Journal of Lie Theory
Volume **8** (1998) 163–172

Crofton formulae and geodesic distance in hyperbolic spaces

Guyan Robertson*

Communicated by A. Valette

Abstract. The geodesic distance between points in real hyperbolic space is a hypermetric, and hence is a kernel negative type. The proof given here uses an integral formula for geodesic distance, in terms of a measure on the space of hyperplanes. An analogous integral formula, involving the space of horospheres, is given for complex hyperbolic space. By contrast geodesic distance in a projective space is not of negative type.

Introduction

Motivated by problems in harmonic analysis on homogeneous spaces, J. Faraut and K. Harzallah showed in [7] that the geodesic distance d on a real or complex hyperbolic space of dimension $n \geq 2$ is a kernel of negative type. Equivalently, $\sqrt{d}$ is a Hilbert space distance [HV: Chapter 5]. On the other hand, in the real hyperbolic plane, there is an explicit Crofton integral formula for the length L of a rectifiable curve C in terms of a measure μ on the space of geodesics [18, Section 3], [9, Section 4.4]. The formula has the form $\int n(\gamma) d\mu(\gamma) = 2L$, where $n(\gamma)$ is the number of times the geodesic γ meets C and the integral is over the space of all geodesics. (M.W. Crofton proved the corresponding formula for the euclidean plane in 1868.)

This paper proves a Crofton formula for geodesic distances in real hyperbolic space of dimension n, from which one obtains an explicit geometric proof of the result of Faraut and Harzallah, in the real case. In this case the measure μ is defined on the space of totally geodesic submanifolds of codimension one. There is also a Crofton formula in the complex case. However, the measure is defined on the space of horospheres, and it is not clear how to deduce that d is of negative type.

Hyperbolic spaces $H^n_{\mathbb{F}}$ can be defined over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and (in the case $n = 2$) the octonions $\mathbb{O}$. In each case the space is two point homogeneous: if $d(x, y) = d(x', y')$ then there is an isometry g of $H^n_{\mathbb{F}}$ such that $gx = x'$, $gy = y'$ [11, Chapter IX, Proposition 5.1, p.355]. It turns out that this property, together

* This research was supported by the Australian Research Council.

ISSN 0949–5932 / \$2.50

with the existence of a certain invariant measure is at the heart of the Crofton formulae. We note that the noncompact riemannian manifolds which are two point homogeneous are exactly the euclidean spaces and hyperbolic spaces [11].

The Crofton formula that we prove for $H^n_{\mathbb{R}}$ has the consequence that the geodesic distance d is a *hypermetric* in the sense of [1]. It follows that any finite subset of $H^n_{\mathbb{R}}$, endowed with the metric $\sqrt{d}$, embeds isometrically in a euclidean sphere (Corollary 3.2).

Finally, we complete the investigation of Faraut and Harzallah [7] for symmetric spaces of rank one, by showing explicitly that the geodesic distance on a projective space is not of negative type, although it can be expressed by means of a Crofton formula.

1. Motivation and preliminaries

In the course of the proof of [17, Proposition 1.4], a geometric argument was given to show that the euclidean distance $d(x,y)$ between points $x, y \in \mathbb{R}^n$ is a kernel of negative type. In retrospect, the basis of that argument is seen to be the classical Crofton formula asserting that $d(x,y)$ equals the measure of the set of euclidean hyperplanes which meet the line segment $[xy]$. The relevant measure is the appropriately normalized measure on the space of hyperplanes which is invariant under isometries of $\mathbb{R}^n$. This measure lifts to a measure μ on the space of half spaces of $\mathbb{R}^n$. Then $d(x,y) = \mu(S_x \triangle S_y)$, where S_x denotes the set of half spaces which contain x. It is natural to try to extend this argument to the rank one symmetric spaces considered by J. Faraut and K. Harzallah [7]. It turns out that the only such spaces for which the geodesic distance is a kernel of negative type are real or complex hyperbolic spaces, and euclidean spheres.

Let $\mathbb{F}$ be one of the (skew-) fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Regard $\mathbb{F}^{n+1}$ as a right vector space over $\mathbb{F}$ and define a hermitian form on $\mathbb{F}^{n+1}$ by means of the formula

$$\langle z, w\rangle = -\overline{z^0}w^0 + \overline{z^1}w^1 + \cdots + \overline{z^n}w^n.$$

The hyperbolic space $H^n_{\mathbb{F}}$ is the image in the projective space $P^n_{\mathbb{F}}$ of the set of negative vectors $\{z \in \mathbb{F}^{n+1} : \langle z, z\rangle < 0\}$.

It is convenient to use the same notation for a negative vector $x \in \mathbb{F}^{n+1}$ and its equivalence class in $P^n_{\mathbb{F}}$. The geodesic distance between points x, y in X is given by $\cosh d(x,y) = \frac{|\langle x,y\rangle|}{(\langle x,x\rangle\langle y,y\rangle)^{\frac{1}{2}}}$.

The octionic hyperbolic plane $H^2_{\mathbb{O}}$ requires a more involved definition [15].

2. Real hyperbolic space

Denote by X the hyperbolic space $H^n_{\mathbb{R}}$ [CG]. Then X is the image in the projective space $\mathbb{RP}^n$ of the set of negative vectors

$$\left\{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = -(x^0)^2 + (x^1)^2 + \ldots (x^n)^2 < 0\right\}.$$

The orthogonal group $G = O(1,n)$ is the subgroup of $GL(n+1,\mathbb{R})$ which preserves the form $\langle\cdot,\cdot\rangle$, and G acts isometrically and transitively on X. The stabilizer in G of the point $x_0 = (1,0,\dots,0) \in X$ is the compact group $K = O(1)\times O(n)$. Thus X is isomorphic to the topological homogeneous space G/K. The hyperbolic space $H_{\mathbb{R}}^{n-1}$ embeds naturally into $X = H_{\mathbb{R}}^{n}$ as the subspace S_0 consisting of all points with last coordinate equal to zero.

Every totally geodesic submanifold of codimension one in X is a G–translate of S_0 [5, Proposition 2.5.1]. It is convenient simply to refer to such a submanifold as a hyperplane. The space $\mathfrak{S}$ of all hyperplanes may therefore be identified with the topological homogeneous space $G/G(S_0)$, where $G(S_0)$ is the subgroup of G consisting of elements which leave S_0 globally invariant. By [5, Lemma 4.2.1], we have $G(S_0) \cong O(1,n-1)\times O(1)$. The groups G and $G(S_0)$ are both unimodular [11, Chapter X, Proposition 1.4]. (A direct proof is given in [4, Proposition C.4.11].) It follows [16, Chapter 3, p.140, Corollary 4] that there is a nonzero positive G-invariant measure $\mu_{\mathfrak{S}}$ on the space $\mathfrak{S}$ of hyperplanes.

We also consider the space $\mathfrak{H}$ of half spaces in X. These are the G–translates of the half–space H_0 consisting of points with last coordinate positive. The group $G(H_0) \cong O(1,n-1)$ is unimodular and there is a corresponding invariant measure $\mu_{\mathfrak{H}}$ on $\mathfrak{H} = G/G(H_0)$.

There is a natural double covering $\pi : \mathfrak{H} \to \mathfrak{S}$ and so by uniqueness of the measures (up to a positive multiple) [16, Chapter 2, p.95, Corollary] we may assume that $\mu_{\mathfrak{S}} = \pi^* \circ \mu_{\mathfrak{H}}$.

Let $[xy]$ denote the (unique) geodesic between points $x, y \in X$. We first prove the following Crofton formula.

Proposition 2.1. *There is a constant $k > 0$ such that, if $x, y \in X$ then*

$$\mu_{\mathfrak{S}}\{S \in \mathfrak{S} : S \cap [xy] \neq \emptyset\} = kd(x,y).$$

For the purposes of the proof, we introduce the notation

$$m(x,y) = \mu_{\mathfrak{S}}\{S \in \mathfrak{S} : S \cap [xy] \neq \emptyset\}.$$

Lemma 2.2. *If $x, y \in X$ then*

(a) $\{S \in \mathfrak{S} : S \cap [xy] \neq \emptyset\}$ *is compact;*

(b) $m(x,y) < \infty$.

PROOF: (a) Let $g, h \in G$. We claim that the point gx_0 lies on the hyperplane hS_0 if and only if $gK \cap hG(S_0) \neq \emptyset$. The crucial fact used in the proof is that $G(S_0)$ acts transitively on S_0. Therefore

$$\begin{aligned} gx_0 \in hS_0 &\iff h^{-1}gx_0 \in S_0 \\ &\iff h^{-1}gx_0 = g_0x_0, \text{ for some } g_0 \in G(S_0) \\ &\iff g_0^{-1}h^{-1}g \in K, \text{ for some } g_0 \in G(S_0) \\ &\iff h^{-1}g = g_0k_0, \text{ for some } g_0 \in G(S_0), k_0 \in K \\ &\iff gK \cap hG(S_0) \neq \emptyset. \end{aligned}$$

This proves our assertion. Furthermore, since $[xy]$ is compact, there exists a compact subset $J \subset G$ such that $Jx_0 = [xy]$, by [16, p.137, Lemma 1]. A hyperplane hS_0 meets $[xy]$ if and only if $JK \cap hG(S_0) \neq \emptyset$, that is $h \in JKG(S_0)$. The set of such hyperplanes hS_0 is therefore compact, being the image of the compact set JK under the quotient map $G \to G/G(S_0)$.

(b) This follows immediately from (a). ∎

Proof of Proposition 2.1. We must prove that $m(x,y) = kd(x,y)$ where $k > 0$ is constant. This is based on the following facts: (a) X is two-point homogeneous with respect to the action of G; (b) the measure $\mu_{\mathfrak{S}}$ is G-invariant. It follows that $m(x,y) = m(x',y')$ whenever $d(x,y) = d(x',y')$. Moreover, by considering a large number of pairwise disjoint geodesic segments of equal length in some fixed geodesic segment $[ab]$, we see that $m(x,y) \to 0$ as $d(x,y) \to 0$. In particular $m(x,x) = 0$. If we divide $[xy]$ into s segments $[x_i x_{i+1}]$ of equal length $(1 \leq i \leq s)$, then $m(x,y) = sm(x_1,x_2)$. In this way, if $d(x',y') = qd(x,y)$ where q is rational, then $m(x',y') = qm(x,y)$. Now $m(x,y) < \infty$, by Lemma 2, and so by continuity there is a constant $k \geq 0$ such that $m(x,y) = kd(x,y)$ for all $x, y \in X$. We must check that $k > 0$. For this, it is enough to show that $m(x,y) > 0$ for some x, y. Take $x = (1,0,0.\dots,\epsilon)$, $y = (1,0,0.\dots,-\epsilon)$. Then the hyperplane S_0 meets $[xy]$ transversally at the interior point $x_0 = (1,0,\dots,0)$ of the geodesic segment $[xy]$. There is an open neighbourhood $\widetilde{V}$ of the identity in G such that gS_0 meets $[xy]$ at an interior point for all $g \in \widetilde{V}$. The image V of $\widetilde{V}$ in $G/G(S_0)$ is an open set which is contained in the set of all hyperplanes meeting $[xy]$. Since V has positive measure, this implies that $m(x,y) > 0$, as required. ∎

Remark 2.3. Unlike the case $n = 2$ in [18, Section 3], [9, Section 4.4], the above proof of the Crofton formula in hyperbolic space is not constructive. However it gives a geometric explanation of why the result is true.

We now re-interpret Proposition 1, replacing $\mu_{\mathfrak{S}}$ by $\mu_{\mathfrak{H}}$, the invariant measure on the space of all half-spaces. This will allow us to use the method of [17] to show that the metric on X is of negative type. Given $x \in X$, let Σ_x denote the set of half-spaces containing X.

Lemma 2.4. *If $x, y \in X$ then*

$$\mu_{\mathfrak{H}}(\Sigma_x \bigtriangleup \Sigma_y) = \mu_{\mathfrak{S}} \{S \in \mathfrak{S} : S \cap [xy] \neq \emptyset\}.$$

Proof: The double covering $\pi : \mathfrak{H} \to \mathfrak{S}$ is given by $\pi(H) = \partial H$, the boundary of H. Since a hyperplane S is totally geodesic we have either $\#([xy] \cap S) \leq 1$ or $[xy] \subseteq S$. Therefore

$$\Sigma_x \bigtriangleup \Sigma_y = \pi^{-1} \{S \in \mathfrak{S} : S \cap [xy] \neq \emptyset \text{ and } [xy] \nsubseteq S\}.$$

Now $\mu_{\mathfrak{S}} \{S \in \mathfrak{S} : [xy] \subseteq S\} = 0$. The result follows, since $\mu_{\mathfrak{S}} = \pi^* \circ \mu_{\mathfrak{H}}$. ∎

Corollary 2.5. *If $x, y \in X$ then $d(x,y) = k\mu_{\mathfrak{H}}(\Sigma_x \triangle \Sigma_y)$ where $k > 0$ is constant. Hence d is a kernel of negative type.*

PROOF: The first assertion is immediate from Lemma 2.4. The second assertion follows by embedding X into $L^2(\mathfrak{H}, \mu_{\mathfrak{H}})$ via $x \mapsto v_x$ where $v_x = \chi_x - \chi_{x_0}$, and χ_x is the characteristic function of Σ_x. Then $\mu_{\mathfrak{H}}(\Sigma_x \triangle \Sigma_y) = \|v_x - v_y\|_2^2$ and so $\sqrt{d}$ is a Hilbert space distance. (c.f. [17, Proposition 1.1].) ∎

Remark 2.6. Faraut and Harzallah show [7] that the distance function on real or complex hyperbolic space is a kernel of negative type. See [10, Chapitre 6, Théorème 21]. The corresponding result for quaternionic Hilbert space is false, because the group of isometries $Sp(1,n)$ has Kazhdan's property (T) [7, Théorème 6.4]. It would be interesting to have a direct proof of this fact, avoiding the use of property (T).

Remark 2.7. Corollary 2.5 is stronger than the result of Faraut and Harzallah. It asserts that the distance d is a measure definite kernel, in the sense of [17], a concept which is in general strictly stronger than that of negative type. See the next section.

Remark 2.8. Since $G(H_0)$ is a closed noncompact subgroup of G, it follows from Moore's ergodicity Theorem [20, Theorem 2.2.6] that any lattice subgroup Γ of G acts ergodically on $\mathfrak{H} = G/G(H_0)$. Note that the group Γ does not have property (T). Fix an element $x \in X$ then $\mu_{\mathfrak{H}}(\Sigma_x \triangle g\Sigma_x) = \mu_{\mathfrak{H}}(\Sigma_x \triangle \Sigma_{gx})$ is a positive constant multiple of $d(x, gx)$, and hence unbounded. The action of Γ in [17, Theorem 2.1] may therefore be chosen to be ergodic. Whether one can find an ergodic action with similar properties for all groups without property (T) is an open question.

Remark 2.9. Consider any Coxeter system (W, S), where W is a Coxeter group and S is the canonical generating set. Let Δ denote the associated Coxeter complex. By a result of J. Tits [3, Chap. IV, p.41], the natural distance $d(c, c')$ between chambers c and c' is equal to the number of walls of Δ separating c and c'. This is precisely the Crofton formula for distance, relative to counting measure on the space of walls.

Let $\mathcal{H}$ denote the space of half spaces ("roots") of Δ with counting measure. If c is a chamber of Δ, let H_c be the set of roots containing c. Then $2d(c, c') = \#(H_c \triangle H_{c'})$. (See the proof of [10, Proposition 6.14].) The argument proceeds as before, showing that d is a kernel of negative type.

The same type of argument is applied in [2, Proof of Theorem 3] to a polygonal complex X, which is locally finite, simply connected and of type $(4,4)$ or $(6,3)$. Let $\mathfrak{B}$ denote the set consisting of all barycentres of faces of X or of edges of X which are not adjacent to faces. An unbounded negative definite kernel N of the above form is defined on $\mathfrak{B}$. The kernel N is equivalent to the geodesic distance between elements of $\mathfrak{B}$ and is invariant under automorphisms of X. This is used in [2] to show that a properly discontinuous group of automorphisms of X does not have property (T).

3. The hypermetric property for real hyperbolic space

A semimetric space (X,d) is said to be *hypermetric* if it satisfies the following property: for each finite subset $\{x_1,x_2\dots,x_m\}$ of X and integers $\{t_1,t_2\dots,t_m\}$ such that $\sum_{i=1}^m t_i = 1$, we have $\sum_{i,j=1}^m t_it_jd(x_i,x_j) \leq 0$. The corresponding statement, with $t_1,t_2\dots,t_m$ real numbers satisfying $\sum_{i=1}^m t_i = 0$, says that d is a kernel negative type. It is easy to see that if d has the hypermetric property then d is a kernel negative type [1, 1.2].

Suppose that $(\mathfrak{W},\mu)$ is a measure space and that a semimetric d on a set X is defined by the formula $d(x,y) = \mu(S_x \triangle S_y)$, where for each $x\in X$, S_x is a measurable set. It was proved in [14, Theorem 3.1] that d is then a hypermetric. The next result is therefore an immediate consequence of Corollary 2.5.

Corollary 3.1. *The geodesic distance d on $H^n_{\mathbb{R}}$ is a hypermetric.*

Finite hypermetric spaces have been characterized up to isometry in [1], and there is a detailed exposition of their properties in [6]. In view of the fact that the space $(H^n_{\mathbb{R}},d)$ has constant negative curvature, the next result may seem slightly surprising.

Corollary 3.2. *Let $\{x_1,x_2\dots,x_m\}$ be any finite subset of $H^n_{\mathbb{R}}$, endowed with the metric $\sqrt{d}$. Then $(\{x_1,x_2\dots,x_m\},\sqrt{d})$ embeds isometrically in a euclidean sphere in $\mathbb{R}^p$, where $p\geq \log_2 m$.*

Proof: Since d is of negative type (being hypermetric) it follows from [10, Proposition 5.14] that $(\{x_1,x_2\dots,x_m\},\sqrt{d})$ embeds isometrically in $\mathbb{R}^p$, for some p. The fact that the image is contained in a euclidean sphere is then a consequence of [1, Lemme 1.12]. The estimate $p\geq \log_2 m$ is provided by [1, Proposition 1.18]. ∎

4. Complex hyperbolic space

Complex hyperbolic space $H^n_{\mathbb{C}}$ is constructed in a manner similar to that of $H^n_{\mathbb{R}}$ [5]. The bihermitian form $\langle z,w\rangle = -\overline{z^0}w^0+\overline{z^1}w^1+\dots+\overline{z^n}w^n$ on $\mathbb{C}^n$ defines a set of negative vectors in $\mathbb{C}^{n+1}$, defined by the condition $\langle z,z\rangle <0$, which projects to the subspace $H^n_{\mathbb{C}}$ of $\mathbb{CP}^n$.

One might try to mimic the arguments of the preceding section in the complex case. The space $H^n_{\mathbb{C}}$ is two point homogeneous with respect to the isometry group $U(1,n)$. Natural analogues of hyperplanes are the equidistant hypersurfaces [8, Section 4], also known as spinal surfaces in [15]. These are, by definition, subspaces of the form $S=\{z\in H^n_{\mathbb{C}} : d(z,z_1)=d(z,z_2)\}$, where $z_1,z_2\in H^n_{\mathbb{C}}$. There is a natural invariant measure on the space of equidistant hypersurfaces, but we cannot prove an analogue of Lemma 2.2, because the group $G(S_0)$ of elements of $U(1,n)$ which leave a given equidistant hypersurface S_0 globally invariant does not act transitively on S_0 [15, Section 3.2]. An additional problem is that equidistant hypersurfaces are not totally geodesic and a geodesic segment can meet an equidistant hypersurface more than once, without being contained in the hypersurface. However, even in the complex case, it is possible to prove a Crofton formula if we consider horospheres instead of equidistant hypersurfaces.

Horospheres in complex hyperbolic space are described geometrically in [8, Section 1]. From a group-theoretic point of view they may be described as follows [12, Chapter II.1]. Let $G = U(1,n)$, so that $H^n_{\mathbb{C}} = G/K$ where $K = U(1) \times U(n)$. Let $G = KAN$ be the Iwasawa decomposition and M the centralizer of A in K. Thus MAN is a minimal parabolic subgroup. Let x_0 be the origin K in G/K and $\xi_0 = Nx_0$ Then ξ_0 is a horosphere and the subgroup of G which maps ξ_0 to itself equals MN. The homogeneous space G/MN is the space of horocycles. Note that N is isomorphic to the complex Heisenberg group [12, Chapter 2, p.215]. The groups G and MN are therefore unimodular [11, Chapter X, Proposition 1.4], and there is a G-invariant measure on G/MN. Since MN acts transitively on ξ_0, the set of horospheres which meet a geodesic segment $[xy]$ is compact, as in Lemma 2.2. Consider the expression $m(x,y) = \int_{G/MN} n(h)d\mu(h)$, where $n(h)$ is the number of times a horocycle $h \in G/MN$ meets the geodesic segment $[xy]$. Since $H^n_{\mathbb{C}}$ is two-point homogeneous, and μ is G-invariant, $m(x,y)$ depends only on $d(x,y)$. The same argument as in the proof of Proposition 2.1 establishes the following Crofton formula.

Proposition 4.1. *There is a constant $k > 0$ such that, for all $x, y \in H^n_{\mathbb{C}}$,*

$$\int_{G/MN} n[x,y](h)d\mu(h) = kd(x,y)$$

where $n[x,y](h)$ is the number of times that h meets $[xy]$.

Remark 4.2. Because it is possible to have $1 < n[x,y](h) < \infty$, it is not clear how to use this result to prove that $d(x,y)$ is of negative type, as we did for the real case in Corollary 2.5. In fact, the example in the next section indicates that a Crofton formula alone is not enough. Note that a Crofton formula involving horospheres is clearly also valid in real hyperbolic space.

5. Projective space

The purpose of this section is twofold. Firstly we give an example to show that the existence of a Crofton formula for a distance function does not in general imply that the distance is of negative type. Secondly, this example completes the classification of riemannian symmetric spaces of rank one for which the geodesic distance is of negative type.

The projective spaces $P^n_{\mathbb{F}}$ of dimension $n \geq 2$ over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and (in the case $n = 2$) the octonions $\mathbb{O}$, are compact two-point homogeneous spaces.

Proposition 5.1. *The geodesic distance d on $P^n_{\mathbb{F}}$ is not of negative type.*

The projective spaces all contain $P^2_{\mathbb{R}}$ as a geodesic subspace, and so it is enough to consider the case of $P^2_{\mathbb{R}}$. As usual, the same notation is used for a vector $x \in \mathbb{R}^3$ and its equivalence class in $P^2_{\mathbb{R}}$. The geodesic distance between points x, y in $P^2_{\mathbb{R}}$ is given by $\cos d(x,y) = \frac{|(x,y)|}{((x,x)(y,y))^{\frac{1}{2}}}$, where (x,y) is the usual inner product on $\mathbb{R}^3$.

Proposition 5.2. *The geodesic distance d on $P^2_{\mathbb{R}}$ is not of negative type.*

PROOF: We show that the metric d is not of negative type by exhibiting points $x_1, x_2 \dots, x_6 \in P^2_{\mathbb{R}}$ and real numbers $t_1, t_2 \dots, t_6$ satisfying $\sum_{i=1}^{6} t_i = 0$ such that $\sum_{i,j=1}^{6} t_i t_j d(x_i, x_j) > 0$. In fact we choose $(t_1, t_2 \dots, t_6) = (1, 1, 1, -1, -1, -1)$ and for notational convenience put $(x_1, x_2, x_3, x_4, x_5, x_6) = (p_1, p_2, p_3, q_1, q_2, q_3)$. It is enough to choose these points so that

$$\sum d(p_i, p_j) + \sum d(q_i, q_j) > \sum d(p_i, q_j). \tag{1}$$

We make the following choices. Let $p_1 = (1, 0, 1)$, $p_2 = (1, 0, -1)$, $p_3 = (0, 1, 0)$, and $q_1 = (0, 1, 1)$, $q_2 = (0, 1, -1)$, $q_3 = (1, 0, 0)$. Then $d(p_i, p_j) = \pi/2$, $d(q_i, q_j) = \pi/2$, for $i, j \in \{1, 2, 3\}$. Moreover

$$d(p_i, q_j) = \begin{cases} \pi/3 & \text{if both } i, j \in \{1, 2\}, \\ \pi/4 & \text{if exactly one of } i, j \text{ equals } 3, \\ \pi/2 & \text{if } i = 3 \text{ and } j = 3. \end{cases}$$

We therefore obtain $\sum d(p_i, p_j) + \sum d(q_i, q_j) = 3\pi/2 + 3\pi/2 = 3\pi$ and $\sum d(p_i, q_j) = 4\pi/3 + 4\pi/4 + \pi/2 = 17\pi/6$, which proves the inequality (1). ■

EXAMPLE. There is a Crofton formula for geodesic distance d on the space $P^n_{\mathbb{R}}$, but as we have just seen, this distance is not of negative type. The proof of a Crofton formula for $d(x, y)$ in terms of totally geodesic hypersurfaces which meet the segment $[xy]$ is exactly the same as for real hyperbolic space in Section 2., except that by compactness there is no need for an analogue of Lemma 2.2. The reason that the analogue of Corollary 2.5 fails is that a totally geodesic hypersurface does not separate $P^n_{\mathbb{R}}$ into two parts.

Remark 5.3. Two point homogeneous riemannian manifolds have been classified completely [11, Chapter IX §5]. They are the euclidean spaces, the circle, and the symmetric spaces of rank one. The symmetric spaces of rank one for which the geodesic distance is of negative type are the spheres and the real or complex hyperbolic spaces. This follows from the results of [7], together with our result for projective spaces. Note that it is easy to give a new proof that the geodesic distance on a sphere is of negative type along the lines of Corollary 2.5, using a measure on the space of half-spaces.

Remark 5.4. There is a naturally occurring invariant metric of negative type on the projective space $P^n_{\mathbb{F}}$ [13]. For there is an embedding of $P^n_{\mathbb{F}}$ as the set of primitive idempotents in the formally real Jordan algebra $\mathfrak{A}$ of hermitian $n \times n$ matrices over $\mathbb{F}$. The trace form defines a euclidean distance d_E on $\mathfrak{A}$ which induces a metric of negative type on $P^n_{\mathbb{F}}$. It follows from [13] that $P^n_{\mathbb{F}}$ is also two point homogeneous relative to this metric.

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Department of Mathematics
University of Newcastle
NSW 2308
AUSTRALIA
guyan@maths.newcastle.edu.au

Received June 18, 1997
and in final form July 27, 1997