5], [8] and in particular [1].

1. Iwasawa subgroup

Throughout this paper we keep the following assumptions and notations, which are the same as in [3].

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V is an n-dimensional simple Euclidean Jordan algebra with identity e. xy denotes the Jordan product of x and y. $\mathrm{tr}(x)$ and $\det(x)$ are defined as in [3]. We also write $\Delta(x)$ instead of $\det(x)$. The inner product on V is given by $(x|y) = \mathrm{tr}(xy)$. $L(x)$ is the linear map of V defined by $L(x)y = xy$. An element $c \in V$ is idempotent if $c^2 = c$. The only eigenvalues of $L(c)$ are 1, $\frac{1}{2}$, and 0. The corresponding eigenspaces are denoted by $V(c,1)$, $V(c,\frac{1}{2})$ and $V(c,0)$. We fix a Jordan frame $\{c_1, \cdots, c_r\}$, where r is the rank of V. Then we have the Peirce decomposition

$$V = \bigoplus_{j \le k} V_{jk}$$

where

$$\begin{aligned} V_{ii} &= V(c_i, 1) = \mathbf{R}c_i, \\ V_{ij} &= V(c_i, \frac{1}{2}) \cap V(c_j, \frac{1}{2}). \end{aligned}$$

$d = \dim V_{ij}$, which does not depend on i and j, is called the degree of V. Let

$$P(x) = 2L(x)^2 - L(x^2)$$

be the quadratic representation, and write

$$x \Box y = L(xy) - [L(x), L(y)].$$

For given j and for $z^{(j)} \in \bigoplus_{k=j+1}^{r} V_{jk}$ the Frobenius transform $\tau(z^{(j)})$ is defined by

$$\tau(z^{(j)}) = \exp(2z^{(j)} \Box c_j).$$

Let Ω be the symmetric cone which consists of elements x in V such that $L(x)$ is positive definite. $G(\Omega)$ denotes the automorphism group of Ω and G is the identity component of $G(\Omega)$. G has Iwasawa decomposition $G = NAK$, where

$$\begin{aligned} K &= \{g \in G : \, ge = e\}, \\ A &= \{P(a) : \, a = \sum_{j=1}^{r} a_j c_j, \, a_j > 0\}, \\ N &= \{\tau(z^{(1)}) \cdots \tau(z^{(r-1)}) : \, z^{(j)} \in \bigoplus_{k=j+1}^{r} V_{jk}\} \end{aligned}$$

are compact, diagonal and strict triangular respectively. A normalizes N and

$$P(a)\tau(z^{(j)}) = \tau(\tilde{z}^{(j)})P(a) \tag{1.1}$$

where

$$\begin{aligned} z^{(j)} &= \sum_{j<k} z_{jk}, \quad z_{jk} \in V_{jk}, \\ \tilde{z}^{(j)} &= \sum_{j<k} \tilde{z}_{jk}, \quad \tilde{z}_{jk} \in V_{jk}, \\ \tilde{z}_{jk} &= \frac{a_k}{a_j} z_{jk}. \end{aligned}$$

$T = NA$ is a semi-direct product. We will use another parametrization of the triangular subgroup T. Set

$$V_+ = \{u = \sum_{j=1}^{r} u_j c_j + \sum_{j<k} u_{jk} : u_j > 0,\, u_{jk} \in V_{jk}\}.$$

For $u \in V_+$, we define

$$t(u) = P(b_1)\tau(u^{(1)})P(b_2)\cdots\tau(u^{(r-1)})P(b_r),$$

where

$$b_j = c_1 + \cdots + c_{j-1} + u_j c_j + c_{j+1} + \cdots + c_r,$$
$$u^{(j)} = \sum_{k=j+1}^{r} u_{jk}.$$

Then

$$T = \{t(u) : u \in V_+\}.$$

Using (1.1), it is easy to determine the left and right Haar measures of T. The left Haar measure of T is given by

$$d\mu_l(t(u)) = 2^r \prod_{j=1}^{r} u_j^{-d(j-1)-1} du,$$

and the right Haar measure of T is given by

$$d\mu_r(t(u)) = 2^r \prod_{j=1}^{r} u_j^{-d(r-j)-1} du.$$

T acts simply and transitively on Ω. If

$$x = \sum_{j=1}^{r} x_j c_j + \sum_{j<k} x_{jk}$$

is the Peirce decomposition of $x = t(u)e$, then

$$x_j = u_j^2 + \frac{1}{2}\sum_{k=1}^{j-1} \|u_{kj}\|^2,$$
$$x_{jk} = u_j u_{jk} + 2\sum_{l=1}^{j-1} u_{lj} u_{lk}.$$

We identify Ω with T by identification of $x = t(u)e$ and $t(u)$. Then we have

$$dx = 2^r \prod_{j=1}^{r} u_j^{d(r-j)+1} du,$$
$$\Delta(x) = \prod_{j=1}^{r} u_j^2.$$

Therefore

$$\Delta(x)^{-\frac{n}{r}}dx = d\mu_l(t(u)),$$

which gives the G-invariant measure on Ω.

Let V^C denote the complexification of V. $T_\Omega = V + i\Omega$ is the tube domain over Ω in V^C. $G(T_\Omega)$ denotes the holomorphic automorphism group of T_Ω and $G(T_\Omega)^0$ is the identity component of $G(T_\Omega)$. The Iwasawa decomposition of $G(T_\Omega)^0$ is given by $G(T_\Omega)^0 = \underline{N}A\underline{K}$, where

$$\begin{aligned}
&\underline{K} = \{g \in G(T_\Omega)^0 : g(ie) = ie\} \supset K,\\
&\underline{N} = N^+N,\\
&N^+ = \{\tau_u : z \mapsto z + u, \quad u \in V\} \cong V.
\end{aligned}$$

Therefore,

$$G(T_\Omega)^0 = N^+T\underline{K}.$$

We call it the partial Iwasawa decomposition as in Terras' book [11]. T normalizes N^+ as

$$t(v)\tau_u = \tau_{t(v)u}t(v), \quad u \in V,\, v \in V_+.$$

$P = \underline{N}A = N^+T$ is called the Iwasawa subgroup. P is a nonunimodular group. Using the parametrization (u, v) for $\tau_u t(v) \in P$, the left Haar measure of P is given by

$$d\mu_l(u,v) = 2^r \prod_{j=1}^{r} v_j^{-d(r+j-2)-3} du\, dv = \prod_{j=1}^{r} v_j^{-d(r-1)-2} du\, d\mu_l(t(v)),$$

and the right Haar measure of P is given by

$$d\mu_r(u,v) = 2^r \prod_{j=1}^{r} v_j^{-d(r-j)-1} du\, dv = du\, d\mu_r(t(v)).$$

P acts on T_Ω simply and transitively. We identify T_Ω with P by identification of $\tau_u t(v)(ie)$ and $\tau_u t(v)$. If $x + iy = \tau_u t(v)(ie) = u + it(v)e$, then

$$\Delta(y)^{-\frac{2n}{r}}dx\, dy = d\mu_l(u,v),$$

which is the $G(T_\Omega)^0$-invariant measure on T_Ω. Note that

$$\operatorname{Det}(g) = \Delta(ge)^{\frac{n}{r}}, \quad g \in G.$$

P has a natural unitary representation on $L^2(V)$ defined by

$$\pi_{(u,v)} : f(x) \mapsto \Delta(t(v)e)^{-\frac{n}{2r}} f(t(v)^{-1}x - t(v)^{-1}u).$$

We shall decompose $L^2(V)$ into the direct sum of irreducible invariant closed subspaces under π.

2. The decomposition of $L^2(V)$

In order to decompose $L^2(V)$, we need to identify the non-degenerate T-orbits of V under the contragredient action of T, which is given by $x \mapsto t(v)'^{\,-1}x$ where $t(v)'$ denotes the transpose of $t(v)$. First we prove

Lemma 1. *(1) Suppose* $z_{ij} \in V_{ij}$, $w_{kl} \in V_{kl}$, $i<j$, $k<l$, $i \neq l$, $k \neq j$, *then*

$$[z_{ij}\Box c_i, w_{kl}\Box c_k] = 0.$$

(2) Suppose $z_{ij} \in V_{ij}$, *then*

$$(z_{ij}\Box c_i)' = z_{ij}\Box c_j.$$

Proof. (a) To prove (1), we use the facts

$$\begin{aligned} &V_{ij}\cdot V_{jk} \subset V_{ik}, && \text{if} \quad i \neq k,\\ &V_{ij}\cdot V_{kl} = \{0\}, && \text{if} \quad \{i,j\}\cap\{k,l\} = \emptyset,\\ &xy = \frac{1}{2}(x|y)(c_i+c_j), && \text{if} \quad x,y \in V_{ij}\end{aligned}$$

(cf [3], Theorem IV.2.1 (iii) and Proposition IV.1.4 (i)). We also use the matrix of $z\Box c$ with respect to the Peirce decomposition, when c is idempotent in V and $z \in V(c,\frac{1}{2})$ (see [3], proof of Lemma VI.3.1). Let

$$x = \sum_{j=1}^{r} x_j c_j + \sum_{j<k} x_{jk}, \quad x_{jk} \in V_{jk}.$$

We compute separately in four cases.

1) If $k=i$, $l=j$, $i<j$, then

$$(z_{ij}\Box c_i)(w_{ij}\Box c_i)x = \frac{x_i}{4}(z_{ij}|w_{ij})c_j = (w_{ij}\Box c_i)(z_{ij}\Box c_i)x.$$

2) If $k=i$, $l \neq j$, $i<j,l$, then

$$(z_{ij}\Box c_i)(w_{il}\Box c_i)x = \frac{x_i}{2} z_{ij} w_{il} = (w_{il}\Box c_i)(z_{ij}\Box c_i)x.$$

3) If $k \neq i$, $l=j$, $i,k<j$, then

$$(z_{ij}\Box c_i)(w_{kj}\Box c_k)x = \frac{1}{2}(z_{ij}|x_{ik}w_{kj})c_j = \frac{1}{2}(w_{kj}|x_{ik}z_{ij})c_j = (w_{kj}\Box c_k)(z_{ij}\Box c_i)x,$$

where the second equality is due to the associativity of the inner product.

4) If $k \neq i, j,\ l \neq i, j,\ i < j,\ k < l$, we may assume $i < k$, then

$$(z_{ij}\Box c_i)(w_{kl}\Box c_k)x = z_{ij}(x_{ik}w_{kl}) = w_{kl}(x_{ik}z_{ij}) = (w_{kl}\Box c_k)(z_{ij}\Box c_i)x,$$

where the second equality follows from the Lemma V.3.2 in [3].

(b) Take $x = z_{ij},\ y = c_i + c_j$ in the identity

$$[L(x), L(y^2)] + 2[L(y), L(xy)] = 0$$

(cf [3]. Proposition II.1.1), we obtain

$$[L(c_i), L(z_{ij})] = [L(z_{ij}), L(c_j)].$$

It follows that

$$(z_{ij}\Box c_i)' = c_i\Box z_{ij} = z_{ij}\Box c_j. \qquad \blacksquare$$

Let $z_{jk} \in V_{jk}\ (j < k)$ and put

$$z^{(j)} = \sum_{k=j+1}^{r} z_{jk}, \quad z_{(k)} = \sum_{j=1}^{k-1} z_{jk}.$$

Put

$$\tau'(z_{(k)}) = \exp(2z_{(k)}\Box c_k).$$

If $z_{ij} \in V_{ij},\ w_{kl} \in V_{kl},\ i < j,\ k < l,\ i \neq l,\ k \neq j$, Lemma 1 implies that

$$\tau(z_{ij})\tau(w_{kl}) = \tau(w_{kl})\tau(z_{ij})$$

and

$$\tau(z_{ij})' = \tau'(z_{ij}).$$

Thus $\tau'(z_{ij})$ is a dual Frobenius transform. Also, by Lemma 1,

$$\tau(z^{(j)}) = \tau(z_{j,j+1})\cdots\tau(z_{j,r}),$$
$$\tau'(z_{(k)}) = \tau'(z_{1,k})\cdots\tau'(z_{k-1,k}).$$

Therefore, for

$$u = \sum_{j=1}^{r} u_j c_j + \sum_{j<k} u_{jk}, \quad u_j > 0,\ u_{jk} \in V_{jk},$$

we have, by also using (1.1),

$$\begin{aligned} t(u) &= P(b_1)\tau(u^{(1)})P(b_2)\cdots\tau(u^{(r-1)})P(b_r) \\ &= P(b_1)\tau(u_{12})P(b_2)\tau(u_{13})\tau(u_{23})\cdots P(b_{r-1})\tau(u_{1r})\cdots\tau(u_{r-1,r})P(b_r). \\ t(u)' &= P(b_r)\tau'(u_{r-1,r})\tau'(u_{r-2,r})\cdots\tau'(u_{1r})P(b_{r-1})\cdots P(b_2)\tau'(u_{12})P(b_1) \\ &= P(b_r)\tau'(u_{(r)})P(b_{r-1})\cdots\tau'(u_{(2)})P(b_1) \end{aligned}$$

where

$$u_{(k)} = \sum_{j=1}^{k-1} u_{jk}.$$

For $j = 1, \cdots, r$, let $V^{(j)}$ be the subalgebra $V(c_1 + \cdots + c_j, 1)$ of V and $W^{(j)}$ be the subalgebra $V(c_{r-j+1} + \cdots + c_r, 1)$ of V. P_j and P_j^* denote the orthogonal projections onto $V^{(j)}$ and $W^{(j)}$ respectively. $\det_{(j)}$ and $\det^*_{(j)}$ are the determinants relative to $V^{(j)}$ and $W^{(j)}$ respectively. We define

$$\Delta_j(x) = \det_{(j)}(P_j x),$$
$$\Delta_j^*(x) = \det^*_{(j)}(P_j^* x).$$

Furthermore, for $\mathbf{s} = (s_1, \cdots, s_r)$. We let

$$\Delta_{\mathbf{s}}(x) = \Delta_1(x)^{s_1 - s_2} \cdots \Delta_{r-1}(x)^{s_{r-1} - s_r} \Delta_r(x)^{s_r},$$
$$\Delta_{\mathbf{s}}^*(x) = \Delta_1^*(x)^{s_1 - s_2} \cdots \Delta_{r-1}^*(x)^{s_{r-1} - s_r} \Delta_r^*(x)^{s_r}.$$

For $x \in V, t(u) \in T$, we have

$$\Delta_{\mathbf{s}}^*(t(u)'x) = u_1^{2s_r} \cdots u_r^{2s_1} \Delta_{\mathbf{s}}^*(x) = \Delta_{\mathbf{s}}^*(t(u)'e)\Delta_{\mathbf{s}}^*(x). \tag{2.1}$$

In particular, $\Delta_{\mathbf{s}}^*$ is invariant under the Frobenius transform $\tau'(z_{(k)})$ (cf [3], Proposition VII.1.5).

Set

$$E = \{\varepsilon = \sum_{j=1}^{r} \varepsilon_j c_j : \varepsilon_j = 1 \text{ or } i\},$$
$$\Omega_\varepsilon = \{x \in V : x = t(u)'P(\varepsilon)e, \, u \in V_+\}.$$

Lemma 2. *(1) The Ω_ε's are disjoint and simply transitive orbits under the contragredient action of T. (2) $\bigcup_{\varepsilon \in E} \Omega_\varepsilon$ is a set with a complementary of measure zero.*

Proof. (a) Suppose that

$$t(u)'P(\varepsilon)e = t(v)'P(\delta)e, \quad u, v \in V_+, \varepsilon, \delta \in E.$$

Write

$$g = P(\delta)t(v)'^{-1}t(u)'P(\varepsilon).$$

Since $t(u), t(v)$ are triangular and $P(\varepsilon), P(\delta)$ are diagonal, g is triangular. On the other hand, since $ge = e$, from the Proposition VIII.2.4 in [3] g is an automorphism of V^C and $g' = g^{-1}$. Therefore g is diagonal. Because $t(u), t(v)$ have positive diagonal elements and $P(\varepsilon), P(\delta)$ have diagonal elements $1, -1$ or i, it is concluded that $u = v, \varepsilon = \delta$.

(b) Set

$$B = \{x \in V : \Delta_k^*(x) \neq 0, k = 1, \cdots, r\}.$$

Obviously, $V \setminus B$ is a zero measure set. We will prove that $B = \bigcup_{\varepsilon \in E} \Omega_\varepsilon$. It is easy to see that $B \supset \bigcup_{\varepsilon \in E} \Omega_\varepsilon$. Assume that

$$x = \sum_{j=1}^{r} x_j c_j + \sum_{j<k} x_{jk} \in B.$$

By [3], Theorem VI.3.5 we can write

$$x = \tau'(z_{(r)}) \cdots \tau'(z_{(2)}) \sum_{j=1}^{r} a_j c_j$$

where

$$\begin{aligned} z_{(k)} &= \sum_{j=1}^{k-1} z_{jk} \in \bigoplus_{j=1}^{k-1} V_{jk}, \\ a_j &= \frac{\Delta^*_{r-j+1}(x)}{\Delta^*_{r-j}(x)} \neq 0, \quad j = 1, \cdots, r-1, \\ a_r &= \Delta^*_1(x) \neq 0. \end{aligned}$$

Set

$$\begin{aligned} \varepsilon_j &= \begin{cases} 1, & \text{if} \quad a_j > 0, \\ i, & \text{if} \quad a_j < 0, \end{cases} \\ u_j &= \sqrt{|a_j|}, \\ u_{jk} &= u_k z_{jk}. \end{aligned}$$

Then, by (1.1),

$$x = t(u)' P(\varepsilon) e. \qquad \blacksquare$$

Remark. Clearly, $\Omega_e = \Omega, \Omega_{ie} = -\Omega$. Ω_ε is a connected open set in V because Ω_ε is homeomorphic to V_+. But Ω_ε may not be convex neither K-invariant in general.

A simple example of Lemma 2 can be given as follows. Let V be the space $\mathrm{Sym}(m, \mathbf{R})$ of all $m \times m$ symmetric matrices and $c_j = \mathrm{diag}(0, \cdots, 0, 1, 0, \cdots, 0)$. An element t in T has the following form: $tx = uxu'$, where u is a lower triangular matrix with positive diagonal elements. Let Σ denote the set of all diagonal matrices with diagonal elements ± 1. $\Omega_\sigma (\sigma \in \Sigma)$ consists of all matrices of form $u'\sigma u$. Then Ω_σ's are disjoint and simply transitive orbits under the adjoint action of T and $\bigcup_{\sigma \in \Sigma} \Omega_\sigma$ is a total measure set. Now we are ready to decompose $L^2(V)$. Set

$$H_\varepsilon = \{ f \in L^2(V) : \mathrm{supp} \hat{f} \subseteq \mathrm{Cl}(\Omega_\varepsilon) \}.$$

Proposition 1. *Each of H_ε is an irreducible invariant closed subspace of $L^2(V)$ under π and*

$$L^2(V) = \bigoplus_{\varepsilon \in E} H_\varepsilon. \tag{2.2}$$

Proof. (2.2) follows from Lemma 2. Because

$$(\pi_{(u,v)}f)\hat{}(y) = \Delta(t(v)e)^{\frac{n}{2r}} e^{-i(u|y)} \hat{f}(t(v)'y),$$

it is easy to see that H_ε is invariant under π. We need to prove that H_ε is irreducible. Let W be a non-zero invariant closed subspace of H_ε under π and W^+ the orthogonal complement of W in H_ε. Taking a function $g \in W$, not identically zero, if $f \in W^+$, then

$$\langle f, \pi_{(u,v)}g\rangle_{L^2(V)} = \int_V f(x)\overline{\pi_{(u,v)}g(x)}\, dx = 0, \quad u \in V,\, v \in V_+.$$

Write

$$\tilde{g}(x) = \overline{g(-x)},$$
$$g_{t(v)}(x) = \Delta(t(v)e)^{-\frac{n}{2r}} g(t(v)^{-1}x).$$

We have

$$\langle f, \pi_{(u,v)}g\rangle_{L^2(V)} = f * \tilde{g}_{t(v)}(u). \tag{2.3}$$

Therefore,

$$(f * \tilde{g}_{t(v)})\hat{}(y) = \Delta(t(v)e)^{\frac{n}{2r}} \hat{f}(y)\overline{\hat{g}(t(v)'y)} = 0, \quad a.e.\ y \in V. \tag{2.4}$$

Set

$$S_1 = \mathrm{supp}\hat{f} \cap \Omega_\varepsilon,$$
$$S_2 = \mathrm{supp}\hat{g} \cap \Omega_\varepsilon.$$

S_1^d and S_2^d consist of points of density of S_1 and S_2 respectively. S_2^d is a positive measure set since g is not identically zero. If S_1^d has positive measure, by Lemma 2, there exists $t(v_0) \in T$ such that $S = S_1^d \cap t(v_0)'^{\,-1} S_2^d$ has positive measure. But

$$(f * \tilde{g}_{t(v)})\hat{}(y) \neq 0, \quad y \in S,$$

which contradicts (2.4). Therefore f is identically zero. This proves that H_ε is irreducible. ∎

Remark. For F in $H^2(T_\Omega)$, the Hardy space on T_Ω, the following limit exists,

$$\lim_{y\to 0, y\in\Omega} F(\cdot + iy) = f, \quad \text{in } L^2(V).$$

Then

$$H^2(V) = \{f \in L^2(V) : \text{ there exists } F \in H^2(T_\Omega) \text{ such that } f = \lim F\}$$

is called the Hardy space on V. It is easy to see that $H_e = H^2(V)$ and $H_{ie} = \overline{H^2(V)}$.

3. The admissibility condition

The restriction of π on H_ε is square-integrable, *i.e.*, there exists a function $\phi(\neq 0)$ in H_ε such that

$$C_\phi = \frac{1}{\|\phi\|^2_{L^2(V)}} \int_P |\langle \phi, \pi_{(u,v)}\phi\rangle_{L^2(V)}|^2 d\mu_l(u,v) < \infty. \tag{3.1}$$

(3.1) is called the admissibility condition and ϕ is called an admissible wavelet. We want to give a characterization of the admissibility conditioin in terms of Fourier transform and Jordan algebra, which does not involve any group representation.

Lemma 3. *Suppose $x = t(u)'P(\varepsilon)e$ in Ω_ε. If*

$$x = \sum_{j=1}^r x_j c_j + \sum_{j<k} x_{jk}$$

is the Peirce decomposition of x, then

$$x_j = \varepsilon_j^2 u_j^2 + \frac{1}{2}\sum_{k=j+1}^r \varepsilon_k^2 \|u_{jk}\|^2,$$

$$x_{jk} = \varepsilon_k^2 u_k u_{jk} + 2\sum_{l=k+1}^r \varepsilon_l^2 u_{jl} u_{kl}.$$

Lemma 3 can be proved in a similar way as in [3], Proposition VI.3.8.

For the transformation $x = t(u)'P(\varepsilon)e$, by Lemma 3, it is easy to compute that

$$\begin{aligned} dx &= 2^r \prod_{j=1}^r u_j^{d(j-1)+1} du \\ &= \prod_{j=1}^r u_j^{2d(j-1)+2} d\mu_l(t(u)). \end{aligned}$$

Let

$$\underline{\mathbf{s}} = (1 + d(r-1), 1 + d(r-2), \cdots, 1).$$

By (2.1),

$$\Delta^*_{\underline{\mathbf{s}}}(x) = \Delta^*_{\underline{\mathbf{s}}}(t(u)'e)\Delta^*_{\underline{\mathbf{s}}}(P(\varepsilon)e).$$

Therefore,

$$|\Delta^*_{\underline{\mathbf{s}}}(x)| = \Delta^*_{\underline{\mathbf{s}}}(t(u)'e).$$

and we have

$$|\Delta^*_{\underline{\mathbf{s}}}(x)|^{-1} dx = d\mu_l(t(u)). \tag{3.2}$$

We denoted by AW_ε the set of all admissible wavelets in H_ε.

Theorem 1. *Suppose $\phi(\neq 0)$ in H_ε. Then $\phi \in AW_\varepsilon$ if and only if*

$$C_\phi = \int_{\Omega_\varepsilon} |\hat{\phi}(y)|^2 |\Delta^*_{\underline{\mathbf{s}}}(y)|^{-1} dy < \infty.$$

Proof. Using (2.3), we have

$$\begin{aligned}
C_\phi &= \frac{1}{\|\phi\|^2_{L^2(V)}} \int_P |\langle \phi, \pi_{(u,v)}\phi\rangle_{L^2(V)}|^2 d\mu_l(u,v) \\
&= \frac{1}{\|\phi\|^2_{L^2(V)}} \int_T \left(\int_V |\phi * \tilde{\phi}_{t(v)}(u)|^2 du \right) \prod_{j=1}^{r} v_j^{-d(r-1)-2}\, d\mu_l(t(v)) \\
&= \frac{1}{(2\pi)^n} \frac{1}{\|\phi\|^2_{L^2(V)}} \int_T \left(\int_{\Omega_\varepsilon} |\hat{\phi}(y)\overline{\hat{\phi}(t(v)'y)}|^2 dy \right) d\mu_l(t(v)) \\
&= \frac{1}{(2\pi)^n} \frac{1}{\|\phi\|^2_{L^2(V)}} \int_{\Omega_\varepsilon} |\hat{\phi}(y)|^2 \left(\int_T |\hat{\phi}(t(v)'y)|^2 d\mu_l(t(v)) \right) dy.
\end{aligned}$$

For $y \in \Omega_\varepsilon$, there exists $v^1 \in V_+$ such that $y = t(v^1)'P(\varepsilon)e$. Using (3.2) we obtain

$$\begin{aligned}
&\int_T |\hat{\phi}(t(v)'y)|^2 d\mu_l(t(v)) \\
&= \int_T \left| \hat{\phi}\Big((t(v^1)t(v))'P(\varepsilon)e \Big) \right|^2 d\mu_l(t(v)) \\
&= \int_T |\hat{\phi}(t(v)'P(\varepsilon)e)|^2 d\mu_l(t(v)) \\
&= \int_{\Omega_\varepsilon} |\hat{\phi}(y)|^2 |\Delta^*_{\underline{\mathbf{s}}}(y)|^{-1} dy.
\end{aligned}$$

The proof of Theorem 1 is completed. ∎

Suppose ϕ and ψ are admissible wavelets. We define the "inner product"of ϕ and ψ by

$$\langle \phi, \psi \rangle_{AW} = \int_V \hat{\phi}(y)\overline{\hat{\psi}(y)} |\Delta^*_{\underline{\mathbf{s}}}(y)|^{-1} dy.$$

Remark. If $\phi \in AW_\varepsilon$, $\psi \in AW_\delta$, $\varepsilon \neq \delta$, then $\langle \phi, \psi \rangle_{AW} = 0$. For $f \in H_\varepsilon$, $\phi \in AW_\varepsilon$, we define the wavelet transform of f with respect to ϕ by

$$W_\phi f(u,v) = \langle f, \pi_{(u,v)}\phi \rangle_{L^2(V)}.$$

Theorem 2. *Suppose $f, g \in H_\varepsilon$, $\phi, \psi \in AW_\varepsilon$. Then*

$$\langle W_\phi f, W_\psi g \rangle_{L^2(P,\, d\mu_l)} = \langle \psi, \phi \rangle_{AW} \langle f, g \rangle_{L^2(V)}.$$

In particular,

$$\|W_\phi f\|^2_{L^2(P,\,d\mu_l)} = C_\phi \|f\|^2_{L^2(V)}.$$

Theorem 2 can be proved in a similar way as Theorem 1. From the theory of square-integrable representation of nonunimodular groups (cf [2]), Theorem 1 and Theorem 2 are equivalent.

We are going to construct a family of admissible wavelets which is complete and orthonormal with respect to $\langle \cdot, \cdot \rangle_{AW}$.

Let $\{c^l_{jk} : l = 1, \cdots, d\}$ be an orthonormal basis of V_{jk}. The set of indices $\mathcal{A}$ is defined by

$$\mathcal{A} = \{\alpha \in V : \alpha = \sum_{j=1}^{r} \alpha_j c_j + \sum_{j<k} \sum_{l=1}^{d} \alpha^l_{jk} c^l_{jk},\ \alpha_j, \alpha^l_{jk} \text{ are nonnegative integers}\}.$$

Let $L^{(\mu)}_m(s)$ be the Laguerre polynomials defined by

$$L^{(\mu)}_m(s) = \sum_{j=0}^{m} \binom{m+\mu}{m-j} \frac{(-s)^j}{j!} = \frac{1}{m!} e^s s^{-\mu} \big(\frac{d}{ds}\big)^m (e^{-s} s^{m+\mu}), \quad \mu > -1,$$

and $H_m(s)$ be the Hermite polynomials defined by

$$H_m(s) = \sum_{j=0}^{[m/2]} (-1)^j \frac{m!}{j!(m-2j)!} (2s)^{m-2j} = (-1)^m e^{s^2} \big(\frac{d}{ds}\big)^m (e^{-s^2}).$$

The Laguerre polynomials and the Hermite polynomials satisfy the following orthogonal relations respectively.

$$\int_0^\infty e^{-s} s^\mu L^{(\mu)}_m(s) L^{(\mu)}_k(s)\, ds = \Gamma(\mu+1) \binom{m+\mu}{m} \delta_{mk},$$

$$\int_{-\infty}^\infty e^{-s^2} H_m(s) H_k(s)\, ds = \pi^{\frac{1}{2}} 2^m m! \delta_{mk}.$$

And they are complete in $L^2(\mathbf{R}^+, e^{-s} s^\mu ds)$ and $L^2(\mathbf{R}, e^{-s^2} ds)$ respectively (cf [10], Chapter V). We also need following notations. For

$$\alpha = \sum_{j=1}^{r} \alpha_j c_j + \sum_{j<k} \sum_{l=1}^{d} \alpha^l_{jk} c^l_{jk} \in \mathcal{A},$$

we set

$$|\alpha| = \sum_{j=1}^{r} \alpha_j + \sum_{j<k} \sum_{l=1}^{d} \alpha^l_{jk},$$

$$\alpha! = \prod_{j=1}^{r} \alpha_j! \prod_{j<k} \prod_{l=1}^{d} \alpha^l_{jk}!,$$

$$\alpha^0 = (\alpha_1, \cdots, \alpha_r),$$

$$|\alpha^0| = \sum_{j=1}^{r} \alpha_j,$$

$$\alpha^0! = \prod_{j=1}^{r} \alpha_j!.$$

For $\mathbf{s}\in\mathbf{C}^r$, $\lambda\in\mathbf{C}$, we write

$$\mathbf{s}+\lambda=(s_1+\lambda,\cdots,s_r+\lambda).$$

The gamma function of the symmetric cone Ω is defined by

$$\begin{aligned}\Gamma_\Omega(\mathbf{s})&=\int_\Omega e^{-\operatorname{tr}(x)}\Delta_{\mathbf{s}}(x)\Delta(x)^{-\frac{n}{r}}\,dx\\&=(2\pi)^{\frac{n-r}{2}}\prod_{j=1}^r\Gamma\big(s_j-(j-1)\frac{d}{2}\big),\quad \operatorname{Re}s_j>(j-1)\frac{d}{2},\ j=1,\cdots,r.\end{aligned}$$

Assume that

$$v=\sum_{j=1}^r v_jc_j+\sum_{j<k}\sum_{l=1}^d v_{jk}^lc_{jk}^l\in V_+,$$

we define

$$\psi_\alpha(v)=C\prod_{j=1}^r e^{-v_j^2}v_j^{\nu-\frac{n}{r}}L_{\alpha_j}^{(\mu_j)}(2v_j^2)\prod_{j<k}\prod_{l=1}^d e^{-\frac{1}{2}(v_{jk}^l)^2}H_{\alpha_{jk}^l}(v_{jk}^l)$$

where

$$\begin{aligned}C&=2^{\frac{1}{2}(\nu r-n+|\alpha|-|\alpha^0|)}\alpha^0!(\alpha!)^{-\frac{1}{2}}\Gamma_\Omega(\alpha^0+\nu-\frac{n}{r})^{-\frac{1}{2}},\\ \mu_j&=\nu-\frac{n}{r}-1-\frac{d}{2}(j-1),\quad \nu>1+d(r-1).\end{aligned}$$

We regard $\psi_\alpha(v)$ as the functions on group T. From the orthogonal relations and completeness of the Laguerre polynomials and Hermite polynomials, we conclude that $\{\psi_\alpha(v):\ \alpha\in\mathcal{A}\}$ is an orthonormal basis of $L^2(T,d\mu_l)$.

Now we define a family of functions ϕ_α^ε by

$$\hat{\phi}_\alpha^\varepsilon(y)=\begin{cases}\psi_\alpha(v), & y=t(v)'P(\varepsilon)e,\\ 0, & y\notin\Omega_\varepsilon.\end{cases}$$

Then $\phi_\alpha^\varepsilon\in H_\varepsilon$ is admissible and $\{\phi_\alpha^\varepsilon:\ \varepsilon\in E,\alpha\in\mathcal{A}\}$ is an complete orthonormal system with respect to $\langle\cdot,\cdot\rangle_{AW}$.

4. The decomposition of $L_\nu^2(T_\Omega)$

The weighted L^2-space on the symmetric tube domain T_Ω is defined by

$$L_\nu^2(T_\Omega)=\{F:\ \|F\|_\nu^2=\int_{T_\Omega}|F(x+iy)|^2\Delta(y)^{\nu-\frac{2n}{r}}\,dx\,dy<\infty\}.$$

$\mathcal{H}^2_\nu(T_\Omega)$, the weighted Bergman space, is the subspace of all holomorphic functions in $L^2_\nu(T_\Omega)$, *i.e.*,

$$\mathcal{H}^2_\nu(T_\Omega) = \{F \in L^2_\nu(T_\Omega) : F \text{ is holomorphic on } T_\Omega\}.$$

We assume that $\nu > 1 + d(r-1)$ so that $\mathcal{H}^2_\nu(T_\Omega) \neq \{0\}$. We want to decompose $L^2_\nu(T_\Omega)$ into the direct sum of subspaces such that the first component is exactly the weighted Bergman space $\mathcal{H}^2_\nu(T_\Omega)$.

Suppose $f \in H_\varepsilon$, $\phi \in AW_\varepsilon$. For $\nu > 1 + d(r-1)$, we define the weighted wavelet transform W^ν_ϕ by

$$F(u + it(v)e) = W^\nu_\phi f(u + it(v)e) = C_\phi^{-\frac{1}{2}} W_\phi f(u,v) \Delta(t(v)e)^{-\frac{\nu}{2}}.$$

Set

$$\mathcal{H}^\varepsilon_\alpha = \{F = W^\nu_{\phi^\varepsilon_\alpha} f : f \in H_\varepsilon\}.$$

Proposition 2. *For $\nu > 1 + d(r-1)$, we have*

$$L^2_\nu(T_\Omega) = \bigoplus_{\varepsilon \in E, \alpha \in \mathcal{A}} \mathcal{H}^\varepsilon_\alpha \tag{4.1}$$

and

$$\mathcal{H}^e_0 = \mathcal{H}^2_\nu(T_\Omega).$$

Proof. By Theorem 2, W^ν_ϕ is an isometric operator from H_ε into $L^2_\nu(T_\Omega)$ and $\mathcal{H}^\varepsilon_\alpha$'s are mutually orthogonal subspaces of $L^2_\nu(T_\Omega)$. We need to prove

$$L^2_\nu(T_\Omega) \subset \bigoplus_{\varepsilon \in E, \alpha \in \mathcal{A}} \mathcal{H}^\varepsilon_\alpha.$$

Suppose $F \in L^2_\nu(T_\Omega)$. Write $F_v(u) = F(u + it(v)e)$. For $v \in V_+$ almost every where, $F_v \in L^2(V)$. We let $G(v,y) = \hat{F}_v(y)$. Fix $y = t(v^1)' P(\varepsilon) e \in \Omega_\varepsilon$, then $G(v,y)$, regarded as the function on T, is in $L^2(T, \Delta(t(v)e)^{\nu - \frac{n}{r}} d\mu_l(t(v)))$. Since

$$\{\Delta(t(v)e)^{-\frac{\nu}{2} + \frac{n}{2r}} \hat{\phi}^\varepsilon_\alpha(t(v)'y) : \alpha \in \mathcal{A}\}$$

is an orthonormal basis of $L^2(T, \Delta(t(v)e)^{\nu - \frac{n}{r}} d\mu_l(t(v)))$, we get

$$G(v,y) = \sum_{\alpha \in \mathcal{A}} a_\alpha(y) \Delta(t(v)e)^{-\frac{\nu}{2} + \frac{n}{2r}} \hat{\phi}^\varepsilon_\alpha(t(v)'y), \quad y \in \Omega_\varepsilon.$$

We define the functions f^ε_α by

$$\hat{f}^\varepsilon_\alpha(y) = \begin{cases} a_\alpha(y), & y \in \Omega_\varepsilon, \\ 0, & y \notin \Omega_\varepsilon. \end{cases}$$

It is easy to see that $f^\varepsilon_\alpha \in H_\varepsilon$. Therefore we have

$$F(u + it(v)e) = \sum_{\varepsilon \in E, \alpha \in \mathcal{A}} W^\nu_{\phi^\varepsilon_\alpha} f^\varepsilon_\alpha(u + it(v)e).$$

This proves (4.1).

Now we assume that

$$F(u+it(v)e) = W^{\nu}_{\phi_0^e} f(u+it(v)e), \quad f \in H_e.$$

Note that

$$\operatorname{tr}(y) = \sum_{j=1}^{r} y_j = \sum_{j=1}^{r} v_j^2 + \frac{1}{2}\sum_{j<k}\sum_{l=1}^{d} |v_{jk}^l|^2,$$

$$\Delta(y) = \prod_{j=1}^{r} v_j^2, \qquad y = t(v)'e,$$

we have

$$\hat{\phi}_0^e(y) = \begin{cases} \Gamma_\Omega(\nu - \frac{n}{r})^{-\frac{1}{2}} 2^{\frac{\nu r - n}{2}} e^{-\operatorname{tr} y} \Delta(y)^{\frac{\nu}{2} - \frac{n}{2r}}, & y \in \Omega, \\ 0, & y \notin \Omega. \end{cases}$$

Therefore,

$$\begin{aligned} F(u+it(v)e) &= \Delta(t(v)e)^{-\frac{\nu}{2}-\frac{n}{2r}} \int_V f(x)\overline{\phi(t(v)^{-1}x - t(v)^{-1}u)}\, dx \\ &= (2\pi)^{-n}\Delta(t(v)e)^{-\frac{\nu}{2}+\frac{n}{2r}} \int_\Omega \hat{f}(y) e^{i(u|y)} \overline{\hat{\phi}(t(v)'y)}\, dy \\ &= (2\pi)^{-n}\Gamma_\Omega(\nu - \frac{n}{r})^{-\frac{1}{2}} \int_\Omega e^{i(u+it(v)e|y)} \hat{f}(y)\Delta(2y)^{\frac{\nu}{2}-\frac{n}{2r}}\, dy, \end{aligned}$$

where we make use of the equality

$$\Delta(t(v)e) = \operatorname{Det}(t(v))^{\frac{r}{n}} = \operatorname{Det}(t(v)')^{\frac{r}{n}} = \Delta(t(v)'e).$$

We define the map $\mathcal{F}_\nu$ by

$$g(y) = \mathcal{F}_\nu f(y) = (2\pi)^{-\frac{n}{2}}\Gamma_\Omega(\nu - \frac{n}{r})^{-\frac{1}{2}} \hat{f}(y)\Delta(2y)^{-\frac{\nu}{2}+\frac{n}{2r}}$$

and the map $\mathcal{L}_\nu$ by

$$F(u+it(v)e) = \mathcal{L}_\nu g(u+it(v)e) = (2\pi)^{-\frac{n}{2}} \int_\Omega e^{i(u+it(v)e|y)} g(y)\Delta(2y)^{\nu - \frac{n}{r}}\, dy.$$

It is obvious that $\mathcal{F}_\nu$ is an isomorphism from $H^2(V)$ onto $L^2_\nu(\Omega) = L^2(\Omega, \Delta(2y)^{\nu-\frac{n}{r}} dy)$. $\mathcal{L}_\nu$ is an isomorphism from $L^2_\nu(\Omega)$ onto $\mathcal{H}^2_\nu(T_\Omega)$ (cf [3]). We see that $W^{\nu}_{\phi_0^e} = \mathcal{L}_\nu \circ \mathcal{F}_\nu$ and

$$\|F\|^2_{L^2_\nu(T_\Omega)} = \Gamma_\Omega(\nu - \frac{n}{r})\|g\|^2_{L^2_\nu(\Omega)} = \|f\|^2_{L^2(V)}.$$

Clearly, $\mathcal{H}^e_0$ is exactly the weighted Bergman space $\mathcal{H}^2_\nu(T_\Omega)$. ■

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