Journal of Lie Theory
Volume **8** (1998) 433–440

Berezin Transforms and Group Representations

Takaaki Nomura

Communicated by D. Poguntke

Abstract. We study group-invariant Berezin transforms from a general standpoint.

Introduction

In this note we shall study group-invariant Berezin transforms from a general standpoint. To be more specific, let X be a locally compact space endowed with a Radon measure μ. Let $\mathfrak{H}$ be a closed subspace of $L^2(X, d\mu)$ consisting of continuous functions and we assume that $\mathfrak{H}$ has a reproducing kernel κ. Then the *Berezin symbol* $\sigma(A)$ of a bounded linear operator A on $\mathfrak{H}$ is the function on X given by $\sigma(A)(x) := (Ae_x \mid e_x)$, where $e_x := \kappa(\cdot, x)/\kappa(x,x)^{1/2} \in \mathfrak{H}$. The Toeplitz operator $\sigma^*(f)$ with symbol f (f being a bounded function on X) is the operator $\sigma^*(f)h = P(fh)$ ($h \in \mathfrak{H}$) on $\mathfrak{H}$, where P is the orthogonal projection operator $L^2(X, d\mu) \to \mathfrak{H}$. We note that the linear map σ^* is adjoint to σ in a suitable sense when both operators are considered on appropriate Hilbert spaces, and it is this case which is studied in this article. The *Berezin transform* B associated to $\mathfrak{H}$ is, by definition, the composite $B := \sigma \circ \sigma^*$. We are primarily interested in the situation where a locally compact group G acts on X and $\mathfrak{H}$ carries a unitary representation π of G arising from this action. We show in Theorem 4 within this framework that not only is the Berezin transform B a G-invariant operator but also B resides in the space of a subrepresentation of the tensor product representation $\pi \otimes \pi^\dagger$ ($\pi^\dagger$ being the conjugate of π) transferred to a subrepresentation of the quasi-regular representation of G on $L^2(X_0, \kappa(x,x)d\mu)$, where

$$X_0 := \{x \in X \;;\; \kappa(x,x) \neq 0\}$$

and the measure $\kappa(x,x)d\mu$ turns out to be G-invariant. The relationship between Berezin transforms and this type of tensor product representations was observed by [7], [6] and [4] in their individual cases and our theorem says that the relationship holds in quite a general context.

ISSN 0949–5932 / \$2.50

Our idea in this note is based on the observation made in [6] that when $\mathfrak{H}$ is the Fock or the Bergman space, the Berezin transform comes also from the diagonalization operator for the tensor product of the space of holomorphic functions and the space of their conjugates. The diagonalization operator that we consider in our general context is the operator M defined by

$$M(A)(x) = \frac{1}{\kappa(x,x)}\,\xi(x)\overline{\eta(x)} \qquad (x \in X_0)$$

for decomposable $A = \xi \otimes \eta \in \mathfrak{H} \otimes \mathfrak{H}^\dagger$, where $\mathfrak{H}^\dagger$ is the Hilbert space conjugate to $\mathfrak{H}$. Identifying $\mathfrak{H} \otimes \mathfrak{H}^\dagger$ canonically with the Hilbert space of the Hilbert-Schmidt operators on $\mathfrak{H}$, we show in Proposition 1 that M is a bounded operator $\mathfrak{H} \otimes \mathfrak{H}^\dagger \to L^2(X_0, \kappa(x,x)d\mu)$ with $\|M\| \leqq 1$. Since obtaining an expression for the adjoint operator M^* is a plain matter (Lemma 2) and since M coincides with σ, we thus get another way of showing that B is a bounded operator on $L^2(X_0, \kappa(x,x)d\mu)$ (cf. [2] and [8]). After clarifying the relations of κ, M and B to the group representation (Lemma 3, Theorem 4), the note is concluded with two examples. The first example deals with the well-known case of the Fock representation of the Heisenberg group. The second one concerns irreducible representations $(\pi, \mathfrak{H})$ of compact Lie groups K in spaces of polynomials. We will generalize Theorem 1.2 of [3] in a completely analogous way in Theorem 5: the Berezin transform related to $\mathfrak{H}$ restricts itself to the K-invariant functions as the one-dimensional orthogonal projection operator onto the constant functions. However, the proof given here is an application of Theorem 4 and much simpler.

This note is an outgrowth of a discussion with Professor Detlev Poguntke, to whom the present author is grateful for his interest in the previous works [3] and [4].

1. Diagonalization operator

Let X be a locally compact Hausdorff space. We assume throughout this note that X is second countable for simplicity, though this assumption is not absolutely necessary. Let μ be a Radon measure on X. Thus μ is a regular Borel measure such that any compact subset of X has a finite μ-measure. The inner product of the usual L^2-space $L^2(X, d\mu)$ will be denoted as $(\cdot\,|\,\cdot)$. Let $\mathfrak{H}$ be a closed subspace of $L^2(X, d\mu)$ consisting of continuous functions. Moreover we make the following assumption about $\mathfrak{H}$.

Assumption. $\mathfrak{H}$ has a continuous reproducing kernel κ. In other words, there exists a continuous function κ on $X \times X$ such that $\kappa(\cdot, x) \in \mathfrak{H}$ and $h(x) = (h\,|\,\kappa(\cdot, x))$ for all $h \in \mathfrak{H}$ and $x \in X$.

Let $\mathfrak{H}^\dagger$ be the Hilbert space conjugate-linearly isomorphic to $\mathfrak{H}$ such that the underlying real Hilbert space is identical with that of $\mathfrak{H}$. We form the algebraic tensor product space $\mathfrak{H} \otimes_{\mathrm{alg}} \mathfrak{H}^\dagger$ carrying the canonical inner product. The completion yields a Hilbert space $\mathfrak{H} \otimes \mathfrak{H}^\dagger$ called the Hilbert space tensor product of $\mathfrak{H}$ and $\mathfrak{H}^\dagger$. It is well-known that $\mathfrak{H} \otimes \mathfrak{H}^\dagger$ can be identified with the

Hilbert space $\mathbf{B}_2(\mathfrak{H})$ of the Hilbert-Schmidt operators on $\mathfrak{H}$, the inner product of $\mathbf{B}_2(\mathfrak{H})$ being $(A\,|\,B)_{\mathrm{HS}} := \mathrm{tr}\,(B^*A)$. Let

$$N := \{x \in X \; ; \; \kappa(x,x) = 0\}$$

and we put $X_0 := X \setminus N$. It is clear that X_0 is open, so that X_0 itself is locally compact and second countable.

Proposition 1. *Let*

$$M\left(\sum \xi_j \otimes \eta_j\right)(x) := \frac{1}{\kappa(x,x)} \sum \xi_j(x)\overline{\eta_j(x)}$$

for $\sum \xi_j \otimes \eta_j \in \mathfrak{H} \otimes_{\mathrm{alg}} \mathfrak{H}^\dagger$ *and* $x \in X_0$. *Then the right hand side is independent of the expression* $A = \sum \xi_j \otimes \eta_j$ *of* A, *and* M *extends to a bounded linear operator* $\mathfrak{H} \otimes \mathfrak{H}^\dagger \to L^2(X_0, \kappa(x,x)d\mu)$ *with* $\|M\| \leqq 1$.

Proof. The identification of $\mathfrak{H} \otimes \mathfrak{H}^\dagger$ with $\mathbf{B}_2(\mathfrak{H})$ is given by $\sum \xi_j \otimes \eta_j \equiv \sum (\cdot\,|\,\eta_j)\xi_j$. Thus if $A = \sum \xi_j \otimes \eta_j \in \mathfrak{H} \otimes_{\mathrm{alg}} \mathfrak{H}^\dagger$ is considered as a finite rank operator on $\mathfrak{H}$, then

$$M(A)(x) = \frac{1}{\kappa(x,x)} \sum (\xi_j\,|\,\kappa_x)(\kappa_x\,|\,\eta_j) = \frac{1}{\kappa(x,x)}\,(A\kappa_x\,|\,\kappa_x), \tag{1}$$

where we have put $\kappa_x(y) = \kappa(y,x)$ for simplicity. Hence $M(A)$ is independent of the expression $A = \sum \xi_j \otimes \eta_j$ of A.

Now we can assume that $\{\eta_j\}$ is orthonormal in the expression $A = \sum \xi_j \otimes \eta_j$ by applying the Gram-Schmidt process to $\{\eta_j\}$ if necessary. The Schwarz inequality shows

$$|M(A)(x)| \leqq \frac{1}{\kappa(x,x)} \left(\sum |\xi_j(x)|^2\right)^{1/2} \left(\sum |\eta_j(x)|^2\right)^{1/2}.$$

Since the Bessel inequality gives

$$\sum |\eta_j(x)|^2 = \sum |(\eta_j\,|\,\kappa_x)|^2 \leqq \|\kappa_x\|^2 = \kappa(x,x),$$

we then obtain

$$|M(A)(x)| \leqq \frac{1}{\kappa(x,x)^{1/2}} \left(\sum |\xi_j(x)|^2\right)^{1/2}.$$

Therefore we arrive at

$$\begin{aligned}\int_{X_0} |M(A)(x)|^2\,\kappa(x,x)\,d\mu(x) &\leqq \sum_j \int_X |\xi_j(x)|^2\,d\mu(x) = \sum_j \|\xi_j\|^2 \\ &= \Big\|\sum_j \xi_j \otimes \eta_j\Big\|^2 = \|A\|^2,\end{aligned}$$

so that the proof of Proposition 1 is completed. ∎

From now on we put

$$d\mu_0(x) = \kappa(x,x)\, d\mu(x)$$

for simplicity. We denote by $(\cdot\,|\,\cdot)_0$ the inner product of $L^2(X_0, d\mu_0)$ or of $L^2(X, d\mu_0)$ to avoid a possible confusion with that of $L^2(X, d\mu)$. Let $M^* : L^2(X_0, d\mu_0) \to \mathfrak{H} \otimes \mathfrak{H}^\dagger$ be the adjoint operator of the bounded linear operator $M : \mathfrak{H} \otimes \mathfrak{H}^\dagger \to L^2(X_0, d\mu_0)$. Let P be the orthogonal projection operator $L^2(X, d\mu) \to \mathfrak{H}$.

Lemma 2. *Under the identification of $\mathfrak{H} \otimes \mathfrak{H}^\dagger$ with $\mathbf{B}_2(\mathfrak{H})$, one has for $f \in L^2(X_0, d\mu_0) \cap L^\infty(X_0)$*

$$M^*(f)h = P(fh) \qquad (h \in \mathfrak{H}),$$

where f is extended to a function on X by setting $f(x) = 0$ for $x \in N$. Thus for such f, $M^*(f)$ is the Toeplitz operator on $\mathfrak{H}$ with symbol f.

Proof. By definition we have

$$(M(A)\,|\,f)_0 = (A\,|\,M^*(f))_{\mathrm{HS}} \qquad (A \in \mathfrak{H} \otimes \mathfrak{H}^\dagger,\ f \in L^2(X_0, d\mu_0)). \tag{2}$$

Take $h_1 \in \mathfrak{H}$, $h_2 \in \mathfrak{H}^\dagger$ and consider $A := h_1 \otimes h_2$. Then, if $f \in L^2(X_0, d\mu_0) \cap L^\infty(X_0)$, we have

$$(M(A)\,|\,f)_0 = \int_X h_1(x)\overline{h_2(x)}\,\overline{f(x)}\, d\mu = (h_1\,|\,fh_2) = (h_1\,|\,P(fh_2)). \tag{3}$$

The right hand side of (2) for $A = h_1 \otimes h_2$ is rewritten as

$$\begin{aligned} (A\,|\,M^*(f))_{\mathrm{HS}} &= \mathrm{tr}\left(M^*(f)^*[(\cdot\,|\,h_2)h_1]\right) \\ &= (M^*(f)^*h_1\,|\,h_2) = (h_1\,|\,M^*(f)h_2). \end{aligned} \tag{4}$$

From (2), (3) and (4), it follows that $M^*(f)h = P(fh)$ for any $h \in \mathfrak{H}$. ■

For every $x \in X$, let e_x be the element of $\mathfrak{H}$ defined by

$$e_x := \begin{cases} \dfrac{\kappa_x}{\|\kappa_x\|} & (x \in X_0), \\ 0 & (x \in N). \end{cases}$$

For any bounded linear operator A on $\mathfrak{H}$, the *Berezin symbol* $\sigma(A)$ is, by definition, the function on X given by

$$\sigma(A)(x) = (Ae_x\,|\,e_x) \qquad (x \in X).$$

The formula (1) says that $M(A)(x) = \sigma(A)(x)$ for all $x \in X_0$. Thus the *Berezin transform* $B := \sigma\sigma^*$ on $L^2(X_0, d\mu_0)$ associated to $\mathfrak{H}$ coincides with MM^*. It is clear that B is a positive selfadjoint operator and Proposition 1 implies $\|B\| \leqq 1$.

2. Group-invariant Berezin transform

We now consider the situation where a locally compact group G acts on X continuously and $\mathfrak{H}$ carries a unitary representation of G arising naturally from the G-action. Thus we have a continuous map $G \times X \ni (g, x) \mapsto gx \in X$. In order to specify the representations that we have in mind, we assume that there is a continuous function J on $G \times X$ with values in $\mathbb{C} \setminus \{0\}$ such that

$$\begin{cases} J(e, x) = 1 & (\text{for all } x \in X), \\ J(g_1 g_2, x) = J(g_1,\ g_2 x) J(g_2, x) & (g_1, g_2 \in G,\ x \in X), \end{cases} \tag{5}$$

where e denotes the unit element of G. We assume further that we have a unitary representation π of G on $L^2(X, d\mu)$ given by

$$\pi(g) f(x) = J(g^{-1}, x)^{-1} f(g^{-1}x) \qquad (g \in G,\ x \in X) \tag{6}$$

and that the closed subspace $\mathfrak{H}$ is $\pi(G)$-invariant. While (5) ensures the homomorphy of π, the unitarity condition $\|\pi(g)f\|^2 = \|f\|^2$ $(f \in L^2(X, d\mu))$ is obviously equivalent to

$$d\mu(gx) = |J(g, x)|^{-2} d\mu(x) \qquad (g \in G,\ x \in X). \tag{7}$$

Lemma 3. *One has*

$$\kappa(gx, gy) = J(g, x)\kappa(x, y)\overline{J(g, y)} \qquad (g \in G,\ x, y \in X).$$

In particular, the set X_0 as well as N is G-invariant, and the measure $d\mu_0$ is G-invariant.

Proof. By the reproducing property, we have for any $h \in \mathfrak{H}$

$$\begin{aligned} (h \,|\, \kappa_{gy}) = h(gy) &= J(g, y)\pi(g^{-1})h(y) \\ &= J(g, y)(\pi(g^{-1})h \,|\, \kappa_y) = J(g, y)(h \,|\, \pi(g)\kappa_y), \end{aligned}$$

where it should be noted that $\pi(g^{-1})h \in \mathfrak{H}$ for any $g \in G$ by our assumption. Hence it holds that $\kappa_{gy} = \overline{J(g, y)}\pi(g)\kappa_y$, from which the first part of the lemma follows immediately. Using (7) one obtains the invariance of $d\mu_0$. ∎

Lemma 3 implies that we have another unitary representation ρ of G on $L^2(X_0, d\mu_0)$ given by

$$\rho(g) f(x) = f(g^{-1}x).$$

Recall the diagonalization operator $M : \mathfrak{H} \otimes \mathfrak{H}^\dagger \to L^2(X_0, d\mu_0)$.

Theorem 4. (i) *The closed subspace* $\overline{\mathrm{Range}\,(M)}$ *is invariant under* $\rho(G)$ *and carries a representation of* G *equivalent to* $\pi \otimes \pi^\dagger|_{\mathrm{Ker}\,(M)^\perp}$.
(ii) B *is a* G*-invariant operator, that is,* $B\rho(g) = \rho(g)B$ *for all* $g \in G$. *Moreover,* $\overline{\mathrm{Range}\,(M)}$ *is invariant under* B *and one has* $\mathrm{Ker}\,(B) = \mathrm{Range}\,(M)^\perp$.

Proof. (i) We first show that M is an intertwining operator for the representations $\pi \otimes \pi^\dagger$ and ρ, that is,

$$M(\pi(g) \otimes \pi^\dagger(g)) = \rho(g)M \qquad (g \in G). \tag{8}$$

Indeed if $A = \sum \xi_j \otimes \eta_j \in \mathfrak{H} \otimes_{\mathrm{alg}} \mathfrak{H}^\dagger$, then

$$\begin{aligned} M(\pi(g) \otimes \pi^\dagger(g)A)(x) &= \frac{1}{\kappa(x,x)} \sum \pi(g)\xi_j(x)\,\overline{\pi(g)\eta_j(x)} \\ &= \frac{|J(g^{-1},x)|^{-2}}{\kappa(x,x)} \sum \xi_j(g^{-1}x)\overline{\eta_j(g^{-1}x)}. \end{aligned}$$

Since we have $\kappa(x,x)|J(g^{-1},x)|^2 = \kappa(g^{-1}x,\, g^{-1}x)$ by Lemma 3, the last term is equal to $M(A)(g^{-1}x) = \rho(g)M(A)(x)$, so that we get (8). In particular, $\mathrm{Range}\,(M)$ is $\rho(G)$-invariant. Let $M = W|M|$ be the polar decomposition of the bounded operator M, that is, $|M| := (M^*M)^{1/2}$ and W is a partial isometry with the initial space $\mathrm{Ker}\,(M)^\perp$ and the final space $\overline{\mathrm{Range}\,(M)}$ (see [5, VI.2.7] for example). Noting that $\mathrm{Ker}\,(M)^\perp$ is $(\pi \otimes \pi^\dagger)(G)$-invariant, we see that W gives rise to a unitary intertwining operator for $\pi \otimes \pi^\dagger\,|_{\mathrm{Ker}\,(M)^\perp}$ and $\rho\,|_{\overline{\mathrm{Range}\,(M)}}$.
(ii) Since both $\pi \otimes \pi^\dagger$ and ρ are unitary we also have

$$(\pi(g) \otimes \pi^\dagger(g))M^* = M^*\rho(g) \qquad (g \in G).$$

Hence we get $B\rho(g) = \rho(g)B$ for all $g \in G$. From $(Bf\,|\,f)_0 = \|M^*(f)\|^2_{\mathrm{HS}}$ ($f \in L^2(X_0, d\mu_0)$), we see that

$$\mathrm{Ker}\,(B) = \mathrm{Ker}\,(M^*) = \mathrm{Range}\,(M)^\perp.$$

Since it is evident that $\mathrm{Range}\,(M)$ is stable under B, the proof of Theorem 4 is now complete. ∎

3. Examples

(a) Let $X = \mathbb{C}^n$. Fixing $\lambda > 0$, we consider the normalized Gaussian measure $d\mu(z) := (\lambda/\pi)^n e^{-\lambda\|z\|^2} dm(z)$ on $\mathbb{C}^n$, where dm is the euclidean measure on $\mathbb{C}^n$. The Fock space $\mathfrak{F}$ of square μ-integrable entire functions on $\mathbb{C}^n$ is a closed subspace of $L^2(\mathbb{C}^n, d\mu)$ with the reproducing kernel κ given by $\kappa(z,w) = e^{\lambda z \cdot \overline{w}}$, where $z \cdot \overline{w}$ denotes the canonical Hermitian inner product of $\mathbb{C}^n$. Take the $(2n+1)$-dimensional Heisenberg group $G = \mathbb{C}^n \times \mathbb{R}$ with product

$$(z,t)(z',t') = (z+z',\ t+t' - \mathrm{Im}\, z \cdot \overline{z}') \qquad (z,z' \in \mathbb{C}^n,\ t,t' \in \mathbb{R}).$$

It is clear that G acts on $\mathbb{C}^n$ by $(z,t)\cdot w = z+w$. Put

$$J((z,t),w) := e^{i\lambda t}e^{\lambda w\cdot\overline{z}}e^{\lambda\|z\|^2/2} \qquad ((z,t)\in G,\ w\in\mathbb{C}^n).$$

Then it is easy to verify that J satisfies the conditions in (5) and (7). Thus we have a unitary representation π of G on $L^2(\mathbb{C}^n, d\mu)$ given by the formula (6). Moreover it is obvious that $\mathfrak{F}$ is invariant under $\pi(G)$. In this case, the space $\mathfrak{F}^\dagger$ can be identified with the space $\overline{\mathfrak{F}}$ of complex conjugates of the functions in $\mathfrak{F}$. Hence the Hilbert space tensor product $\mathfrak{F}\otimes\mathfrak{F}^\dagger \equiv \mathfrak{F}\otimes\overline{\mathfrak{F}}$ is regarded as the Hilbert space of continuous functions F on $\mathbb{C}^n\times\mathbb{C}^n$ such that $F(\cdot,w)\in\mathfrak{F}$ for any $w\in\mathbb{C}^n$ and $F(z,\cdot)\in\overline{\mathfrak{F}}$ for any $z\in\mathbb{C}^n$. Under this identification, the operator M in Proposition 1 is rewritten as $MF(z) = e^{-\lambda\|z\|^2}F(z,z)$. We remark that in this case we have $X_0 = X$ and M is injective with dense range (note that the G-invariant measure $d\mu_0$ equals $(\lambda/\pi)^n dm$). Finally it is well-known [2], [8] that $B = \exp(-(1/4\lambda)T)$, where T denotes the positive selfadjoint operator defined by the minus Laplacian $-\varDelta$ on $\mathbb{C}^n$ and the family $\{\exp(-tT)\}_{t>0}$ is the one-parameter semigroup of operators generated by T.

(b) Let K be a compact Lie group acting linearly on a finite-dimensional real vector space X. Equipping X with a K-invariant inner product, we consider the normalized Gaussian measure $d\mu(x) := \pi^{-n/2}e^{-\|x\|^2}dm(x)$, where $n = \dim X$. We have a unitary representation π of K on $L^2(X, d\mu)$ given by

$$\pi(k)f(x) := f(k^{-1}x).$$

It is evident that the space $\mathcal{P}(X)$ of polynomial functions on X is contained in $L^2(X,d\mu)$ and $\pi(K)$-invariant. Let $\mathfrak{H}\neq\{0\}$ be a (necessarily finite-dimensional) $\pi(K)$-irreducible subspace of $\mathcal{P}(X)$. Since $\mathfrak{H}$ is finite-dimensional, the space $\mathfrak{H}$, considered as a closed subspace of $L^2(X,d\mu)$, has a reproducing kernel κ. Let $p_1,\dots,p_d$ $(d := \dim\mathfrak{H})$ be an orthonormal basis of $\mathfrak{H}$. Then we have $\kappa(x,y) = \sum p_j(x)\overline{p_j(y)}$. Since the functions p_j are polynomials, it is clear from this expansion that the set N is μ-null. Thus we have the Berezin transform B on $L^2(X,d\mu_0)$ associated to $\mathfrak{H}$, where $d\mu_0(x) := \pi^{-n/2}\kappa(x,x)e^{-\|x\|^2}dm(x)$. We note that in this case κ is K-invariant:

$$\kappa(kx,ky) = \kappa(x,y) \qquad (k\in K,\ x,y\in X).$$

Theorem 5. *B acts on the closed subspace $L^2(X,d\mu_0)^K$ of K-invariant functions as the orthogonal projection operator onto the one-dimensional subspace of constant functions. In particular, one has $\|B\| = 1$.*

Proof. Since $\mathfrak{H}$ is $\pi(K)$-irreducible, the multiplicity of the trivial representation of K in the tensor product $\mathfrak{H}\otimes\mathfrak{H}^\dagger$ is equal to one. Under the identification of $\mathfrak{H}\otimes\mathfrak{H}^\dagger$ with $\mathbf{B}(\mathfrak{H})$, the unique (up to scalar multiples) K-invariant in $\mathfrak{H}\otimes\mathfrak{H}^\dagger$ is the identity operator I. Now we see immediately from the formula (1) that $M(I)$ equals the constant function $\mathbf{1}$ on X with value 1. Thus Theorem 4 says that the restriction of B to $L^2(X,d\mu_0)^K$ resides in the one-dimensional subspace $\mathbb{C}\mathbf{1}$. Since it is evident by Lemma 2 that $M^*(\mathbf{1}) = I$, we have $B\mathbf{1} = \mathbf{1}$. This completes the proof of Theorem 5. ∎

References

[1] Aronszajn, N., *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[2] Berezin. F. A., *Quantization*, Math. USSR Izv. **8** (1974), 1109–1165.

[3] Fujita, E., and T. Nomura, *Spectral decompositions of Berezin transformations on* $\mathbb{C}^n$ *related to the natural* $\mathrm{U}(n)$*-action*, J. Math. Kyoto Univ. **36** (1996), 877–888.

[4] —, *Berezin transforms on the* 2×2 *matrix space related to the* $\mathrm{U}(2)\times\mathrm{U}(2)$*-action*, Integral Equations Operator Theory, to appear..

[5] Kato, T., "Perturbation theory for linear operators, 2nd ed.," Springer-Verlag Berlin, etc. 1976.

[6] Ørsted, B., and G. Zhang, *Weyl quantization and tensor products of Fock and Bergman spaces*, Indiana Univ. Math. J. **43** (1994), 551–583.

[7] Peetre, J., and G. Zhang, *A weighted Plancherel formula III. The case of the hyperbolic matrix ball*, Collect. Math. **43** (1992), 273–301.

[8] Unterberger, A., and H. Upmeier, *The Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (1994), 563–597.

Department of Mathematics
Faculty of Science,
Kyoto University
Kitashirakawa Oiwake-cho 606-8502,
Sakyo-ku, Kyoto, Japan
nomura@kusm.kyoto-u.ac.jp

Received February 20, 1998
and in final form April 24, 1998