Journal of Lie Theory
Volume **8** (1998) 293–309

On the Geometry of the Virasoro-Bott group

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Communicated by W.A.F. Ruppert

Abstract. We consider a natural Riemannian metric on the infinite dimensional manifold of all embeddings from a manifold into a Riemannian manifold, and derive its geodesic equation in the case $\mathrm{Emb}(\mathbb{R},\mathbb{R})$ which turns out to be Burgers' equation. Then we derive the geodesic equation, the curvature, and the Jacobi equation of a right invariant Riemannian metric on an infinite dimensional Lie group, which we apply to $\mathrm{Diff}(\mathbb{R})$, $\mathrm{Diff}(S^1)$, and the Virasoro-Bott group. Many of these results are well known, the emphasis is on conciseness and clarity.

1. Introduction

We consider a natural Riemannian metric on the infinite dimensional manifold of all embeddings from a manifold into a Riemannian manifold, derive its geodesic equation in the case $\mathrm{Emb}(\mathbb{R},\mathbb{R})$ which turns out to be Burgers' equation. For the general case see [9]. Then we give a careful exposition of the derivation of the geodesic equation, the curvature, and the Jacobi equation of a right invariant Riemannian metric on an infinite dimensional Lie group. This is a careful presentation and extension of results in [1], [2], [3]. The formulas obtained in this way are then applied to $\mathrm{Diff}(\mathbb{R})$, $\mathrm{Diff}(S^1)$, and the Virasoro-Bott group, where the geodesic equation is the Korteweg-de Vries equations. This is due to [8], [10], [23], and also [22]. A fast overview on the geometry of the Virasoro-Bott group can also be found in [24]. The emphasis of this paper is on a unified setting for these results, and on conciseness and clarity. Thanks to Hermann Schichl and to the referee for detailed checks of the computations.

1991 *Mathematics Subject Classification.* 58D05, 58F07, Secondary 35Q53.

Key words and phrases. diffeomorphism group, connection, Jacobi field, symplectic structure, KdV equation.

P.W.M. was supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 10037 PHY'.

T.R. acknowledges the partial support of NSF Grant DMS-9503273 and DOE contract DE-FG03-95ER25245-A000.

ISSN 0949–5932 / \$2.50

2. The general setting and a motivating example

2.1. The principal bundle of embeddings. Let M and N be smooth finite dimensional manifolds, connected and second countable without boundary, such that $\dim M \leq \dim N$. The space $\operatorname{Emb}(M,N)$ of all embeddings (immersions which are homeomorphisms on their images) from M into N is an open submanifold of $C^\infty(M,N)$ which is stable under the right action of the diffeomorphism group of M. Here $C^\infty(M,N)$ is a smooth manifold modeled on spaces of sections with compact support $\Gamma_c(f^*TN)$. In particular the tangent space at f is canonically isomorphic to the space of vector fields along f with compact support in M. If f and g differ on a non-compact set then they belong to different connected components of $C^\infty(M,N)$. See [19] and [14]. Then $\operatorname{Emb}(M,N)$ is the total space of a smooth principal fiber bundle with structure group the diffeomorphism group of M; the base is called $B(M,N)$, it is a Hausdorff smooth manifold modeled on nuclear (LF)-spaces. It can be thought of as the "nonlinear Grassmannian" of all submanifolds of N which are of type M. This result is based on an idea implicitly contained in [25], it was fully proved in [5] for compact M and for general M in [18]. The clearest presentation is in [19], section 13. If we take a Hilbert space H instead of N, then $B(M,H)$ is the classifying space for $\operatorname{Diff}(M)$ if M is compact, and the classifying bundle $\operatorname{Emb}(M,H)$ carries also a universal connection. This is shown in [21].

2.2. If (N,g) is a Riemannian manifold then on the manifold $\operatorname{Emb}(M,N)$ there is a naturally induced weak Riemannian metric given, for $s_1, s_2 \in \Gamma_c(f^*TN)$ and $\varphi \in \operatorname{Emb}(M,N)$, by

$$G_\phi(s_1,s_2) = \int_M g(s_1,s_2)\operatorname{vol}(\phi^*g), \quad \phi \in \operatorname{Emb}(M,N),$$

where $\operatorname{vol}(g)$ denotes the volume form on N induced by the Riemannian metric g and $\operatorname{vol}(\phi^*g)$ the volume form on M induced by the pull back metric ϕ^*g. The covariant derivative and curvature of the Levi-Civita connection induced by G were investigated in [4] if $N = \mathbb{R}^{\dim M+1}$ (endowed with the standard inner product) and in [9] for the general case. We shall not reproduce the general formulae here.

This weak Riemannian metric is invariant under the action of the diffeomorphism group $\operatorname{Diff}(M)$ by composition from the right and hence it induces a Riemannian metric on the base manifold $B(M,N)$.

2.3. Example. Let us consider the special case $M = N = \mathbb{R}$, that is, the space $\operatorname{Emb}(\mathbb{R},\mathbb{R})$ of all embeddings of the real line into itself, which contains the diffeomorphism group $\operatorname{Diff}(\mathbb{R})$ as an open subset. The case $M = N = S^1$ is treated in a similar fashion and the results of this paper are also valid in this situation, where $\operatorname{Emb}(S^1,S^1) = \operatorname{Diff}(S^1)$. For our purposes, we may restrict attention to the space of orientation-preserving embeddings, denoted by $\operatorname{Emb}^+(\mathbb{R},\mathbb{R})$. The weak Riemannian metric has thus the expression

$$G_f(h,k) = \int_{\mathbb{R}} h(x)k(x)|f'(x)|\,dx, \quad f \in \operatorname{Emb}(\mathbb{R},\mathbb{R}), \quad h,k \in C_c^\infty(\mathbb{R},\mathbb{R}).$$

We shall compute the geodesic equation for this metric by variational calculus. The energy of a curve f of embeddings is

$$E(f) = \tfrac{1}{2}\int_a^b G_f(f_t, f_t)dt = \tfrac{1}{2}\int_a^b\int_{\mathbb{R}} f_t^2 f_x\,dxdt.$$

If we assume that $f(x,t,s)$ is a smooth function and that the variations are with fixed endpoints, then the derivative with respect to s of the energy is

$$\begin{aligned}
\frac{\partial}{\partial s}\Big|_0 E(f(\quad,s)) &= \frac{\partial}{\partial s}\Big|_0 \tfrac{1}{2}\int_a^b\int_{\mathbb{R}} f_t^2 f_x\,dxdt \\
&= \tfrac{1}{2}\int_a^b\int_{\mathbb{R}}(2f_t f_{ts} f_x + f_t^2 f_{xs})dxdt \\
&= -\tfrac{1}{2}\int_a^b\int_{\mathbb{R}}(2f_{tt} f_s f_x + 2f_t f_s f_{tx} + 2f_t f_{tx} f_s)dxdt \\
&= -\int_a^b\int_{\mathbb{R}}\left(f_{tt} + 2\frac{f_t f_{tx}}{f_x}\right) f_s f_x dxdt,
\end{aligned}$$

so that the geodesic equation with its initial data is:

$$\begin{aligned}
f_{tt} &= -2\frac{f_t f_{tx}}{f_x}, \quad f(\quad,0)\in \mathrm{Emb}^+(\mathbb{R},\mathbb{R}), \quad f_t(\quad,0)\in C_c^\infty(\mathbb{R},\mathbb{R}) \\
&=: \Gamma_f(f_t,f_t),
\end{aligned} \tag{1}$$

where the Christoffel symbol $\Gamma : \mathrm{Emb}(\mathbb{R},\mathbb{R})\times C_c^\infty(\mathbb{R},\mathbb{R})\times C_c^\infty(\mathbb{R},\mathbb{R}) \to C_c^\infty(\mathbb{R},\mathbb{R})$ is given by symmetrisation:

$$\Gamma_f(h,k) := -\frac{hk_x + h_x k}{f_x} = -\frac{(hk)_x}{f_x}\,. \tag{2}$$

For vector fields X, Y on $\mathrm{Emb}(\mathbb{R},\mathbb{R})$ the covariant derivative is given by the expression $\nabla_X^{\mathrm{Emb}} Y = dY(X) - \Gamma(X,Y)$. The Riemannian curvature $R(X,Y)Z = (\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})Z$ is then determined in terms of the Christoffel form by

$$\begin{aligned}
&R_f(h,k)\ell \\
&\quad = -d\Gamma(f)(h)(k,\ell) + d\Gamma(f)(k)(h,\ell) + \Gamma_f(h,\Gamma_f(k,\ell)) - \Gamma_f(k,\Gamma_f(h,\ell)) \\
&\quad = -\frac{h_x(k\ell)_x}{f_x^2} + \frac{k_x(h\ell)_x}{f_x^2} + \frac{\left(h\frac{(k\ell)_x}{f_x}\right)_x}{f_x} - \frac{\left(k\frac{(h\ell)_x}{f_x}\right)_x}{f_x} \\
&\quad = \frac{1}{f_x^3}\Big(f_{xx}h_x k\ell - f_{xx}hk_x\ell + f_x hk_{xx}\ell - f_x h_{xx}k\ell + 2f_x hk_x\ell_x - 2f_x h_x k\ell_x\Big)
\end{aligned} \tag{3}$$

The geodesic equation can be solved in the following way: if instead of the obvious framing we change variables to $T\,\mathrm{Emb} = \mathrm{Emb}\times C_c^\infty \ni (f,h) \mapsto (f, hf_x^2) =: (f,F)$ then the geodesic equation becomes $F_t = \frac{\partial}{\partial t}(f_t f_x^2) = f_x^2(f_{tt} + 2\frac{f_t f_{tx}}{f_x}) = 0$, so that $F = f_t f_x^2$ is constant in t, or $f_t(x,t)f_x(x,t)^2 = f_t(x,0)f_x(x,0)^2$. From here, a standard separation of variables argument gives the solution; it blows up in finite time for most initial conditions.

Now let us consider the trivialisation of $T\,\mathrm{Emb}(\mathbb{R},\mathbb{R})$ by right translation (this is most useful for $\mathrm{Diff}(\mathbb{R})$). The derivative of the inversion $\mathrm{Inv} : g \mapsto g^{-1}$ is given by

$$T_g(\mathrm{Inv})h = -T(g^{-1})\circ h \circ g^{-1} = \frac{h\circ g^{-1}}{g_x \circ g^{-1}} \quad \text{for} \quad g \in \mathrm{Emb}(\mathbb{R},\mathbb{R}),\, h \in C_c^\infty(\mathbb{R},\mathbb{R}).$$

Defining

$$u := f_t \circ f^{-1}, \quad \text{or, in more detail,} \quad u(x,t) = f_t(f(\quad,t)^{-1}(x),t),$$

we have

$$\begin{aligned} u_x &= (f_t \circ f^{-1})_x = (f_{tx}\circ f^{-1})\frac{1}{f_x \circ f^{-1}} = \frac{f_{tx}}{f_x}\circ f^{-1}, \\ u_t &= (f_t \circ f^{-1})_t = f_{tt}\circ f^{-1} + (f_{tx}\circ f^{-1})(f^{-1})_t \\ &= f_{tt}\circ f^{-1} + (f_{tx}\circ f^{-1})\frac{1}{f_x\ f^{-1}}(f_t\ f^{-1}) \end{aligned}$$

which, by (1) and the first equation becomes

$$u_t = f_{tt}\circ f^{-1} - \left(\frac{f_{tx}f_t}{f_x}\right)\circ f^{-1} = -3\left(\frac{f_{tx}f_t}{f_x}\right)\circ f^{-1} = -3u_x u.$$

The geodesic equation on $\mathrm{Emb}(\mathbb{R},\mathbb{R})$ in right trivialization, that is, in Eulerian formulation, is hence

$$u_t = -3u_x u\,, \tag{4}$$

which is just Burgers' equation.

3. Right invariant Riemannian metrics on Lie groups

3.1. Geodesics of a right invariant metric on a Lie group. Let G be a Lie group which may be infinite dimensional, with Lie algebra $\mathfrak{g}$. Let $\mu : G\times G \to G$ be the multiplication, let μ_x be left translation and μ^y be right translation, given by $\mu_x(y) = \mu^y(x) = xy = \mu(x,y)$. We also need the right Maurer-Cartan form $\kappa = \kappa^r \in \Omega^1(G,\mathfrak{g})$, given by $\kappa_x(\xi) := T_x(\mu^{x^{-1}})\cdot\xi$. It satisfies the right Maurer-Cartan equation $d\kappa - \frac{1}{2}[\kappa,\kappa]_\wedge = 0$, where $[\ ,\ ]_\wedge$ denotes the wedge product of $\mathfrak{g}$-valued forms on G induced by the Lie bracket. Note that $\frac{1}{2}[\kappa,\kappa]_\wedge(\xi,\eta) = [\kappa(\xi),\kappa(\eta)]$.

Let $\langle\ ,\ \rangle : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ be a positive definite bounded (weak) inner product. Then

$$G_x(\xi,\eta) = \langle T(\mu^{x^{-1}})\cdot\xi, T(\mu^{x^{-1}})\cdot\eta\rangle = \langle\kappa(\xi),\kappa(\eta)\rangle \tag{1}$$

is a right invariant (weak) Riemannian metric on G, and any (weak) right invariant bounded Riemannian metric is of this form, for suitable $\langle\ ,\ \rangle$.

Let $g : [a,b] \to G$ be a smooth curve. The velocity field of g, viewed in the right trivializations, coincides with the right logarithmic derivative $T(\mu^{g^{-1}}) \cdot \partial_t g = \kappa(\partial_t g) = (g^*\kappa)(\partial_t)$, where $\partial_t = \frac{\partial}{\partial t}$. The energy of the curve $g(t)$ is given by

$$E(g) = \tfrac{1}{2}\int_a^b G_g(g', g')dt = \tfrac{1}{2}\int_a^b \langle (g^*\kappa)(\partial_t), (g^*\kappa)(\partial_t)\rangle\, dt.$$

For a variation $g(t,s)$ with fixed endpoint we have then, using the right Maurer-Cartan equation and integration by parts,

$$\begin{aligned}
\partial_s E(g) &= \tfrac{1}{2}\int_a^b 2\langle \partial_s (g^*\kappa)(\partial_t),\ (g^*\kappa)(\partial_t)\rangle\, dt \\
&= \int_a^b \langle \partial_t (g^*\kappa)(\partial_s) - d(g^*\kappa)(\partial_t, \partial_s),\ (g^*\kappa)(\partial_t)\rangle\, dt \\
&= \int_a^b \left(-\langle (g^*\kappa)(\partial_s),\ \partial_t (g^*\kappa)(\partial_t)\rangle - \langle [(g^*\kappa)(\partial_t), (g^*\kappa)(\partial_s)],\ (g^*\kappa)(\partial_t)\rangle\right) dt \\
&= -\int_a^b \langle (g^*\kappa)(\partial_s),\ \partial_t (g^*\kappa)(\partial_t) + \operatorname{ad}((g^*\kappa)(\partial_t))^\top((g^*\kappa)(\partial_t))\rangle\, dt
\end{aligned}$$

where $\operatorname{ad}((g^*\kappa)(\partial_t))^\top : \mathfrak{g} \to \mathfrak{g}$ is the adjoint of $\operatorname{ad}((g^*\kappa)(\partial_t))$ with respect to the inner product $\langle\ ,\ \rangle$. In infinite dimensions one also has to check the existence of this adjoint. In terms of the right logarithmic derivative $u : [a,b] \to \mathfrak{g}$ of $g : [a,b] \to G$, given by $u(t) := g^*\kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}}) \cdot g'(t)$, the geodesic equation has the expression

$$u_t = -\operatorname{ad}(u)^\top u\,. \tag{2}$$

This is, of course, just the Euler-Poincaré equation for right invariant systems using the Lagrangian given by the kinetic energy (see [15], section 13) and the above derivation is done directly without invoking this theorem.

3.2. The covariant derivative. Our next aim is to derive the Riemannian curvature and for that we develop the basis-free version of Cartan's method of moving frames in this setting, which also works in infinite dimensions. The right trivialization, or framing, $(\kappa, \pi_G) : TG \to \mathfrak{g} \times G$ induces the isomorphism $R : C^\infty(G,\mathfrak{g}) \to \mathfrak{X}(G)$, given by $R(X)(x) := R_X(x) := T_e(\mu^x) \cdot X(x)$, for $X \in C^\infty(G,\mathfrak{g})$ and $x \in G$. Here $\mathfrak{X}(G) := \Gamma(TG)$ denote the Lie algebra of all vector fields. For the Lie bracket and the Riemannian metric we have

$$\begin{aligned}
&[R_X, R_Y] = R(-[X,Y]_\mathfrak{g} + dY \cdot R_X - dX \cdot R_Y), \\
&R^{-1}[R_X, R_Y] = -[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X), \\
&G_x(R_X(x), R_Y(x)) = \langle X(x), Y(x)\rangle\,,\ x \in G.
\end{aligned} \tag{1}$$

In the sequel we shall compute in $C^\infty(G,\mathfrak{g})$ instead of $\mathfrak{X}(G)$. In particular, we shall use the convention

$$\nabla_X Y := R^{-1}(\nabla_{R_X} R_Y) \quad \text{for } X, Y \in C^\infty(G,\mathfrak{g}).$$

to express the Levi-Civita covariant derivative.

Lemma. *Assume that for all $\xi \in \mathfrak{g}$ the adjoint $\operatorname{ad}(\xi)^\top$ with respect to the inner product $\langle\ ,\ \rangle$ exists and that $\xi \mapsto \operatorname{ad}(\xi)^\top$ is bounded. Then the Levi-Civita covariant derivative of the metric 2.1(1) exists and is given for any $X, Y \in C^\infty(G,\mathfrak{g})$ in terms of the isomorphism R by*

$$\nabla_X Y = dY.R_X + \tfrac{1}{2}\operatorname{ad}(X)^\top Y + \tfrac{1}{2}\operatorname{ad}(Y)^\top X - \tfrac{1}{2}\operatorname{ad}(X)Y. \tag{2}$$

Proof. Easy computations show that this formula satisfies the axioms of a covariant derivative, that relative to it the Riemannian metric is covariantly constant, since

$$R_X\langle Y, Z\rangle = \langle dY.R_X, Z\rangle + \langle Y, dZ.R_X\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle,$$

and that it is torsion free, since

$$\nabla_X Y - \nabla_Y X + [X,Y]_\mathfrak{g} - dY.R_X + dX.R_Y = 0. \quad \blacksquare$$

For $\xi \in \mathfrak{g}$ define $\alpha(\xi) : \mathfrak{g} \to \mathfrak{g}$ by $\alpha(\xi)\eta := \operatorname{ad}(\eta)^\top \xi$. With this notation, the previous lemma states that for all $X \in C^\infty(G,\mathfrak{g})$ the covariant derivative of the Levi-Civita connection has the expression

$$\nabla_X = R_X + \tfrac{1}{2}\operatorname{ad}(X)^\top + \tfrac{1}{2}\alpha(X) - \tfrac{1}{2}\operatorname{ad}(X). \tag{3}$$

3.3. The curvature. First note that we have the following relations:

$$\begin{aligned} &[R_X, \operatorname{ad}(Y)] = \operatorname{ad}(R_X(Y)), \qquad [R_X, \alpha(Y)] = \alpha(R_X(Y)),\\ &[R_X, \operatorname{ad}(Y)^\top] = \operatorname{ad}(R_X(Y))^\top, \quad [\operatorname{ad}(X)^\top, \operatorname{ad}(Y)^\top] = -\operatorname{ad}([X,Y]_\mathfrak{g})^\top. \end{aligned} \tag{1}$$

The Riemannian curvature is then computed by

(2)
$$\begin{aligned} \mathcal{R}(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)}\\ &= [R_X + \tfrac{1}{2}\operatorname{ad}(X)^\top + \tfrac{1}{2}\alpha(X) - \tfrac{1}{2}\operatorname{ad}(X), R_Y + \tfrac{1}{2}\operatorname{ad}(Y)^\top + \tfrac{1}{2}\alpha(Y) - \tfrac{1}{2}\operatorname{ad}(Y)]\\ &\quad - R_{-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)} - \tfrac{1}{2}\operatorname{ad}(-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X))^\top\\ &\quad - \tfrac{1}{2}\alpha(-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X)) + \tfrac{1}{2}\operatorname{ad}(-[X,Y]_\mathfrak{g} + R_X(Y) - R_Y(X))\\ &= -\tfrac{1}{4}[\operatorname{ad}(X)^\top + \operatorname{ad}(X), \operatorname{ad}(Y)^\top + \operatorname{ad}(Y)]\\ &\quad + \tfrac{1}{4}[\operatorname{ad}(X)^\top - \operatorname{ad}(X), \alpha(Y)] + \tfrac{1}{4}[\alpha(X), \operatorname{ad}(Y)^\top - \operatorname{ad}(Y)]\\ &\quad + \tfrac{1}{4}[\alpha(X), \alpha(Y)] + \tfrac{1}{2}\alpha([X,Y]_\mathfrak{g}). \end{aligned}$$

If we plug in all definitions and use 4 times the Jacobi identity we get the following expression

$$\begin{aligned} \langle 4\mathcal{R}(X,Y)Z, U\rangle &= +2\langle [X,Y],[Z,U]\rangle - \langle [Y,Z],[X,U]\rangle + \langle [X,Z],[Y,U]\rangle\\ &\quad - \langle Z, [U,[X,Y]]\rangle + \langle U, [Z,[X,Y]]\rangle - \langle Y, [X,[U,Z]]\rangle - \langle X, [Y,[Z,U]]\rangle\\ &\quad + \langle \operatorname{ad}(X)^\top Z, \operatorname{ad}(Y)^\top U\rangle + \langle \operatorname{ad}(X)^\top Z, \operatorname{ad}(U)^\top Y\rangle + \langle \operatorname{ad}(Z)^\top X, \operatorname{ad}(Y)^\top U\rangle\\ &\quad - \langle \operatorname{ad}(U)^\top X, \operatorname{ad}(Y)^\top Z\rangle - \langle \operatorname{ad}(Y)^\top Z, \operatorname{ad}(X)^\top U\rangle - \langle \operatorname{ad}(Z)^\top Y, \operatorname{ad}(X)^\top U\rangle\\ &\quad - \langle \operatorname{ad}(U)^\top X, \operatorname{ad}(Z)^\top Y\rangle + \langle \operatorname{ad}(U)^\top Y, \operatorname{ad}(Z)^\top X\rangle. \end{aligned}$$

3.4. Jacobi fields, I. We compute first the Jacobi equation directly via variations of geodesics. So let $g : \mathbb{R}^2 \to G$ be smooth, $t \mapsto g(t,s)$ a geodesic for each s. Let again $u = \kappa(\partial_t g) = (g^*\kappa)(\partial_t)$ be the velocity field along the geodesic in right trivialization which satisfies the geodesic equation $u_t = -\operatorname{ad}(u)^\top u$. Then $y := \kappa(\partial_s g) = (g^*\kappa)(\partial_s)$ is the Jacobi field corresponding to this variation, written in the right trivialization. From the right Maurer-Cartan equation we then have:

$$\begin{aligned} y_t &= \partial_t (g^*\kappa)(\partial_s) = d(g^*\kappa)(\partial_t,\partial_s) + \partial_s(g^*\kappa)(\partial_t) + 0 \\ &= [(g^*\kappa)(\partial_t),(g^*\kappa)(\partial_s)]_{\mathfrak{g}} + u_s \\ &= [u,y] + u_s. \end{aligned}$$

Using the geodesic equation, the definition of α, and the fourth relation in 3.3.(1), this identity implies

$$\begin{aligned} u_{st} = u_{ts} = \partial_s u_t &= -\partial_s(\operatorname{ad}(u)^\top u) = -\operatorname{ad}(u_s)^\top u - \operatorname{ad}(u)^\top u_s \\ &= -\operatorname{ad}(y_t + [y,u])^\top u - \operatorname{ad}(u)^\top(y_t + [y,u]) \\ &= -\alpha(u) y_t - \operatorname{ad}([y,u])^\top u - \operatorname{ad}(u)^\top y_t - \operatorname{ad}(u)^\top([y,u]) \\ &= -\operatorname{ad}(u)^\top y_t - \alpha(u)y_t + [\operatorname{ad}(y)^\top, \operatorname{ad}(u)^\top]u - \operatorname{ad}(u)^\top \operatorname{ad}(y)u\,. \end{aligned}$$

Finally we get the Jacobi equation as

$$\begin{aligned} y_{tt} &= [u_t, y] + [u, y_t] + u_{st} \\ &= \operatorname{ad}(y)\operatorname{ad}(u)^\top u + \operatorname{ad}(u) y_t - \operatorname{ad}(u)^\top y_t \\ &\qquad - \alpha(u) y_t + [\operatorname{ad}(y)^\top, \operatorname{ad}(u)^\top]u - \operatorname{ad}(u)^\top \operatorname{ad}(y) u\,, \end{aligned}$$

$$y_{tt} = [\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)^\top]u - \operatorname{ad}(u)^\top y_t - \alpha(u) y_t + \operatorname{ad}(u) y_t\,. \tag{1}$$

3.5. Jacobi fields, II. Let y be a Jacobi field along a geodesic g with right trivialized velocity field u. Then y should satisfy the analogue of the finite dimensional Jacobi equation

$$\nabla_{\partial_t}\nabla_{\partial_t} y + \mathcal{R}(y,u)u = 0$$

We want to show that this leads to same equation as 3.4.(1). First note that from 3.2.(2) we have

$$\nabla_{\partial_t} y = y_t + \tfrac12 \operatorname{ad}(u)^\top y + \tfrac12 \alpha(u) y - \tfrac12 \operatorname{ad}(u) y$$

so that, using $u_t = -\operatorname{ad}(u)^\top u$, we get:

$$\begin{aligned} \nabla_{\partial_t}\nabla_{\partial_t} y &= \nabla_{\partial_t}\Big(y_t + \tfrac12 \operatorname{ad}(u)^\top y + \tfrac12\alpha(u)y - \tfrac12 \operatorname{ad}(u)y\Big) \\ &= y_{tt} + \tfrac12 \operatorname{ad}(u_t)^\top y + \tfrac12 \operatorname{ad}(u)^\top y_t + \tfrac12 \alpha(u_t) y \\ &\quad + \tfrac12 \alpha(u) y_t - \tfrac12 \operatorname{ad}(u_t) y - \tfrac12 \operatorname{ad}(u) y_t \\ &\quad + \tfrac12 \operatorname{ad}(u)^\top\Big(y_t + \tfrac12 \operatorname{ad}(u)^\top y + \tfrac12\alpha(u)y - \tfrac12 \operatorname{ad}(u)y\Big) \\ &\quad + \tfrac12 \alpha(u)\Big(y_t + \tfrac12 \operatorname{ad}(u)^\top y + \tfrac12\alpha(u)y - \tfrac12 \operatorname{ad}(u)y\Big) \\ &\quad - \tfrac12 \operatorname{ad}(u)\Big(y_t + \tfrac12 \operatorname{ad}(u)^\top y + \tfrac12\alpha(u)y - \tfrac12 \operatorname{ad}(u)y\Big) \end{aligned}$$

$$
\begin{aligned}
&= y_{tt} + \operatorname{ad}(u)^\top y_t + \alpha(u) y_t - \operatorname{ad}(u) y_t \\
&\quad - \tfrac{1}{2}\alpha(y)\operatorname{ad}(u)^\top u - \tfrac{1}{2}\operatorname{ad}(y)^\top \operatorname{ad}(u)^\top u - \tfrac{1}{2}\operatorname{ad}(y)\operatorname{ad}(u)^\top u \\
&\quad + \tfrac{1}{2}\operatorname{ad}(u)^\top\Big(\tfrac{1}{2}\alpha(y)u + \tfrac{1}{2}\operatorname{ad}(y)^\top u + \tfrac{1}{2}\operatorname{ad}(y)u\Big) \\
&\quad + \tfrac{1}{2}\alpha(u)\Big(\tfrac{1}{2}\alpha(y)u + \tfrac{1}{2}\operatorname{ad}(y)^\top u + \tfrac{1}{2}\operatorname{ad}(y)u\Big) \\
&\quad - \tfrac{1}{2}\operatorname{ad}(u)\Big(\tfrac{1}{2}\alpha(y)u + \tfrac{1}{2}\operatorname{ad}(y)^\top u + \tfrac{1}{2}\operatorname{ad}(y)u\Big).
\end{aligned}
$$

In the second line of the last expression we use

$$
-\tfrac{1}{2}\alpha(y)\operatorname{ad}(u)^\top u = -\tfrac{1}{4}\alpha(y)\operatorname{ad}(u)^\top u - \tfrac{1}{4}\alpha(y)\alpha(u)u
$$

and similar forms for the other two terms to get:

$$
\begin{aligned}
\nabla_{\partial_t}\nabla_{\partial_t} y &= y_{tt} + \operatorname{ad}(u)^\top y_t + \alpha(u) y_t - \operatorname{ad}(u) y_t \\
&\quad + \tfrac{1}{4}[\operatorname{ad}(u)^\top, \alpha(y)]u + \tfrac{1}{4}[\operatorname{ad}(u)^\top, \operatorname{ad}(y)^\top]u + \tfrac{1}{4}[\operatorname{ad}(u)^\top, \operatorname{ad}(y)]u \\
&\quad + \tfrac{1}{4}[\alpha(u), \alpha(y)]u + \tfrac{1}{4}[\alpha(u), \operatorname{ad}(y)^\top]u + \tfrac{1}{4}[\alpha(u), \operatorname{ad}(y)]u \\
&\quad - \tfrac{1}{4}[\operatorname{ad}(u), \alpha(y)]u - \tfrac{1}{4}[\operatorname{ad}(u), \operatorname{ad}(y)^\top + \operatorname{ad}(y)]u,
\end{aligned}
$$

where in the last line we also used $\operatorname{ad}(u)u = 0$. We now compute the curvature term using 3.3.(2):

$$
\begin{aligned}
\mathcal{R}(y,u)u &= -\tfrac{1}{4}[\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)^\top + \operatorname{ad}(u)]u \\
&\quad + \tfrac{1}{4}[\operatorname{ad}(y)^\top - \operatorname{ad}(y), \alpha(u)]u + \tfrac{1}{4}[\alpha(y), \operatorname{ad}(u)^\top - \operatorname{ad}(u)]u \\
&\quad + \tfrac{1}{4}[\alpha(y), \alpha(u)] + \tfrac{1}{2}\alpha([y,u])u \\
&= -\tfrac{1}{4}[\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)^\top]u - \tfrac{1}{4}[\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)]u \\
&\quad + \tfrac{1}{4}[\operatorname{ad}(y)^\top, \alpha(u)]u - \tfrac{1}{4}[\operatorname{ad}(y), \alpha(u)]u + \tfrac{1}{4}[\alpha(y), \operatorname{ad}(u)^\top - \operatorname{ad}(u)]u \\
&\quad + \tfrac{1}{4}[\alpha(y), \alpha(u)]u + \tfrac{1}{2}\operatorname{ad}(u)^\top \operatorname{ad}(y)u.
\end{aligned}
$$

Summing up we get

$$
\begin{aligned}
\nabla_{\partial_t}\nabla_{\partial_t} y + \mathcal{R}(y,u)u &= y_{tt} + \operatorname{ad}(u)^\top y_t + \alpha(u) y_t - \operatorname{ad}(u) y_t \\
&\quad - \tfrac{1}{2}[\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)^\top]u \\
&\quad + \tfrac{1}{2}[\alpha(u), \operatorname{ad}(y)]u + \tfrac{1}{2}\operatorname{ad}(u)^\top \operatorname{ad}(y)u.
\end{aligned}
$$

Finally we need the following computation using 3.3.(1):

$$
\begin{aligned}
\tfrac{1}{2}[\alpha(u), \operatorname{ad}(y)]u &= \tfrac{1}{2}\alpha(u)[y,u] - \tfrac{1}{2}\operatorname{ad}(y)\alpha(u)u \\
&= \tfrac{1}{2}\operatorname{ad}([y,u])^\top u - \tfrac{1}{2}\operatorname{ad}(y)\operatorname{ad}(u)^\top u \\
&= -\tfrac{1}{2}[\operatorname{ad}(y)^\top, \operatorname{ad}(u)^\top]u - \tfrac{1}{2}\operatorname{ad}(y)\operatorname{ad}(u)^\top u.
\end{aligned}
$$

Inserting we get the desired result:

$$
\begin{aligned}
\nabla_{\partial_t}\nabla_{\partial_t} y + \mathcal{R}(y,u)u &= y_{tt} + \operatorname{ad}(u)^\top y_t + \alpha(u) y_t - \operatorname{ad}(u) y_t \\
&\quad - [\operatorname{ad}(y)^\top + \operatorname{ad}(y), \operatorname{ad}(u)^\top]u.
\end{aligned}
$$

3.6. The weak symplectic structure on the space of Jacobi fields. Let us assume now that the geodesic equation in $\mathfrak{g}$

$$u_t = -\operatorname{ad}(u)^\top u$$

admits a unique solution for some time interval, depending smoothly on the choice of the initial value $u(0)$. Furthermore we assume that G is a regular Lie group (see [13], 5.3) so that each smooth curve u in $\mathfrak{g}$ is the right logarithmic derivative of a smooth curve g in G which depends smoothly on u, so that $u = (g^*\kappa)(\partial_t)$. Furthermore we have to assume that the Jacobi equation along u admits a unique solution for some time, depending smoothly on the initial values $y(0)$ and $y_t(0)$. These are non-trivial assumptions: in [13], 2.4 there are examples of ordinary linear differential equations 'with constant coefficients' which violate existence or uniqueness. These assumptions have to be checked in the special situations. Then the space $\mathcal{J}_u$ of all Jacobi fields along the geodesic g described by u is isomorphic to the space $\mathfrak{g}\times\mathfrak{g}$ of all initial data.

There is the well known symplectic structure on the space $\mathcal{J}_u$ of all Jacobi fields along a fixed geodesic with velocity field u, see e.g. [11], II, p.70. It is given by the following expression which is constant in time t:

$$\begin{aligned}\sigma(y,z) :&= \langle y, \nabla_{\partial_t} z\rangle - \langle \nabla_{\partial_t} y, z\rangle \\ &= \langle y, z_t + \tfrac12 \operatorname{ad}(u)^\top z + \tfrac12 \alpha(u) z - \tfrac12 \operatorname{ad}(u) z\rangle \\ &\qquad - \langle y_t + \tfrac12 \operatorname{ad}(u)^\top y + \tfrac12\alpha(u) y - \tfrac12 \operatorname{ad}(u) y, z\rangle \\ &= \langle y, z_t\rangle - \langle y_t, z\rangle + \langle [u,y], z\rangle - \langle y, [u,z]\rangle - \langle [y,z], u\rangle \\ &= \langle y, z_t - \operatorname{ad}(u) z + \tfrac12\alpha(u) z\rangle - \langle y_t - \operatorname{ad}(u) y + \tfrac12 \alpha(u) y, z\rangle.\end{aligned}$$

It is worth while to check directly from the Jacobi field equation 3.4.(1) that $\sigma(y,z)$ is indeed constant in t. Clearly σ is a weak symplectic structure on the relevant vector space $\mathcal{J}_u \cong \mathfrak{g}\times\mathfrak{g}$, i.e., σ gives an injective (but in general not surjective) linear mapping $\mathcal{J}_u \to \mathcal{J}_u^*$. This is seen most easily by writing

$$\sigma(y,z) = \langle y, z_t - \Gamma_g(u,z)\rangle|_{t=0} - \langle y_t - \Gamma_g(u,y), z\rangle|_{t=0}$$

which is induced from the standard symplectic structure on $\mathfrak{g}\times\mathfrak{g}^*$ by applying first the automorphism $(a,b)\mapsto(a, b-\Gamma_g(u,a))$ to $\mathfrak{g}\times\mathfrak{g}$ and then by injecting the second factor $\mathfrak{g}$ into its dual $\mathfrak{g}^*$.

For regular (infinite dimensional) Lie groups variations of geodesics exist, but there is no general theorem stating that they are uniquely determined by $y(0)$ and $y_t(0)$. For concrete regular Lie groups, this needs to be shown directly.

4. The diffeomorphism group of the circle revisited

4.1. Geodesics and curvature. We consider again the Lie groups $\operatorname{Diff}(\mathbb{R})$ and $\operatorname{Diff}(S^1)$ with Lie algebras $\mathfrak{X}_c(\mathbb{R})$ and $\mathfrak{X}(S^1)$ where the Lie bracket $[X,Y] = X'Y - XY'$ is the negative of the usual one. For the inner product $\langle X,Y\rangle = \int X(x)Y(x)\,dx$ integration by parts gives

$$\langle [X,Y], Z\rangle = \int_{\mathbb{R}} (X'YZ - XY'Z)dx = \int_{\mathbb{R}} (2X'YZ + XYZ')dx = \langle Y, \operatorname{ad}(X)^\top Z\rangle,$$

which in turn gives rise to

$$\mathrm{ad}(X)^\top Z = 2X'Z + XZ', \tag{1}$$
$$\alpha(X)Z = 2Z'X + ZX', \tag{2}$$
$$(\mathrm{ad}(X)^\top + \mathrm{ad}(X))Z = 3X'Z, \tag{3}$$
$$(\mathrm{ad}(X)^\top - \mathrm{ad}(X))Z = X'Z + 2XZ' = \alpha(X)Z. \tag{4}$$

Equation (4) states that $-\frac{1}{2}\alpha(X)$ is the skew-symmetrization of $\mathrm{ad}(X)$ with respect to to the inner product $\langle\ ,\ \rangle$. From the theory of symmetric spaces one then expects that $-\frac{1}{2}\alpha$ is a Lie algebra homomorphism and indeed one can check that

$$-\tfrac{1}{2}\alpha([X,Y]) = \left[-\tfrac{1}{2}\alpha(X), -\tfrac{1}{2}\alpha(Y)\right]$$

holds for any vector fields X, Y. From (1) we get the same geodesic equation as in 2.3(4), namely Burgers' equation:

$$u_t = -\,\mathrm{ad}(u)^\top u = -3u_x u.$$

Using the above relations and the general curvature formula 3.3.(2), we get

$$\begin{aligned}\mathcal{R}(X,Y)Z &= -X''YZ + XY''Z - 2X'YZ' + 2XY'Z' = -2[X,Y]Z' - [X,Y]'Z \\ &= -\alpha([X,Y])Z. \end{aligned} \tag{5}$$

If we change the framing of the tangent bundle:

$$X = h\circ f^{-1}, \quad X' = \left(\frac{h_x}{f_x}\right)\circ f^{-1}, \quad X'' = \left(\frac{h_{xx}f_x - h_x f_{xx}}{f_x^3}\right)\circ f^{-1},$$

and similarly for $Y = k\circ f^{-1}$ and $Z = \ell\circ f^{-1}$, for $h,k,\ell \in C_c^\infty(\mathbb{R},\mathbb{R})$ or $C^\infty(S^1,\mathbb{R})$, then $(\mathcal{R}(X,Y)Z)\circ f$ given by (5) coincides with formula 2.3.(3) for the curvature.

4.2. Jacobi fields. A Jacobi field y along a geodesic g with velocity field u is a solution of the partial differential equation 3.4.(1), which in our case becomes:

$$\begin{aligned} y_{tt} &= [\mathrm{ad}(y)^\top + \mathrm{ad}(y), \mathrm{ad}(u)^\top]u - \mathrm{ad}(u)^\top y_t - \alpha(u)y_t + \mathrm{ad}(u)y_t \\ &= -3u^2 y_{xx} - 4u y_{tx} - 2u_x y_t \\ u_t &= -3u_x u. \end{aligned} \tag{1}$$

Since the geodesic equation has solutions, locally in time (see the argument in 2.3) it is to be expected that the space of all Jacobi fields exists and is isomorphic to the space of all initial data $(y(0), y_t(0)) \in C^\infty(S^1,\mathbb{R})^2$ or $C_c^\infty(\mathbb{R},\mathbb{R})^2$, respectively. The weak symplectic structure on it is given by 3.6:

$$\begin{aligned}\sigma(y,z) &= \langle y, z_t - \tfrac{1}{2}u_x z + 2uz_x\rangle - \langle y_t - \tfrac{1}{2}u_x y + 2uy_x, z\rangle \\ &= \int_{S^1 \text{or } \mathbb{R}} (yz_t - y_t z + 2u(yz_x - y_x z))\,dx. \end{aligned} \tag{2}$$

5. The Virasoro-Bott group and the Korteweg-de Vries-equation

5.1. Geodesics on the Virasoro-Bott group. For $\varphi \in \mathrm{Diff}^+(S^1)$ let $\varphi' : S^1 \to \mathbb{R}^+$ be the mapping given by $T_x\varphi \cdot \partial_x = \varphi'(x)\partial_x$. Then

$$c : \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1) \to \mathbb{R}$$
$$c(\varphi, \psi) := \int_{S^1} \log(\varphi \circ \psi)' d \log \psi' = \int_{S^1} \log(\varphi' \circ \psi) d \log \psi'$$

satisfies $c(\varphi, \varphi^{-1}) = 0$ and is a smooth group cocycle, called the Bott cocycle. The corresponding central extension group $S^1 \times_c \mathrm{Diff}^+(S^1)$, called the Virasoro-Bott group, is a trivial S^1-bundle $S^1 \times \mathrm{Diff}^+(S^1)$ that becomes a regular Lie group relative to the operations

$$\begin{pmatrix}\varphi \\ \alpha\end{pmatrix}\begin{pmatrix}\psi \\ \beta\end{pmatrix} = \begin{pmatrix}\varphi \circ \psi \\ \alpha\beta\, e^{2\pi i c(\varphi,\psi)}\end{pmatrix}, \quad \begin{pmatrix}\varphi \\ \alpha\end{pmatrix}^{-1} = \begin{pmatrix}\varphi^{-1} \\ \alpha^{-1}\end{pmatrix} \quad \varphi, \psi \in \mathrm{Diff}^+(S^1),\ \alpha, \beta \in S^1.$$

The Lie algebra of this Lie group is the central extension $\mathbb{R} \times_\omega \mathfrak{X}(S^1)$ of $\mathfrak{X}(S^1)$ induced by the Gelfand-Fuchs Lie algebra cocycle $\omega : \mathfrak{X}(S^1) \times \mathfrak{X}(S^1) \to \mathbb{R}$

$$\omega(h, k) = \omega(h)k = \int_{S^1} h' dk' = \int_{S^1} h'k'' dx = \tfrac{1}{2} \int_{S^1} \det \begin{pmatrix} h' & k' \\ h'' & k'' \end{pmatrix} dx,$$

a generator of the 1-dimensional bounded Chevalley cohomology $H^2(\mathfrak{X}(S^1), \mathbb{R})$. Thus the bracket on $\mathbb{R} \times_\omega \mathfrak{X}(S^1)$ is given by

$$\left[\begin{pmatrix}h \\ a\end{pmatrix}, \begin{pmatrix}k \\ b\end{pmatrix}\right] = \begin{pmatrix}h'k - hk' \\ \omega(h,k)\end{pmatrix}, \quad h, k \in \mathfrak{X}(S^1),\ a, b \in \mathbb{R}.$$

Note that the Lie algebra cocycle ω makes sense on the Lie algebra $\mathfrak{X}_c(\mathbb{R})$ of all vector fields with compact support on $\mathbb{R}$, but that it does not integrate to a group cocycle on $\mathrm{Diff}(\mathbb{R})$. The subsequent considerations also make sense on $\mathfrak{X}_c(\mathbb{R})$. Recall also that $H^2(\mathfrak{X}_c(M), \mathbb{R}) = 0$ for each finite dimensional manifold of dimension ≥ 2 (see [7]), which blocks the way to find a higher dimensional analog of the Korteweg – de Vries equation in a way similar to that sketched below.

We shall use the L^2-inner product on $\mathbb{R} \times_\omega \mathfrak{X}(S^1)$:

$$\left\langle \begin{pmatrix}h \\ a\end{pmatrix}, \begin{pmatrix}k \\ b\end{pmatrix} \right\rangle := \int_{S^1} hk\, dx + ab.$$

Integrating by parts we get

$$\begin{aligned}
\left\langle \mathrm{ad}\begin{pmatrix}h \\ a\end{pmatrix}\begin{pmatrix}k \\ b\end{pmatrix}, \begin{pmatrix}\ell \\ c\end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix}h'k - hk' \\ \omega(h,k)\end{pmatrix}, \begin{pmatrix}\ell \\ c\end{pmatrix} \right\rangle \\
&= \int_{S^1} (h'k\ell - hk'\ell + ch'k'')\, dx \\
&= \int_{S^1} (2h'\ell + h\ell' + ch''')k\, dx \\
&= \left\langle \begin{pmatrix}k \\ b\end{pmatrix}, \mathrm{ad}\begin{pmatrix}h \\ a\end{pmatrix}^\top \begin{pmatrix}\ell \\ c\end{pmatrix} \right\rangle, \quad \text{where}
\end{aligned}$$

$$\mathrm{ad}\begin{pmatrix}h \\ a\end{pmatrix}^\top \begin{pmatrix}\ell \\ c\end{pmatrix} = \begin{pmatrix}2h'\ell + h\ell' + ch''' \\ 0\end{pmatrix}.$$

Using matrix notation we get therefore (where $\partial := \partial_x$)

$$\begin{aligned}
\operatorname{ad}\begin{pmatrix} h \\ a \end{pmatrix} &= \begin{pmatrix} h' - h\partial & 0 \\ \omega(h) & 0 \end{pmatrix} \\
\operatorname{ad}\begin{pmatrix} h \\ a \end{pmatrix}^\top &= \begin{pmatrix} 2h' + h\partial & h''' \\ 0 & 0 \end{pmatrix} \\
\alpha\begin{pmatrix} h \\ a \end{pmatrix} = \operatorname{ad}\begin{pmatrix} \ \end{pmatrix}^\top \begin{pmatrix} h \\ a \end{pmatrix} &= \begin{pmatrix} h' + 2h\partial + a\partial^3 & 0 \\ 0 & 0 \end{pmatrix} \\
\operatorname{ad}\begin{pmatrix} h \\ a \end{pmatrix}^\top + \operatorname{ad}\begin{pmatrix} h \\ a \end{pmatrix} &= \begin{pmatrix} 3h' & h''' \\ \omega(h) & 0 \end{pmatrix} \\
\operatorname{ad}\begin{pmatrix} h \\ a \end{pmatrix}^\top - \operatorname{ad}\begin{pmatrix} h \\ a \end{pmatrix} &= \begin{pmatrix} h' + 2h\partial & h''' \\ -\omega(h) & 0 \end{pmatrix}.
\end{aligned}$$

Formula 3.1(2) gives the geodesic equation on the Virasoro-Bott group:

$$\begin{pmatrix} u_t \\ a_t \end{pmatrix} = -\operatorname{ad}\begin{pmatrix} u \\ a \end{pmatrix}^\top \begin{pmatrix} u \\ a \end{pmatrix} = \begin{pmatrix} -3u'u - au''' \\ 0 \end{pmatrix}.$$

Thus a is a constant in time and the geodesic equation is hence the periodic Korteweg-de Vries equation

$$u_t + 3u_x u + a u_{xxx} = 0.$$

Had we worked on $\mathfrak{X}_c(\mathbb{R})$ we would have obtained the usual Korteweg-de Vries equation. The derivation above is direct and does not use the Euler-Poincaré equations; for a derivation of the Korteweg-de Vries equation from this point of view see [15], section 13.8.

5.2. The curvature. The computation of the curvature at the identity element has been done independently by Misiolek [22]; our results of course agree. Here we proceed with a completely general computation that takes advantage of the formalism introduced so far. Inserting the matrices of differential- and integral operators $\operatorname{ad}\binom{h}{a}^\top$, $\alpha\binom{h}{a}$, and $\operatorname{ad}\binom{h}{a}$ etc. given above into formula 3.3(2) and recalling that the matrix is applied to vectors of the form $\binom{\ell}{c}$, where c is a constant, we see that $4\mathcal{R}\left(\binom{h_1}{a_1}, \binom{h_2}{a_2}\right)$ is the following 2×2-matrix whose entries are differential- and integral operators:

$$\begin{pmatrix}
\begin{array}{l} 4(h_1 h_2'' - h_1'' h_2) + 2(a_1 h_2^{(4)} - a_2 h_1^{(4)}) \\ +(8(h_1 h_2' - h_1' h_2) + 10(a_1 h_2''' - a_2 h_1'''))\partial \\ +18(a_1 h_2'' - a_2 h_1'')\partial^2 \\ +(12(a_1 h_2' - a_2 h_1') + 2\omega(h_1, h_2))\partial^3 \\ -h_1'''\omega(h_2) + h_2'''\omega(h_1) \end{array} & \begin{array}{l} 2(h_1''' h_2' - h_1' h_2''') \\ +2(h_1 h_2^{(4)} - h_1^{(4)} h_2) \\ +(a_1 h_2^{(6)} - a_2 h_1^{(6)}) \end{array} \\
\begin{array}{l} \omega(h_2)(4h_1' + 2h_1\partial + a_1\partial^3) \\ -\omega(h_1)(4h_2' + 2h_2\partial + a_2\partial^3) \end{array} & 0
\end{pmatrix}.$$

Therefore, $4\mathcal{R}\left(\binom{h_1}{a_1},\binom{h_2}{a_2}\right)\binom{h_3}{a_3}$ has the following expression

$$\begin{pmatrix} 4(h_1h_2''-h_1''h_2)h_3+2(a_1h_2^{(4)}-a_2h_1^{(4)})h_3 \\ +\left(8(h_1h_2'-h_1'h_2)+10(a_1h_2'''-a_2h_1''')\right)h_3' \\ +18(a_1h_2''-a_2h_1'')h_3''+12(a_1h_2'-a_2h_1')h_3''' \\ +2h_3'''\int_{S^1}h_1'h_2''dx-h_1'''\int_{S^1}h_2'h_3''dx+h_2'''\int_{S^1}h_1'h_3''dx \\ +2a_3(h_1'''h_2'-h_1'h_2''')+2a_3(h_1h_2^{(4)}-h_1^{(4)}h_2)+a_3(a_1h_2^{(6)}-a_2h_1^{(6)}) \\ \\ \int_{S^1}h_3'''(a_1h_2'''-a_2h_1''')dx \\ +\int_{S^1}2h_3'(h_1h_2'''-h_1'''h_2-2h_1'h_2''+2h_1''h_2')dx \end{pmatrix}$$

which coincides with formula (2.3) in Misiolek [22]. This in turn leads to the following expression for the sectional curvature $\left\langle 4\mathcal{R}\left(\binom{h_1}{a_1},\binom{h_2}{a_2}\right)\binom{h_1}{a_1},\binom{h_2}{a_2}\right\rangle =$

$$\begin{aligned}
&=\int_{S^1}\Big(4(h_1h_2''-h_1''h_2)h_1h_2+8(h_1h_2'-h_1'h_2)h_1'h_2\\
&\qquad+2(a_1h_2^{(4)}-a_2h_1^{(4)})h_1h_2+10(a_1h_2'''-a_2h_1''')h_1'h_2\\
&\qquad+18(a_1h_2''-a_2h_1'')h_1''h_2\\
&\qquad+12(a_1h_2'-a_2h_1')h_1'''h_2+2\omega(h_1,h_2)h_1'''h_2\\
&\qquad-h_1'''\omega(h_2,h_1)h_2+h_2'''\omega(h_1,h_1)h_2\\
&\qquad+2(h_1'''h_2'-h_1'h_2''')a_1h_2\\
&\qquad+2(h_1h_2^{(4)}-h_1^{(4)}h_2)a_1h_2\\
&\qquad+(a_1h_2^{(6)}-a_2h_1^{(6)})a_1h_2\\
&\qquad+(4h_1'h_1h_2'''+2h_1h_1'h_2'''+a_1h_1'''h_2'''\\
&\qquad\quad-4h_2'h_1h_1'''-2h_2h_1'h_1'''-a_2h_1'''h_1''')a_2\Big)\,dx\\
&=\int_{S^1}\Big(-4[h_1,h_2]^2+4(a_1h_2-a_2h_1)(h_1h_2^{(4)}-h_1'h_2'''+h_1'''h_2'-h_1^{(4)}h_2)\\
&\qquad-(h_2''')^2a_1^2+2h_1'''h_2'''a_1a_2-(h_1''')^2a_2^2\Big)\,dx\\
&\quad+3\omega(h_1,h_2)^2.
\end{aligned}$$

This formula shows that the sign of the sectional curvature is not constant. Indeed, choosing $h_1(x)=\sin x, h_2(x)=\cos x$ we get $-\pi(8+a_1^2+a_2^2-3\pi)$ which can be positive and negative by choosing the constants a_1,a_2 judiciously.

5.3. Jacobi fields. A Jacobi field $y=\binom{y}{b}$ along a geodesic with velocity field $\binom{u}{a}$ is a solution of the partial differential equation 3.4(1) which in our case looks as follows.

$$\begin{pmatrix} y_{tt} \\ b_{tt} \end{pmatrix} = \left[\operatorname{ad}\begin{pmatrix} y \\ b \end{pmatrix}^\top + \operatorname{ad}\begin{pmatrix} y \\ b \end{pmatrix}, \operatorname{ad}\begin{pmatrix} u \\ a \end{pmatrix}^\top \right] \begin{pmatrix} u \\ a \end{pmatrix}$$
$$- \operatorname{ad}\begin{pmatrix} u \\ a \end{pmatrix}^\top \begin{pmatrix} y_t \\ b_t \end{pmatrix} - \alpha\begin{pmatrix} u \\ a \end{pmatrix}\begin{pmatrix} y_t \\ b_t \end{pmatrix} + \operatorname{ad}\begin{pmatrix} u \\ a \end{pmatrix}\begin{pmatrix} y_t \\ b_t \end{pmatrix}$$
$$= \left[\begin{pmatrix} 3y_x & y_{xxx} \\ \omega(y) & 0 \end{pmatrix}, \begin{pmatrix} 2u_x + u\partial_x & u_{xxx} \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} u \\ a \end{pmatrix}$$
$$+ \begin{pmatrix} -2u_x - 4u\partial_x - a\partial_x^3 & -u_{xxx} \\ \omega(u) & 0 \end{pmatrix} \begin{pmatrix} y_t \\ b_t \end{pmatrix},$$

which leads to

$$\begin{aligned} y_{tt} &= -u(4y_{tx} + 3uy_{xx} + ay_{xxxx}) - u_x(2y_t + 2ay_{xxx}) \\ &\quad - u_{xxx}(b_t + \omega(y,u) - 3ay_x) - ay_{txxx}, \end{aligned} \tag{1}$$

$$b_{tt} = \omega(u, y_t) + \omega(y, 3u_x u) + \omega(y, au_{xxx}). \tag{2}$$

Equation (2) is equivalent to:

$$b_{tt} = \int_{S^1} (-y_{txxx}u + y_{xxx}(3u_x u + au_{xxx}))dx. \tag{2'}$$

Next, let us show that the integral term in equation (1) is constant:

$$b_t + \omega(y,u) = b_t + \int_{S^1} y_{xxx}u\, dx =: B_1. \tag{3}$$

Indeed its t-derivative along the geodesic for u (that is, u satisfies the Korteweg-de Vries equation) coincides with (2′):

$$b_{tt} + \int_{S^1} (y_{txxx}u + y_{xxx}u_t)\, dx = b_{tt} + \int_{S^1} (y_{txxx}u + y_{xxx}(-3u_x u - au_{xxx}))\, dx = 0.$$

Thus $b(t)$ can be explicitly solved from (3) as

$$b(t) = B_0 + B_1 t - \int_a^t \int_{S^1} y_{xxx}u\, dx\, dt. \tag{4}$$

The first component of the Jacobi equation on the Virasoro-Bott group is a genuine partial differential equation. Thus the Jacobi equations are given by the following system:

$$\begin{aligned} y_{tt} &= -u(4y_{tx} + 3uy_{xx} + ay_{xxxx}) - u_x(2y_t + 2ay_{xxx}) \\ &\quad - u_{xxx}(B_1 - 3ay_x) - ay_{txxx}, \\ u_t &= -3u_x u - au_{xxx}, \\ a &= \text{ constant}, \end{aligned} \tag{5}$$

where $u(t,x), y(t,x)$ are either smooth functions in $(t,x) \in I \times S^1$ or in $(t,x) \in I \times \mathbb{R}$, where I is an interval or $\mathbb{R}$, and where in the latter case u, y, y_t have compact support with respect to x.

Choosing $u = c \in \mathbb{R}$, a constant, these equations coincide with (3.1) in Misiolek [22] where it is shown by direct inspection that there are solutions of this equation which vanish at non-zero values of t, thereby concluding that there are conjugate points along geodesics emanating from the identity element of the Virasoro-Bott group on S^1.

5.4. The weak symplectic structure on the space of Jacobi fields on the Virasoro Lie algebra. Since the Korteweg - de Vries equation has local solutions depending smoothly on the initial conditions (and global solutions if $a \neq 0$), we expect that the space of all Jacobi fields exists and is isomorphic to the space of all initial data $(\mathbb{R} \times_\omega \mathfrak{X}(S^1)) \times (\mathbb{R} \times_\omega \mathfrak{X}(S^1))$. The weak symplectic structure is given in section 3.6:

$$\begin{aligned}
\sigma\left(\begin{pmatrix} y \\ b \end{pmatrix}, \begin{pmatrix} z \\ c \end{pmatrix}\right) &= \left\langle \begin{pmatrix} y \\ b \end{pmatrix}, \begin{pmatrix} z_t \\ c_t \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} y_t \\ b_t \end{pmatrix}, \begin{pmatrix} z \\ c \end{pmatrix} \right\rangle + \left\langle \left[\begin{pmatrix} u \\ a \end{pmatrix}, \begin{pmatrix} y \\ b \end{pmatrix}\right], \begin{pmatrix} z \\ c \end{pmatrix} \right\rangle \\
&\quad - \left\langle \begin{pmatrix} y \\ b \end{pmatrix}, \left[\begin{pmatrix} u \\ a \end{pmatrix}, \begin{pmatrix} z \\ b \end{pmatrix}\right] \right\rangle - \left\langle \left[\begin{pmatrix} y \\ b \end{pmatrix}, \begin{pmatrix} z \\ c \end{pmatrix}\right], \begin{pmatrix} u \\ a \end{pmatrix} \right\rangle \\
&= \int_{S^1 \text{or } \mathbb{R}} (yz_t - y_t z + 2u(yz_x - y_x z))\, dx \\
&\quad + b(c_t + \omega(z,u)) - c(b_t + \omega(y,u)) - a\omega(y,z) \\
&= \int_{S^1 \text{or } \mathbb{R}} (yz_t - y_t z + 2u(yz_x - y_x z))\, dx \\
&\quad + bC_1 - cB_1 - a \int_{S^1 \text{or } \mathbb{R}} y' z''\, dx,
\end{aligned} \tag{1}$$

where the constant C_1 relates to c as B_1 does to b, see 5.3.(3) and (4).

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Received June 9, 1997
and in final form December 9, 1997