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Discrete groups of affine isometries

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Abstract. Given a finite set S of isometries of an affine Euclidean space. We ask when the group Γ generated by S is discrete. This includes as a special case the question when the group generated by a finite set of rotations is finite. This latter question is answered in an appendix. In the main part of the paper the general case is reduced to this special case. The result is phrased as a series of tests: Γ is discrete iff S passes all the tests. The testing procedure is algorithmic.

Suppose we are given a finite set S of isometries of some affine Euclidean space. We ask when the group Γ generated by S is discrete. This includes as a special case the question when the group generated by a finite set of rotations is finite. We deal with this special case in an appendix. In the main part of the paper we reduce the general question to this special case. The main result is phrased as a series of tests. Γ is discrete iff S passes all the tests.

The method is based on Bieberbach's theorems characterizing discrete groups of affine isometries. Bieberbach's second theorem says that a group Γ of isometries of an affine Euclidean space $\mathbb{E}$ is discrete (if and) only if there is a Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ such that the restriction homomorphism r from Γ to the group of isometries of $\mathbb{F}$ has finite kernel and a crystallographic group as image.

This theorem suggests the following procedure to check if a given group Γ of affine isometries is discrete. First find a minimal Γ–invariant affine subspace $\mathbb{F}$. This is motivated by the fact that the Γ–invariant affine subspace in Bieberbach's theorem is minimal. Then test if Γ restricted to $\mathbb{F}$ is crystallographic. Finally, check if the restriction homomorphism $r : \Gamma \to \Gamma|\mathbb{F}$ has finite kernel.

Accordingly, the contents of chapters two to four are as follows. In chapter two we describe a minimal Γ–invariant affine subspace $\mathbb{F}$ for any group Γ of affine isometries. In chapter three we present a test for Γ to be crystallographic. Here we suppose that the group $\lambda(\Gamma)$ of linear parts of Γ is finite, a necessary condition by Bieberbach's first theorem. In chapter four we describe ways to check if $r : \Gamma \to \Gamma|\mathbb{F}$ has finite kernel. In chapter five we give examples.

The main result is phrased as a series of tests. The group Γ is discrete iff S passes all the tests. The testing procedure is algorithmic and hence should

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be capable of being computerized. We thus discuss the computational aspects as we go along.

This work was motivated by the following question asked by A. Dress in [6]. Suppose a finite set S of affine isometric involutions of an affine Euclidean space $\mathbb{E}$ is given. When is the group Γ generated by S discrete, respectively finite? Restricting the discussion to a set of generators of this type seems to have more disadvantages than advantages for the general theory.

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1. The Bieberbach theorems

In this chapter we establish notations to be in force for the whole paper and recall the Bieberbach theorems.

Notations 1.1. Let V be a finite dimensional real vector space endowed with a positive definite inner product $\langle\ ,\ \rangle$. Let $\mathbb{E}$ be the corresponding *affine Euclidean space.* So $\mathbb{E}$ has a simply transitive action of V, denoted $(v,x) \longmapsto x+v$, and a metric $d(x,x+v) = \langle v,v\rangle^{\frac{1}{2}}$. Let $G = \mathrm{Iso}(\mathbb{E})$ be the group of isometries of $\mathbb{E}$. Given $x_0 \in \mathbb{E}$, $t \in V$ and U in the orthogonal group O of $(V,\langle\ ,\ \rangle)$ we define an isometry

(1.2 by) $$g = A(x_0,t,U)$$

(1.3) $$g(x_0+v) = x_0+t+Uv\ .$$

Every isometry of $\mathbb{E}$ is of this form. We note for later reference the multiplication formula

(1.4) $$iA(x_0,t_0,U_0)\cdot A(x_0,t_1,U_1) = A(x_0,t_0+U_0t_1\ ,\ U_0U_1)\ ,$$

and the dependence on the base point chosen:

(1.5) $$A(x_0,t_0,U_0) = A(x_1,t_1,U_1)$$

iff $U_0 = U_1$ and $t_1 = t_0 + (U_0-1)(x_1-x_0)$, where, of course, $x_1 - x_0$ is the unique vector $v \in V$ such that $x_1 = x_0 + v$. So the translation t depends on the base point chosen (but cf. 2.1c)), whereas the linear part U does not. We thus have a homomorphism

(1.6) $$\lambda : G \to O$$

defined by

$$\lambda(A(x_0,t,U)) = U\ .$$

We call $\lambda(g)$ the *linear part* or the *differential* of $g \in G$.

The *translations* of $\mathbb{E}$ are the isometries in the kernel of λ. We identify V with the group of translations of $\mathbb{E}$.

For an affine subspace $\mathbb{F}$ of $\mathbb{E}$ we denote by $T\mathbb{F}$ the subspace of V of translations of $\mathbb{F}$, so $T\mathbb{F} = \{v \in V\,;\, x + v \in \mathbb{F} \text{ for every } x \in \mathbb{F}\}$. One can also think of $T\mathbb{F}$ as the tangent space of $\mathbb{F}$. A subgroup Γ of $G = \mathrm{Iso}(\mathbb{E})$ is called *crystallographic* if it is discrete and $\Gamma\backslash\mathbb{E}$ is compact. The following two theorems due to Bieberbach are basic.

Bieberbach's First Theorem. *For a crystallographic group Γ the group $\lambda(\Gamma)$ of linear parts of Γ is finite.*

It follows that $\Gamma \cap V = \ker(\lambda|\Gamma)$ is of finite index in Γ and hence $\Gamma \cap V$ is a discrete subgroup of V with $(\Gamma \cap V) \setminus V$ compact. So there is a basis of the vector space V over the reals which is a set of generators for $\Gamma \cap V$. With respect to this basis $\lambda(\gamma)$ has integer entries for every $\gamma \in \Gamma$.

Bieberbach's Second Theorem. *If Γ is a discrete subgroup of G there is a Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ such that the restriction homomorphism $r : \Gamma \to \mathrm{Iso}(\mathbb{F})$ has finite kernel and a crystallographic subgroup of $\mathrm{Iso}(\mathbb{F})$ as image.*

Note that $\mathbb{F}$ is a minimal Γ–invariant affine subspace of $\mathbb{E}$, i.e. a minimal element in the set of Γ–invariant affine subspaces, partially ordered by inclusion, since the translations in $r(\Gamma)$ span $T\mathbb{F}$ by Bieberbach's first theorem.

The following corollary of the Bieberbach theorems suggests our procedure to check if a given subgroup Γ of G is discrete.

Corollary. *A subgroup Γ of G is discrete iff there is a minimal Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ such that for the restriction homomorphism $r : \Gamma \to \mathrm{Iso}(\mathbb{F})$ the kernel is finite and the image is a crystallographic group on $\mathbb{F}$.*

Proof. Necessity follows from the Bieberbach theorems, as seen above. Conversely, under the conditions of the corollary Γ acts properly discontinuously on $\mathbb{F}$ hence is discrete. ∎

2. A minimal invariant subspace

Let Γ be a subgroup of G, not necessarily discrete. We shall describe explicitly a minimal Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$. It is known that any two such $\mathbb{F}$ are translates of each other by translations which are fixed by $\lambda(\Gamma)$ so that it suffices to answer our questions for one of them.

Of particular importance for our description of $\mathbb{F}$ is the concept of the axis $\mathbb{E}_g$ of an affine isometry g, that is the largest g–invariant affine subspace of $\mathbb{E}$ on which g acts by translations, see 2.1. To find $\mathbb{F}$ we shall use the following two facts:

(i) Any Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ intersects the axis $\mathbb{E}_g$ of every element g of Γ. (ii) For certain affine subspaces $\mathbb{E}_1, \mathbb{E}_2$ of $\mathbb{E}$ the set $\mathbb{E}_{1,2}$ of feet of common perpendiculars from $\mathbb{E}_2$ to $\mathbb{E}_1$ intersects $\mathbb{F}$ if both $\mathbb{E}_1$ and $\mathbb{E}_2$ do. Starting with (i) and using (ii) repeatedly we finally find a minimal Γ–invariant subspace.

We start by defining the notions of *axis* and *translational part*. Fix an element $g \in G$. If $\lambda(g)$ is the linear part of g, let $V^1 = \{v \in V \mid \lambda(g)v = v\}$ be the $+1$–eigenspace of $\lambda(g)$ in V and let $V^{\neq 1}$ be the orthogonal complement of V^1 in V.

2.1 Lemma and Definition

a) *For $g \in G$ there is a unique largest (not only maximal) g–invariant affine subspace $\mathbb{E}_g$ of $\mathbb{E}$ on which g acts as a parallel translation. We call $\mathbb{E}_g$ the* axis *of g.*

b) *The corresponding vector subspace $T\mathbb{E}_g$ is V^1. In particular, $\mathbb{E}_g$ is not empty.*

c) *The translation $\tau(g) := g|\mathbb{F}_g \in T\mathbb{E}_g = V^1$ is called the* translational part *of g.*

d) *If $g = A(x_0, t, U)$ and $t = t_1 + t_2$ is the decomposition of t according to $V = V^1 \oplus V^{\neq 1}$, then $\tau(g) = t_1$.*

e) *Every point $x \in \mathbb{E}$ can be written uniquely in the form $x = y + v$ with $y \in \mathbb{E}_g$ and $v \in V^{\neq 1}$. Then*

$$gx = x + \tau(g) + (\lambda(g) - 1)v$$

Proof. Write $g = A(x_0, t, U)$, so $U = \lambda(g)$. Decompose$t = t_1 + t_2$ according to $V = V^1 \oplus V^{\neq 1}$. Then, by (1.3), $g(x_0 + v) = x_0 + t_1 + t_2 + Uv = x_0 + v + t_1 + (t_2 + (U-1)v)$. There is a vector $v_0 \in V^{\neq 1}$ such that $(U-1)v_0 + t_2 = 0$, since U induces linear endomorphism of $V^{\neq 1}$ and $(U-1)|V^{\neq 1}$ has kernel zero. So $g(x_0 + v_0) = x_0 + v_0 + t_1$ and for $x_1 = x_0 + v_0$ we have $g = A(x_1, t_1, U)$ by (1.5). It follows that g induces a parallel translation by t_1 on $x_1 + V^1$. Every point $x \in \mathbb{E}$ can be written uniquely in the form $x = x_1 + v_1 + v_2$ where $v_1 \in V^1$, $v_2 \in V^{\neq 1}$. Then

$$gx = x + t_1 + (\lambda(g) - 1)v_2 \; .$$

Thus, if x is contained in a g–invariant affine subspace $\mathbb{F}$ such that g acts on $\mathbb{F}$ by a parallel translation, by $\tau \in T\mathbb{F}$ say, then $g^n x = x + n\tau$ and on the other hand $g^n x = x + nt_1 + (\lambda(g) - 1)^n v_2$ which implies $\tau = t_1$ by taking $\lim\limits_{n\to\infty} \frac{g^n x - x}{n}$ and hence $v_2 = 0$. All the claims are now proved. ■

Corollary 2.2. *An affine isometry g has a fixed point iff $\tau(g) = 0$.* ■

Here is our first handle on finding invariant subspaces.

Corollary 2.3. *For every nonempty g-invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ we have $\tau(g) \in T\mathbb{F}$ and $\mathbb{E}_g$ intersects $\mathbb{F}$.*

Proof. We have the following more precise assertion. For the restriction $g|\mathbb{F}$ of g to $\mathbb{F}$ the axis $\mathbb{F}_g$ is $\mathbb{F} \cap \mathbb{E}_g$ for the following reasons: $\mathbb{F}_g \supset \mathbb{F} \cap \mathbb{E}_g$, since g induces a parallel translation on $\mathbb{F} \cap \mathbb{E}_g$; and $\mathbb{F}_g \subset \mathbb{F} \cap \mathbb{E}_g$ since $\mathbb{F}_g$ is a g–invariant subspace of $\mathbb{F}$ on which g induces a parallel translation. ■

The next proposition allows to get further information about invariant subspaces. We need the following lemma, which we state for later reference in greater generality than presently needed.

Lemma 2.4. *Let V be a finite dimensional real Euclidean vector space and let U_1 and U_2 be two vector subspaces. Let P_i, $i = 1, 2$, be the orthogonal projections of V onto U_i. Let A be a finite dimensional subalgebra of End (V) over some subfield k of $\mathbb{R}$. Suppose A contains P_1 and P_2. Then A also contains the orthogonal projection onto $U_1 \cap U_2$. Furthermore, A contains three operators Q_1, Q_2 and Q_3 with the following properties:*

a) *$Q_1 + Q_2 + Q_3 = 1$ where 1 is the identity of V.*

b) *Q_3 is the orthogonal projection onto $(U_1 + U_2)^\perp$.*

c) *The image of Q_i is contained in U_i for $i = 1, 2$.* ■

For the proof let us first recall the following fact from linear algebra.

Lemma 2.5. *Let V be a vector space over a field k and let $T : V \to V$ be a linear map. Let $f \in k[t]$ be a polynomial such that $f(T) = 0$. Suppose $f = f_1 \cdot f_2$ with relatively prime polynomials $f_i \in k[t]$. Then V is the direct sum of the subspaces $V_1 = \ker f_1(T)$ and $V_2 = \ker f_2(T)$, and the projections p_i of V onto V_i are in $k[T]$.* ■

The proof consists in writing $1 = h_1 f_1 + h_2 f_2$ with $h_i \in k[t]$ and checking that the maps $p_i = (h_j f_j)(T)$, $j \neq i$, are the projections onto V_i. ■

Proof of Lemma 2.4. Put $W = U_1 \cap U_2$. Note that $P_i(W^\perp) \subset W^\perp$, since P_i is self adjoint and $P_i(W) \subset W$, in fact $P_i \mid W = 1_W$. Thus the operator $S := P_1 P_2$ maps $W^\perp$ to itself. Furthermore $S \mid W = 1_W$ and $\|S \mid W^\perp\| < 1$ for the following reason: Every orthogonal projection decreases the norm, i.e. $\|Px\| \leq \|x\|$ for every x, and $\|Px\| = \|x\|$ iff $Px = x$. It follows that $\|Sv\| < \|v\|$ for every $0 \neq v \in W^\perp$ since such a vector v is not in U_2 or if in U_2 not in U_1. Thus the norm of $S \mid W^\perp$ is strictly smaller than 1, since this operator norm is the maximum of $\|Sv\|$ where v runs through the compact unit sphere of $W^\perp$. In particular, all the eigenvalues of $S \mid W^\perp$ are of modulus strictly less than 1.

We now show that the orthogonal projection Q of V onto W is in A. The operator S fulfills a polynomial equation $f \in k[t]$, since A is finite dimensional over k. Now write $f = f_1 \cdot f_2$ with relatively prime polynomials $f_1, f_2 \in k[t]$ where f_1 is a power of $(t-1)$. Note that the case $f_1 = 1$ is not excluded. Then $\ker f_1(S) = W$, since $f_1(S)$ maps W to zero and induces an automorphism of $W^\perp$. And $\ker f_2(S) = W^\perp$ since $W \oplus f_2(S) = V$ and $f_2(S)$ maps both W and $W^\perp$ to itself and hence $\ker f_2(S)$ is the sum of its intersections with W and $W^\perp$. Note that if follows that the polynomial $g = (t-1) \cdot f_2$ has the property $g(S) = 0$.

To finish the proof we now apply these results to the two projections $1 - P_i$ onto $U_i^\perp$ and the operator $T = (1 - P_1)(1 - P_2)$. So T induces the identity on $U_1^\perp \cap U_2^\perp = (U_1 + U_2)^\perp$ and maps the orthogonal complement $U_1 + U_2$ to itself. There is a polynomial $g \in k[t]$ of the form $g = (t-1) \cdot f_2$ such that $f(T) = 0$ and f_2 and $t-1$ are relatively prime. Then $\ker(T-1) = (U_1+U_2)^\perp$ and $\ker f_2(T) = U_1 + U_2$, the corresponding projections are the orthogonal projections and belong to A. Let $Q_3 \in A$ be the orthogonal projection onto $(U_1 + U_2)^\perp$. By the proof of 2.5 there is an operator $Q_4 \in A$ such that $(T-1) \cdot Q_4 = 1 - Q_3$, namely $Q_4 = h_1(T)$ in the notations of that proof. Now put $Q_1 = -P_1(1-P_2)Q_4$ and $Q_2 = -P_2 Q_4$. ■

2.6. *The common perpendicular.* Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be two nonempty affine subspaces of $\mathbb{E}$. Let (x_1, x_2) be a pair of points with $x_i \in \mathbb{E}_i$ of minimal distance, so $d(x_1, x_2) \leq d(y_1, y_2)$ for any pair (y_1, y_2) of points with $y_i \in \mathbb{E}_i$. Equivalently, $x_1 - x_2$ is orthogonal to $V_1 + V_2$ where $V_i = T\mathbb{E}_i$. If x_1, x_2 is one such pair all other such pairs are of the form $(x_1 + v_1\,,\, x_2 + v_2)$ where $v_1 = v_2 \in V_1 \cap V_2$. So the difference vector $x_1 - x_2 \in V$ for any such pair depends only on the two subspaces $\mathbb{E}_1$, $\mathbb{E}_2$ and will be called the *common perpendicular* vector from $\mathbb{E}_2$ to $\mathbb{E}_1$ and will be denoted $\operatorname{perp}(\mathbb{E}_2, \mathbb{E}_1)$. So $\operatorname{perp}(\mathbb{E}_2, \mathbb{E}_1)$ is the unique vector of $\mathbb{E}_1 - \mathbb{E}_2 = \{x_1 - x_2 | x_i \in \mathbb{E}_i\}$ orthogonal to $V_1 + V_2$. The first entry x_1 of a pair (x_1, x_2) with $x_i \in \mathbb{E}_i$ and $x_1 - x_2 = \operatorname{perp}(\mathbb{E}_2, \mathbb{E}_1)$ is called the *foot* of a common perpendicular from $\mathbb{E}_2$ to $\mathbb{E}_1$. The set of feet of common perpendiculars from $\mathbb{E}_2$ to $\mathbb{E}_1$ will be denoted $\mathbb{E}_{1,2}$. So $\mathbb{E}_{1,2}$ is an affine subspace of $\mathbb{E}_1$ with $T\mathbb{E}_{1,2} = V_1 \cap V_2$.

Let us return to our subgroup Γ of G. Let A be the subalgebra $\mathbb{R}[\lambda(\Gamma)]$ of $\operatorname{End}(V)$ consisting of all linear combinations of elements of $\lambda(\Gamma)$ with real coefficients.

Proposition 2.7. *Suppose $\mathbb{F}$, $\mathbb{E}_1$ and $\mathbb{E}_2$ are affine subspaces of $\mathbb{E}$ with the following properties: $\mathbb{F}$ is Γ–invariant, both E_i intersect $\mathbb{F}$ and the orthogonal projections P_i of V onto $T\mathbb{E}_i$ are elements of A. Then $\mathbb{E}_{1,2}$ intersects $\mathbb{F}$, the orthogonal projection of V onto $T\mathbb{E}_{1,2}$ is in A and* $\operatorname{perp}(\mathbb{E}_2, \mathbb{E}_1)$ *is in $T\mathbb{F}$.*

Proof. Let P_i be the orthogonal projection of V onto $V_i := T\mathbb{E}_i$. Then the orthogonal projection of V onto $T\mathbb{E}_{1,2} = V_1 \cap V_2$ is in A, by 2.4. Let Q_1, Q_2 and Q_3 be operators as in lemma 2.4. Suppose $a_i \in \mathbb{F} \cap \mathbb{E}_i$. Put $t = a_1 - a_2$, $b_1 = a_1 - Q_1 t$ and $b_2 = a_2 + Q_2 t$. Then t is in the A–module $W := T\mathbb{F}$, hence $Q_1 t \in V_1 \cap W$ and thus $b_1 \in \mathbb{F} \cap \mathbb{E}_1$, and similarly $b_2 \in \mathbb{F} \cap \mathbb{E}_2$. Finally, $b_1 - b_2 = Q_3 t \in (V_1 + V_2)^{\perp}$, so b_1 is the foot of a common perpendicular from $\mathbb{E}_2$ to $\mathbb{E}_1$. We have $\operatorname{perp}(\mathbb{E}_2, \mathbb{E}_1) = b_1 - b_2 \in W$. ∎

We can apply the procedure of the proposition to axes $\mathbb{E}_i$ of single elements or generators of our group Γ and then iterate the procedure with the spaces obtained and thus produce further points in $\mathbb{F}$ and further vectors in $T\mathbb{F}$. We formalize the iteration as follows.

2.8. Let X be a set and let M_X be the free non–associative semigroup on X, called a *free magma* by Serre ([Lie algebras and Lie groups. Benjamin New York 1965] LA 4.1) We have $M_X = \coprod_{n=1}^{\infty} X_n$ where X_n is inductively defined by $X_1 = X$ and $X_n = \coprod_{p+q=n}^{\infty} X_p \times X_q$ for $n \geq 1$ and the multiplication $M_X \times M_X \to M_X$ is defined by the natural inclusion $X_p \times X_q \to X_{p+q}$. The product of the elements m and n in M_X will be denoted by (m, n). We have a unique map M_X to the set of non–empty subsets of X, called *support* and denoted supp with the properties $\operatorname{supp}(x) = \{x\}$ for $x \in X_1$ and $\operatorname{supp}(m, n) = \operatorname{supp}(m) \cup \operatorname{supp}(n)$.

2.9. Now let Γ be the subgroup of G generated by a subset S of G. Define for every element m of the free magma M_S a subspace $\mathbb{E}_m$ of $\mathbb{E}$ as follows. For $\gamma \in X$ let $\mathbb{E}_\gamma$ be the axis of γ, and for m, n in M_S let $\mathbb{E}_{(m,n)}$ be the set $\mathbb{E}_{m,n}$ of feet of common perpendiculars from $\mathbb{E}_n$ to $\mathbb{E}_m$. The family of $\mathbb{E}_m$'s has the following properties:

a) $\mathbb{E}_{(m,n)} \subset \mathbb{E}_m$.

b) $T\mathbb{E}_m = V^{\lambda\langle \mathrm{supp}(m)\rangle}$.
c) For every nonempty Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ we have $\mathbb{E}_m \cap \mathbb{F} \neq \emptyset$ and perp $(\mathbb{E}_m, \mathbb{E}_n) \in T\mathbb{F}$ for m, n in M_Γ.

Here we denote by $\langle Y\rangle$ the subgroup of G generated by the subset Y of G. For a subgroup H of $\lambda(G)$ we denote by V^H the subspace of vectors of V fixed by every element of H. The properties a) – c) follow from 2.6 and 2.7 using the fact that the orthogonal projection of V onto V^H is contained in $\mathbb{R}[H]$. Here is our description of all the minimal Γ-invariant affine subspaces of $\mathbb{E}$.

Theorem 2.10. *Let W be the subspace of V spanned by the set of vectors $\tau(g)$, $g \in \Gamma$, and* perp $(\mathbb{E}_m, \mathbb{E}_n), m, n$ *in M_Γ. An affine subspace $\mathbb{F}$ of $\mathbb{E}$ is a minimal Γ–invariant affine subspace of $\mathbb{E}$ iff $T\mathbb{F} = W$ and $\mathbb{F}$ intersects one of the subspaces $\mathbb{E}_m$ with $m \in M_\Gamma$ and $T\mathbb{E}_m = V^{\lambda(\Gamma)}$. Every Γ–invariant affine subspace of $\mathbb{E}$ contains a space $\mathbb{F}$ of this form.*

The following consequence is well known, see [10], chapter 3, end of § 1.

Corollary 2.11. *Any two minimal Γ–invariant affine subspaces $\mathbb{F}$ of $\mathbb{E}$ are translates of each other by a vector in $V^{\lambda(\Gamma)}$.*

In particular, $\mathbb{F}$ is unique if $V^{\lambda(\Gamma)} = 0$.We restate the theorem for the case that the group Γ is given by a set of generators.

Theorem 2.12. *Let S be a subset of G generating the subgroup Γ of G. There is an $m \in M_S$ such that $V^{\lambda\langle \mathrm{supp}(m)\rangle} = V^{\lambda(\Gamma)}$. Let x be a point contained in a minimal Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$, e.g. $x \in \mathbb{E}_m$. The subspace $W = T\mathbb{F}$ is the smallest $\mathbb{R}[\lambda(\Gamma)]$–submodule of V containing any one of the following two sets of vectors*
a) $\{gx - x\,;\, g \in S\}$.
b) $\{\tau(g), g \in S\} \cup \{\mathrm{perp}\ (\mathbb{E}_m, \mathbb{E}_{(g,m)}), g \in S\}$.

Note that in case $V^{\lambda(\Gamma)} = 0$ the sets $\mathbb{E}_m$ and $\mathbb{E}_{(g,m)}$ consist of one point only, by 2.9 b).

Proofs. Every Γ-invariant affine subspace $\mathbb{F}'$ of $\mathbb{E}$ intersects $\mathbb{E}_m$, by 2.9 c). And $T\mathbb{F}'$ contains $\tau(g)$ for every $g \in \Gamma$, by 2.3, and perp $(\mathbb{E}_m, \mathbb{E}_n)$ for any two m, n in M_Γ, by 2.9 c). Hence $\mathbb{F}'$ contains a subspace $\mathbb{F}$ as in 2.10. To see that such an $\mathbb{F}$ is Γ-invariant, first note that for g and h in G we have

$$\tau(hgh^{-1}) = \lambda(h)\tau(g) \tag{2.13}$$

by 2.1 a), and

$$\mathbb{E}_{hgh^{-1}} = h\mathbb{E}_g \tag{2.14}$$

by the definition of the axes. Hence

$$\mathbb{E}_{\gamma m} = \gamma \mathbb{E}_m \tag{2.15}$$

for $m \in M_\Gamma$ and $\gamma \in \Gamma$, where $m \longmapsto \gamma m$ is the unique morphism of magmas such that $\delta \longmapsto \gamma\delta\gamma^{-1}$ for $\delta \in \Gamma$. So

$$\text{perp}\ (\mathbb{E}_{\gamma m}, \mathbb{E}_{\gamma n}) = \lambda(\gamma)\ \text{perp}\ (\mathbb{E}_m, \mathbb{E}_n)\ . \tag{2.16}$$

It follows that the vector space W of 2.10 is an $\mathbb{R}[\lambda(\Gamma)]$–module. So, to finish the proof of 2.10, it remains to show that $\gamma x_0 \in x_0 + W$ for $\gamma \in \Gamma$ and $x_0 \in \mathbb{E}_m$.

Our hypothesis about m implies that every point x of $\mathbb{E}_m$ is a foot of a perpendicular from $\mathbb{E}_n$ to $\mathbb{E}_m$ for every $n \in M_\Gamma$, since

$$T\mathbb{E}_{m,n} = V^{\lambda\langle \text{supp}(m) \cup\ \text{supp}(n)\rangle} = V^{\lambda(\Gamma)} = V^{\lambda\langle \text{supp}(m)\rangle} = T\mathbb{E}_m$$

by 2.9 b). Hence for $t = \text{perp}\ (\mathbb{E}_m, \mathbb{E}_\gamma)$ we have $x + t \in \mathbb{E}_\gamma$ and thus

$$\gamma x = x + \tau(g) - (\lambda(g) - 1)t \tag{2.17}$$

by 2.1 e), so $\gamma x \in x + W$ for $\gamma \in \Gamma$.

Corollary 2.11 follows from the following facts. Any two minimal Γ–invariant subspaces are parallel since their translation subspaces are the same, namely W. They both intersect every $\mathbb{E}_m$ where $m \in M_\Gamma$ and $T\mathbb{E}_m = V^{\lambda(\Gamma)}$,by 2.9 c), hence they are parallel translates of each other by a vector in $T\mathbb{E}_m = V^{\lambda(\Gamma)}$.

The first claim of 2.12 is clear for dimension reasons. To prove the second one let W_a and W_b be the $\mathbb{R}[\lambda(\Gamma)]$–submodules of V generated by the sets given in a) and b), respectively. Clearly, $x + W_a$ is Γ–invariant and every Γ–invariant affine subspace of $\mathbb{E}$ containing x contains $x + W_a$, thus $\mathbb{F} = x + W_a$ is aminimal Γ–invariant affine subspace of $\mathbb{E}$. Concerning b), we have $W_b \subset W$ by the definition of W in 2.10. For $x_0 \in \mathbb{E}_m$ we have $\gamma x_0 \in x_0 + W_b$ for $\gamma \in S$ by 2.17.

Note here that $\text{perp}\ (\mathbb{E}_1, \mathbb{E}_2) = \text{perp}\ (\mathbb{E}_1, \mathbb{E}_{2,1})$. It follows that $x_0 + W_b$ is γ–invariant for every $\gamma \in S$ and hence for any $\gamma \in \Gamma$. This implies that $x_0 + W_b$ is a minimal Γ–invariant affine subspace, in particular $W_b = W$. Hence $x + W_b$ is a minimal Γ–invariant affine subspace of $\mathbb{E}$ for every point x contained in a minimal Γ–invariant affine subspace of $\mathbb{E}$. ∎

Remark 2.18. W is not generated by one of the two subsets in 2.12 b) alone. E.g. if Γ consists of translations only, then $\mathbb{E}_m = \mathbb{E}$ for every $m \in M_\Gamma$ and hence $\text{perp}\ (\mathbb{E}_m, \mathbb{E}_n) = 0$ always. On the other hand, there are examples of subgroups Γ of G such that every element γ of Γ has a fixed point, but there is no common fixed point. So $\tau(\gamma) = 0$ for every $\gamma \in \Gamma$ but $W \neq 0$. An example of such a group Γ can be obtained as follows. Take two elements g_1, g_2 of G for $\dim \mathbb{E} = 4$ such that for every reduced word w different from the empty word all the eigenvalues of $\lambda(w(g_1, g_2))$ are different from 1 and the points $\text{Fix}\,(g_1)$ and $\text{Fix}\,(g_2)$ are different. Then every element of the group Γ generated by g_1 and g_2 has a fixed point, since $\tau(g) \in V^{\lambda(g)} = \{0\}$. The existence of such a pair (g_1, g_2) can be seen as follows. The group $\mathbb{H}_1$ of quaternions of norm 1 acts on $\mathbb{H}$ by left multiplication. An element $h \in \mathbb{H}_1$ has one eigenvalue equal to 1 iff

$h = 1$. The set of pairs $(h_1, h_2) \in \mathbb{H}_1 \times \mathbb{H}_1$ which freely generate a free subgroup of $\mathbb{H}_1$ hasa complement of measure zero. Hence there are many pairs (g_1, g_2) as above. By contrast, if Γ is discrete, the set of translational parts $\tau(\gamma)$, $\gamma \in \Gamma$ generates W, by the Bieberbach theorems.

2.19. Computational aspects. For an element g of G the axis $\mathbb{E}_g$ can be computed as follows. Suppose g is given in the form $A(x_0, t, U)$. Decompose $t = t_0 + t_1$ according to $V = V^1 \oplus V^{\neq 1}$, where $V^1 = \ker(U-1)$ and $V^{\neq 1}$ is the orthogonal complement of V^1. Then $\mathbb{E}_g$ consists of the points $x_0 + v$ where $t_2 + (U-1)v = 0$, by the proof of 2.1. So $\mathbb{E}_g$ can be computed by solving systems of linear equations. If $\mathbb{E}_1$ and $\mathbb{E}_2$ are two affine subspaces of $\mathbb{E}$ both perp $(\mathbb{E}_1, \mathbb{E}_2)$ and $\mathbb{E}_{1,2}$ can be computed by solving systems of linear equations, see 2.6. After at most $\dim V$ such steps we arrive at $\mathbb{E}_m$ with $\mathbb{E}_m = V^{\lambda(\Gamma)}$. To determine $\mathbb{F}$ take a point $x \in \mathbb{E}_m$ and compute $W = T\mathbb{F}$ as follows. Let $D_0 = \{\gamma x - x; \gamma \in S\}$. Start with a basis $B_0 \subset D_0$ of the $\mathbb{R}$–vector subspace $< D_0 >_{\mathbb{R}}$ of V spanned by D_0. Suppose B_j has been defined. Check for every pair $(\gamma, b), \gamma \in S, b \in B_j$, if $(\gamma)b \in < B_j >_{\mathbb{R}}$. If not define $B_{j+1} = B_j \cup \{\gamma b\}$. By numbering the elements of S and of $\bigcup B_j$ starting with those of B_0 one sees that one arrives at a basis of W after at most $\#S \cdot \dim V$ such checks.

3. When is a group crystallographic?

In the last chapter we described a minimal Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$ for every subgroup Γ of G. According to the Bieberbach theorems our first test for discreteness of Γ will be to check whether the restriction of Γ to $\mathbb{F}$ is crystallographic. Here is our test.

Theorem 3.1. *The group $r(\Gamma)$ is crystallographic for $\mathbb{F}$ iff there is a point x in $\mathbb{E}$ such that the abelian subgroup $D(x)$ generated by $\{\gamma x - x; \gamma \in \Gamma\}$ is discrete.*

In order to apply the theorem for the case that Γ is given by a set S of generators, note that $D(x)$ is the smallest $\mathbb{Z}[\lambda(\Gamma)]$–submodule of V containing $\{\gamma x - x;\ \gamma \in S\}$. The group $D(x)$ is not discrete for arbitrary points $x \in \mathbb{E}$, nor even for arbitrary points $x \in \mathbb{F}$, even if $r(\Gamma)$ is crystallographic. E.g. let $\mathbb{F} = \mathbb{E} = \mathbb{R}$ and $\Gamma = \{x \mapsto \pm x + n; n \in \mathbb{Z}\}$. Then $D(x)$ is discrete iff $x \in \mathbb{Q}$. This is a typical behavior, see 3.3 e). So an obvious question – essential for computations – is how to find a point x as in the theorem. Incidentally, the answer uses the concepts of the preceding chapter. We thus use the notations of chapter two, in particular 2.9.

Addendum 3.2. *Suppose $r(\Gamma)$ is crystallographic. Let $m \in M_\Gamma$ be such that $T\mathbb{E}_m = V^{\lambda(\Gamma)}$. Then $D(x)$ is discrete for $x \in \mathbb{E}_m$, in fact $\Delta = \Gamma \cap V$ is of finite index in $D(x)$ and $T\mathbb{F} = D(x)_{\mathbb{R}} = \Delta_{\mathbb{R}}$.*

The properties of $r(\Gamma)$ are independent of the particular choice of a minimal Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$: Any two such subspaces $\mathbb{F}_1$ and $\mathbb{F}_2$ are translates of each other by a vector in $V^{\lambda(\Gamma)}$, according to 2.11, and this translation commutes with Γ, induces an isometry $\mathbb{F}_1 \to \mathbb{F}_2$ and yields an isomorphism $r_1(\Gamma) \to r_2(\Gamma)$.

In the rest of this chapter we give a proof of 3.1 and 3.2. Let us start by establishing a few elementary and basic facts about $D(x)$.

Lemma 3.3. *Suppose $\lambda(\Gamma) = F$ is a finite group of order f. Let $\Delta = \ker(\lambda|\Gamma) = \Gamma \cap V$ and let x be a point of $\mathbb{E}$.*

a) *$D(x)$ is a $\mathbb{Z}[\lambda(\Gamma)]$–module containing Δ.*

b) *$D(x)$ spans $T\mathbb{F}$ over $\mathbb{R}$ for every $x \in \mathbb{F}$.*

c) *There is a point $x \in \mathbb{E}$ with $D(x) \subset \frac{1}{f}\Delta$.*

d) *For every point x with $D(x) \subset \Delta_{\mathbb{R}}$ the affine space $x + \Delta_{\mathbb{R}}$ is a minimal Γ–invariant affine subspace of $\mathbb{E}$.*

e) *Suppose $D(x) \subset \Delta_{\mathbb{Q}}$ then*

$$\{y; D(y) \subset \Delta_{\mathbb{Q}}\} = x + \Delta_{\mathbb{Q}} + V^F \ .$$

Proof. a) is clear, b) follows from 2.12 a). c): For $x \in \mathbb{E}$ put $\overline{x} = \frac{1}{f}\sum \gamma_i x$, where γ_i runs through a set of representatives of Γ modulo Δ. Note that *affine* combinations $\sum \lambda_i x_i$ of points $x_i \in \mathbb{E}$, that is combinations with $\sum \lambda_i = 1$, are well defined in the affine space $\mathbb{E}$, namely $x_0 + \sum \lambda_i(x_i - x_0)$ is independent of a chosen point $x_0 \in \mathbb{E}$. For another set γ_i' of representatives of Γ modulo Δ we have $\gamma_i' = \gamma_i \delta_i$ with $\delta_i \in \Delta$ and thus the point

$$\overline{y} = \frac{1}{f}\sum \gamma_i' x = \frac{1}{f}(\sum \lambda(\gamma_i)(\delta_i x - x) + \sum \gamma_i x) \in \overline{x} + \frac{1}{f}\Delta$$

is in the $\frac{1}{f}\Delta$–orbit of $\overline{x}$. In particular, for $\gamma \in \Gamma$ the point $\gamma\overline{x} = \frac{1}{f}\sum \gamma\gamma_i x$ is in the $\frac{1}{f}\Delta$–orbit of $\overline{x}$.

d) The affine space $x + \Delta_{\mathbb{R}}$ is Γ–invariant since it contains the orbit of x and $\Delta_{\mathbb{R}}$ is an $\mathbb{R}[\lambda(\Gamma)]$–module. It is minimal since $\Delta \subset \{\gamma x - x\,;\, \gamma \in \Gamma\}$ for every $x \in \mathbb{E}$.

e) That the right hand side is a subset of the left hand side, is clear. The converse inclusion is a consequence of the following lemma. ∎

Lemma 3.4. *Let Γ_1 be a subgroup of Γ. Suppose $F_1 = \lambda(\Gamma_1)$ is finite of order f_1, say. Let D and D' be subgroups of V such that $x + D$ and $x' + D'$ are Γ_1–invariant. Then*

$$f_1 \cdot (x - x') \in D + D' + V^{F_1} \ .$$

Proof. Note that D and D' are F_1–modules since D contains $\gamma(x+d) - x = (\gamma x - x) + \lambda(\gamma)d$ for $d \in D$ and $\gamma \in \Gamma_1$, and similarly for D'. For any two points x and x' in $\mathbb{E}$ we have for $\gamma \in \Gamma$

$$\gamma x' - x' = \gamma x - x + (\lambda(\gamma) - 1)(x' - x) \ .$$

Take x and x' as in the lemma and put $w = x' - x$. Our hypothesis $\gamma x - x \in D$ and $\gamma x' - x' \in D'$ for $\gamma \in \Gamma_1$ imply by summing over a set of representatives of F_1 in Γ_1

$$\sum_{h \in F_1} hw - f_1 \cdot w \in D + D'$$

which implies the lemma in view of $\sum\limits_{h \in F_1} hw \in V^{F_1}$. ∎

We are now ready to prove theorem 3.1.

3.5. Proof of Theorem 3.1. To prove necessity we may assume that $\mathbb{F} = \mathbb{E}$ and that Γ is crystallographic. Then, by Bieberbach's first theorem, $\lambda(\Gamma) = F$ is finite and Δ is discrete. Hence $\frac{1}{f}\Delta$ is a discrete subgroup of V containing $\gamma x - x$ for every $\gamma \in \Gamma$ if x is as in 3.3 c). Conversely, let x be a point of $\mathbb{E}$ such that $D := D(x)$ is discrete. Note that we do not have $x \in \mathbb{F}$ nor $D \subset \Delta_{\mathbb{R}}$, in general, e.g. if $\mathbb{E} = \mathbb{F} \times U$, where Γ acts on $\mathbb{F}$ by translations and as a finite rotation group on the vector space U,and x is any point not in $\mathbb{F}$. The smallest Γ–invariant affine subspace of $\mathbb{E}$ containing x is $x + W$, where $W = D_{\mathbb{R}}$. We may suppose that $\mathbb{E} = x + W$ by restricting Γ to $x + W$ if necessary, since it suffices to prove that $r(\Gamma)$ is crystallographic for *some* minimal Γ–invariant affine subspace $\mathbb{F}$ of $\mathbb{E}$, by the remark following 3.2. The subgroup $\lambda(\Gamma)$ of the (compact) orthogonal group of W leaves the discrete spanning subset D of W invariant, hence is discrete and thus finite. So we can apply 3.3. The group $\Delta = \ker(\lambda|\Gamma) = \Gamma \cap V$ is discrete since $\Delta \subset D$. Let x be a point with $D(x) \subset \frac{1}{f}\Delta$. Such a point exists by 3.3c). Then $\mathbb{F} = y + \Delta_{\mathbb{R}}$ is a minimal Γ–invariant affine subspace of $\mathbb{E}$, by 3.3d), and $r(\Gamma)$ is crystallographic on $\mathbb{F}$, since Γ contains Δ of finite index. ∎

We are now heading for a proof of 3.2. The proof consists in redoing the proof of the last chapter, this time over the rationals rather than the reals, using the hypothesis that $\lambda(\Gamma) = F$ is finite. To prove 3.2 we could actually assume that $\mathbb{E} = \mathbb{F}$ and that Γ is discrete but we need only the following

Assumption. $\lambda(\Gamma)$ *is finite.*

Again, we put $\lambda(\Gamma) = F$, $\#F = f$, $\Delta = \ker(\lambda|\Gamma) = \Gamma \cap V$. We do not suppose that Γ is discrete nor that $\mathbb{F} = \mathbb{E}$. Our first handle towards finding a point x with $D(x)$ discrete is analogous to 2.3:

Corollary 3.6. *Let D be a $\mathbb{Q}[F]$–submodule of V. Any Γ–invariant D–orbit $x + D \subset \mathbb{E}$ intersects the axis $\mathbb{E}_\gamma$ for every $\gamma \in \Gamma$.*

This applies in particular for $D = \Delta_{\mathbb{Q}}$ and any Γ–invariant subset of the form $x + \Delta_{\mathbb{Q}}$ of $\mathbb{E}$, e.g. for x as in 3.3 c)

Proof. Apply lemma 3.4 to the following situation: Let Γ_1 be the subgroup of Γ generated by the element γ, let $x + D$ be Γ–invariant, $x' \in E_\gamma$ and $D' = \mathbb{Z}\tau(\gamma)$. We have $D' \subset D$ in view of the following facts: Δ is contained in D by Γ–invariance of $x + D$, the power γ^f of $\gamma \in \Gamma$ is contained in Δ and γ^f is the translation by $f \cdot \tau(\gamma)$. So $x - x' \in D + V^{\lambda(\gamma)}$ by lemma 3.4. Now use $T\mathbb{E}_\gamma = V^{\lambda(\gamma)}$. ∎

Again, as in chapter 2, we shall prove that every Γ–invariant subset of the form $x + \Delta_{\mathbb{Q}}$ intersects $\mathbb{E}_m$ for every $m \in M_\Gamma$, thus proving 3.2.

Proposition 3.7. *Suppose $\mathbb{E}_1$ and $\mathbb{E}_2$ are affine subspaces of $\mathbb{E}$ such that the orthogonal projections of V onto $T\mathbb{E}_i$, $i = 1, 2$, are in $\mathbb{Q}[F]$. Let D be a $\mathbb{Q}[F]$–submodule of V and let $x + D$ be Γ–invariant. Suppose $x + D$ intersects both $\mathbb{E}_1$ and $\mathbb{E}_2$ then $x + D$ intersects $\mathbb{E}_{1,2}$, the orthogonal projection of V onto $T\mathbb{E}_{1,2}$ is in $\mathbb{Q}[F]$ and* perp $(\mathbb{E}_1, \mathbb{E}_2)$ *is in D.*

Proof. We apply lemma 2.4 for P_i the orthogonal projections of V onto $T\mathbb{E}_i$, $i = 1, 2$, and $A = \mathbb{Q}[F]$. Thus A contains the orthogonal projection of V onto $T\mathbb{E}_1 \cap T\mathbb{E}_2$ and contains operators Q_1, Q_2 and Q_3 such that $Q_1 + Q_2 + Q_3 = 1$, Q_3 is the orthogonal projection onto $(T\mathbb{E}_1 + T\mathbb{E}_2)^\perp$ and the image of Q_i is contained in $T\mathbb{E}_i$ for $i = 1, 2$. Now let $a_i \in \mathbb{E}_i \cap (x + D)$ then $b_1 := a_1 - Q_1(a_1 - a_2) \in \mathbb{E}_1 \cap (x + D)$, $b_2 := a_2 + Q_2(a_1 - a_2) \in \mathbb{E}_2 \cap (x + D)$, $b_1 - b_2 = Q_3(a_1 - a_2) = \operatorname{perp}(\mathbb{E}_2, \mathbb{E}_1) \in D$, $b_1 \in \mathbb{E}_{1,2} \cap (x + D)$ and $b_2 \in \mathbb{E}_{2,1} \cap (x + D)$. ∎

We obtain as in 2.9:

Corollary 3.8. *Every Γ–invariant $\Delta_\mathbb{Q}$–orbit intersects $\mathbb{E}_m$ for every $m \in M_\Gamma$ and* $\operatorname{perp}(\mathbb{E}_m, \mathbb{E}_n) \in \Delta_\mathbb{Q}$ *for every pair m, n of elements of M_Γ.* ∎

3.9. We are now ready to prove 3.2. There is a point $x \in \mathbb{E}$ such that $x + \frac{1}{f}\Delta$ is Γ–invariant, by 3.3 c). For every point $y \in x + \frac{1}{k}\Delta$ we have

$$\gamma y - y = (\gamma x - x) + (\lambda(\gamma) - 1)(y - x) \in \frac{1}{kf}\Delta,$$

hence $D(y) \subset \frac{1}{kf}\Delta$ for $y \in x + \frac{1}{k}\Delta$ where $D(y)$ is as always the abelian subgroup of V generated by $\{\gamma y - y; \gamma \in \Gamma\}$. This holds in particular for a point $y \in \mathbb{E}_m$ with $T\mathbb{E}_m = V^F$ for an appropriate integer k depending on y, by 3.8. Finally, $D(y) = D(y')$ for every point $y' \in \mathbb{E}_m$ since $y - y'$ is fixed by F. The last claim of 3.2 follows from 3.3d). ∎

3.10. Computational aspects. Let S be a finite set of elements of G generating a subgroup Γ of G, let $m \in M_S$ be such that $T\mathbb{E}_m = V^{\lambda(\Gamma)}$, let x be a point of $\mathbb{E}_m$ and let $\mathcal{B} \subset D(x)$ be a basis of the vector space $W = T\mathbb{F}$ where $\mathbb{F} = x + W$ is a minimal Γ–invariant affine subspace of $\mathbb{E}$. In 2.18 it was described how all these things can be computed. With these notations we have

Corollary 3.11. *$r(\Gamma)$ is crystallographic iff the following three conditions hold*

(i) *$\gamma x - x$ is a rational linear combination of $\mathcal{B}$ for every $\gamma \in S$.*
(ii) *With respect to the basis $\mathcal{B}$ of W the linear map $\lambda(\gamma) \mid W = \lambda \circ r(\gamma)$ is represented by a matrix in $GL(d; \mathbb{Q})$, $d = \dim W$, for every $\gamma \in S$.*
(iii) *The subgroup of $O(W)$ generated by $\lambda(\gamma) \mid W = \lambda \circ r(\gamma)$, $\gamma \in S$, is finite.*

Proof. If $r(\Gamma)$ is crystallographic then $\Delta_\mathbb{R} = W \supset D(x) \supset \Delta$ and $D(x)$ contains the subgroup Δ of finite index, hence $D(x)_\mathbb{Q} = \Delta_\mathbb{Q}$ which implies (i) and (ii). Bieberbach's theorems imply (iii). To prove sufficiency let D_0 be the $\mathbb{Z}$–submodule of W generated by $\mathcal{B}$ and $\{\gamma x - x; \gamma \in S\}$. Then D_0 is a lattice in W, by (i). The group $\lambda(\gamma)D_0$ is commensurable with D_0 by (ii) for every $\gamma \in S$ and hence for every $\gamma \in \Gamma$. Then $D_1 = \sum_{\gamma \in \Gamma} \lambda(\gamma) D_0$ is commensurable with D_0, by (iii), and hence a lattice in W in particular the subgroup $D(x)$ of D_1 generated as a $\mathbb{Z}[\lambda(\Gamma)]$–module by $\{\gamma x - x; \gamma \in S\}$ is discrete. ∎

Remark 3.12. Concerning computations, it is easy to check (i) and (ii). As for (iii), the proof above shows that a finite subgroup of $GL(d; \mathbb{Q})$ is conjugate in $GL(d,\mathbb{Q})$ to a subgroup of $GL(d;\mathbb{Z})$ – a basis transformation from a $\mathbb{Z}$–basis of D_0 to a $\mathbb{Z}$–basis of D_1 will give the desired conjugation. But the order of a finite subgroup of $GL(d;\mathbb{Z})$ is bounded by a number depending only on d, by a theorem of Minkowski. We want to point out the following consequences of what was developed in the last two chapters. These results give necessary conditions for a given subgroup Γ of G to be discrete.

Notations will be as above: Γ is a subgroup of G, $\mathbb{F}$ a minimal Γ–invariant affine subspace, $r : \Gamma \to \mathrm{Iso}(\mathbb{F})$ the restriction homomorphism and $\Delta = r(\Gamma) \cap T\mathbb{F}$ the subgroup of translations of $r(\Gamma)$. We will assume that $r(\Gamma)$ is crystallographic. So this applies in particular when Γ is discrete.

Proposition 3.13. *The following elements are contained in $\Delta_{\mathbb{Q}}$:*
a) *the translational part $\tau(\gamma)$ for every $\gamma \in \Gamma$*
b) perp $(\mathbb{E}_m, \mathbb{E}_n)$ *for m, n in M_Γ*
c) *$\gamma x - x$ for $\gamma \in \Gamma$ and $x \in \mathbb{E}_m$, where $m \in M_\Gamma$ and $T(\mathbb{E}_m \cap \mathbb{F}) = (T\mathbb{F})^{\lambda(\Gamma)}$.*

It is not true in general that all these elements are contained in a lattice in $\Delta_{\mathbb{Q}}$, see example 5.11. They are though, if we restrict the length of the elements $m \in M_\Gamma$. Here the length of an element m in the free non-associative semigroup M_X generated by X is defined in the obvious way, i.e. $l(x) = 1$ for $x \in X$ and $l((m,n)) = l(m) + l(n)$.

Proposition 3.14. *For every N there is a lattice Δ_N in $\Delta_{\mathbb{Q}}$ which contains the following elements:*
a) *$\tau(\gamma)$ for every $\gamma \in \Gamma$*
b) perp $(\mathbb{E}_m, \mathbb{E}_n)$ *for $m, n \in M_\Gamma$ and $l(m) \le N$, $l(n) \le N$*
c) *$\gamma x - x$ for $\gamma \in \Gamma$ and $x \in \mathbb{E}_m$, where $m \in M_\Gamma$, $T(\mathbb{E}_m \cap \mathbb{F}) = (T\mathbb{F})^{\lambda(\Gamma)}$ and $l(m) \le N$.*

A geometric consequence is the following

Corollary 3.15. *Suppose Γ is crystallographic. Then for a given number N the set $\{\mathbb{E}_m,\ l(m) \le N\}$ is locally finite. In particular, the set $\{\mathbb{E}_\gamma,\ \gamma \in \Gamma\}$ is locally finite.*

This holds in particular for the set $\mathbb{F}_{r(\gamma)} = \mathbb{E}_\gamma \cap \mathbb{F}$, $\gamma \in \Gamma$, and the derived set $\mathbb{F}_m = \mathbb{E}_m \cap \mathbb{F}$ if Γ is discrete. But it is not true in general for a discrete subgroup Γ of G that the set $\{\mathbb{E}_\gamma,\ \gamma \in \Gamma\}$ is locally finite. For an example see 5.3.

Proof. It suffices to prove 3.14, since 3.13 is an immediate consequence and 3.15 follows in view of the following facts: The distance of $\mathbb{E}_m$ and $\mathbb{E}_n$ is the length of perp $(\mathbb{E}_m, \mathbb{E}_n)$ and thus equal to zero or bounded a way from zero by 3.14, if the lengths of m and n are bounded. But for a point $x \in \mathbb{E}_m$ an affine subspace $\mathbb{E}_n$, $n \in M_\Gamma$, containing x is characterized by the vector subspace $T\mathbb{E}_n = V^{\lambda\langle \mathrm{supp}(n)\rangle}$, see 2.9 b), which is one of a finite collection of subspaces V^H, H a subgroup of the finite group $F = \lambda(\Gamma)$.

To prove 3.14 we may assume that $\mathbb{F} = \mathbb{E}$ and thus that Γ is crystallographic and Δ is a lattice in $V = T\mathbb{E}$, since all our vectors in a)–c) are in $T\mathbb{F}$

by 2.3 and 2.9 c). Again, we put $F = \lambda(\Gamma)$, $f = \#F$. Then for $\gamma \in \Gamma$ we have $\tau(\gamma^f) = f\tau(g)$ and γ^f is a translation, i.e. $\gamma^f \in \Delta$, so $\tau(\gamma) \in \frac{1}{f}\Delta$.

Let $m_0 \in M_\Gamma$ be such that $\lambda\langle \mathrm{supp}(m_0)\rangle = F$ and hence $T\mathbb{E}_{m_0} = V^F$. Then for $x_0 \in \mathbb{E}_{m_0}$ the set $\gamma x - x$, $\gamma \in \Gamma$, is contained in the lattice $D(x_0)$, and $D(x_0) \supset \Delta$, by 3.2. Let $w = \mathrm{perp}\ (\mathbb{E}_{m_0}, \mathbb{E}_\gamma)$. Then

$$\gamma(x_0 + w) = \gamma x_0 + \lambda(\gamma) w$$

and

$$\gamma(x_0 + w) = x_0 + w + \tau(\gamma)$$

since $x_0 + w \in \mathbb{E}_\gamma$, hence

$$\gamma x_0 - x_0 = \tau(\gamma) - (\lambda(\gamma) - 1)w. \tag{3.16}$$

so $(\lambda(\gamma) - 1)w$ is contained in the lattice $\Delta_0 = \frac{1}{f}\Delta + D(x_0)$.

For every $\lambda(\gamma) \in F$ there is a polynomial $h_1 \in \mathbb{Q}[t]$ such that

$$h_1(\lambda(\gamma)) \cdot (\lambda(\gamma) - 1)w = w$$

for every $w \in (V^F)^\perp$, e.g. if we decompose the polynomial $t^f - 1$ into the relatively prime polynomials $t^f - 1 = (t-1) \cdot f_2$, where $f_2 = \sum_{i=0}^{f-1} t^i$, and write $1 = h_1 \cdot (t-1) + h_2 \cdot f_2$ with $h_i \in \mathbb{Q}[t]$, see the proof of lemma 2.5. It follows that there is a lattice $\Delta' \supset \Delta_0$ which contains perp $(\mathbb{E}_{m_0}, \mathbb{E}_\gamma)$ for every $\gamma \in \Gamma$. We may assume that Δ' is a $\mathbb{Z}[F]$–module.

Now choose for every pair H_1, H_2 of subgroups of F operators Q_1, Q_2 and Q_3 in $\mathbb{Q}[F]$ satisfying the conditions of lemma 2.4 for $U_i = V^{H_i}$, $i = 1, 2$, that means $Q_1 + Q_2 + Q_3 = 1$, $Q_i(V) \subset U_i$ for $i = 1, 2$ and Q_3 is the orthogonal projection of V onto $(U_1 + U_2)^\perp$. Suppose M is a positive integer such that M times all these operators for all pairs H_1, H_2 are in $\mathbb{Z}[F]$. Then for γ_1, γ_2 in Γ and $x_0 + w_i \in \mathbb{E}_{\gamma_i}$, $w_i = \mathrm{perp}\ (\mathbb{E}_{m_0}, \mathbb{E}_{\gamma_i})$, we have perp $(\mathbb{E}_{\gamma_1}, \mathbb{E}_{\gamma_2}) = Q_3(w_2 - w_1)$ for Q_3 as above for the two subgroups $H_i = \langle \lambda(\gamma_i)\rangle$, see the proof of 3.7. So the lattice $\frac{1}{M} \cdot \Delta'$ contains perp $(\mathbb{E}_{\gamma_1}, \mathbb{E}_{\gamma_2})$ for every pair γ_1, γ_2 of elements of Γ.

The same argument shows that if we have $a_i = x + w_i \in \mathbb{E}_{m_i}$ for some $x \in \mathbb{E}$ and w_i is contained in the $\mathbb{Z}[F]$–submodule Δ_i of V then perp $(\mathbb{E}_{m_1}, \mathbb{E}_{m_2}) = Q_3(w_2 - w_1) \in \frac{1}{M}(\Delta_1 + \Delta_2)$, $x + w_1 - Q_1(w_1 - w_2) \in \mathbb{E}_{(m_1, m_2)}$ and $x + w_2 + Q_2(w_1 - w_2) \in \mathbb{E}_{(m_2, m_1)}$, see the proof of 3.7. It thus follows that for m_0 as above, firstly, $x_0 + \frac{1}{M^{l(m)}}\Delta'$ intersects $\mathbb{E}_{m_0}$, and secondly, $\mathrm{perp}(\mathbb{E}_{m_1}, \mathbb{E}_{m_2}) \in \frac{1}{M^{l(m_1)+l(m_2)}}\Delta'$. Finally, if $x \in \mathbb{E}_m$, $w = \mathrm{perp}\ (\mathbb{E}_m, \mathbb{E}_\gamma)$, $x + w \in \mathbb{E}_\gamma$, we have

$$\begin{aligned}\gamma(x + w) &= \gamma x + \lambda(\gamma) w \\ &= x + w + \tau(\gamma)\end{aligned}$$

since $x + w \in \mathbb{E}_\gamma$ and hence $\gamma x - x = \tau(\gamma) - (\lambda(\gamma) - 1)w \in \frac{1}{M^{l(m)+1}}\Delta'$, which proves the claim concerning 3.14 c) since if $T\mathbb{E}_m = V^F$, every point x of $\mathbb{E}_m$ is the foot of a perpendicular to $\mathbb{E}_\gamma$, i.e. $\mathbb{E}_m = \mathbb{E}_{(m,\gamma)}$ by 2.9 a) and b). ∎

4. When is a group discrete?

4.1. *Hypotheses and notations for this whole chapter.* $\Gamma = \langle S \rangle$ *is a subgroup of* $G = \mathrm{Iso}(\mathbb{E})$ *given by a finite set* S *of generators,* $\mathbb{F}$ *is a minimal* Γ*–invariant affine subspace of* $\mathbb{E}$ *and* $r\colon \Gamma \to \mathrm{Iso}(\mathbb{F})$ *is the restriction homomorphism. We suppose that* $r(\Gamma)$ *is a crystallographic group on* $\mathbb{F}$*.*

Then Γ is discrete iff the kernel of r is finite. But this seems difficult to check in terms of the generating set S of Γ. We thus propose a different procedure, which is algorithmic. It is based on two ingredients, firstly the observation that the finiteness of the kernel of r is easy to check if B is an abelian group of a certain type and secondly, that Γ contains such a subgroup B of finite index if Γ is discrete.

We start with the following well known fact (s. [11], [1]). We give a short proof based on the Bieberbach theorems which also gives additional information we shall use in the algorithmic procedure.

Proposition 4.2. *If* Γ *is discrete then* Γ *contains a subgroup* B *of finite index such that its group* $\lambda(B)$ *of linear parts is contained in a torus and* $r(B)$ *consists of translations of* $\mathbb{F}$ *only.*

Proof. Let Γ_1 be the normal subgroup of those $\gamma \in \Gamma$ for which $r(\gamma)$ is a translation of $\mathbb{F}$. Then Γ_1 is of finite index in Γ by the corollary to the Bieberbach theorems and Bieberbach's first theorem, and the commutator subgroup of Γ_1 is finite by Bieberbach's second theorem. Hence the commutator subgroups of the following groups are finite: $\lambda(\Gamma_1)$, the closure L_1 of $\lambda(\Gamma_1)$, the connected component L_0 of the identity in L_1. So the Lie algebra of L_0 is abelian. Both the closure L of $\lambda(\Gamma)$ and L_1 are compact, being closed subgroups of the orthogonal group of $T\mathbb{E}$. Furthermore, L/L_1 and L_1/L_0 are finite, so L_0 is the connected component of L and is a torus. Now $r(\gamma\gamma_1\gamma^{-1})$ is the translation $\lambda(\gamma)r(\gamma_1)$ for $\gamma_1 \in \Gamma_1$. So the torus L_0 acts on the lattice $r(\Gamma_1)$ in $T\mathbb{F}$, hence fixes $T\mathbb{F}$. It follows that the normal subgroup $B = \{\gamma \in \Gamma; r(\gamma) \in T\mathbb{F}, \lambda(\gamma) \in L_0\}$ of Γ is abelian and of finite index in Γ. ∎

Note that the proof has shown that the torus L_0 is the connected component of $(\lambda(\Gamma))^-$.

Corollary 4.3. *If* Γ *is discrete then* $L_0 := \overline{(\lambda(\Gamma))}_0$ *is a torus and fixes* $T\mathbb{F}$*. So* $B = \{\gamma \in \Gamma\,;\, r(\gamma) \in T\mathbb{F}, \lambda(\gamma) \in L_0\}$ *is an abelian normal subgroup of* Γ *of finite index.* ∎

If our group Γ is of the special form as B in 4.2 then it is easy to check if r has finite kernel, as follows.

Proposition 4.4. *Suppose a subgroup* B *of* G *leaves the affine subspace* $\mathbb{F}$ *invariant and* $r(B)$ *consists of translations only. Suppose furthermore that* $\lambda(B)$ *is contained in a torus* T*. Then the homomorphism* $(\lambda, r) : B \to T \times T\mathbb{F}$

is injective. Let $\mathfrak{t}$ *be the Lie algebra of* T *and let* $\exp : \mathfrak{t} \to T$ *be its exponential map. Let* A *be the inverse image of the subgroup* $(\lambda, r)B$ *of* $T \times T\mathbb{F}$ *under* $\exp \times \mathrm{id} : \mathfrak{t} \times T\mathbb{F} \to T \times T\mathbb{F}$. *Then*

$$\operatorname{rank} A = \operatorname{rank} \ker(r) + \operatorname{rank} r(B) + \dim T.$$

In particular, if B *is finitely generated then* $\ker(r)$ *is finite iff* $\operatorname{rank} A = \operatorname{rank} r(B) + \dim T$.

Here and in the proof the rank of an abelian group D is $\dim_{\mathbb{Q}} D \otimes \mathbb{Q}$.

Proof. Put $W = T\mathbb{F}$ and $e = \exp \times id : \mathfrak{t} \times W \to T \times W$. For a subgroup A of $T \times W$ the surjection $e : e^{-1}(D) \to D$ has as its kernel the lattice $\ker(e)$ in $\mathfrak{t}$. Thus $\operatorname{rank}\ D = \operatorname{rank}(e^{-1}D) - \dim\ T$. Hence for our group $D = (\lambda, r)B$ we have $\operatorname{rank}(\ker(r)) = \operatorname{rank}\ D - \operatorname{rank}(r(D)) = \operatorname{rank}(e^{-1}(D)) - \dim\ T - \operatorname{rank}(r(D))$. ∎

4.5. Here is now the procedure to determine if Γ is discrete.

Step 1. *Find a finite subset* S_1 *of* Γ *with the following properties*

a) $r(\gamma)$ *is a translation of* $\mathbb{F}$ *for every* $\gamma \in S_1$.

b) *The subgroup of* $T\mathbb{F}$ *generated by* $\{r(\gamma)\,;\, \gamma \in S_1\}$ *is a lattice in* $T\mathbb{F}$

c) $\langle \lambda(\gamma) \rangle^-$ *is connected for every* $\gamma \in S_1$.

We describe below how to find such a set S_1, see 4.13. Concerning c) we use the following notation. For an element U of the orthogonal group $O = O(V)$ of $V = T\mathbb{E}$ let $C(U)$ be the closure of the cyclic group generated by U, so $C(U) = \langle U \rangle^-$. Let $C_0(U)$ be the connected component of 1 in $C(U)$. The order m of the finite cyclic group $C(U)/C_0(U)$ can be determined (see 4.8). For a non zero multiple n of m we have $C(U^n) = C_0(U) = C_0(U^m)$.

The remaining steps of our test if Γ is discrete are as follows.

4.6. Suppose S_1 is a finite subset of Γ with the properties a)–c) of 4.5.

Test (i) *The tori* $C_0(\lambda(\gamma))$, $\gamma \in S_1$, *commute.*

If (i) holds, let T be the subtorus $\prod_{\gamma \in S_1} C_0(\lambda(\gamma))$ of $O(V)$ generated by the $C_0(\lambda(\gamma))$, $\gamma \in S_1$. Let $\mathfrak{t}$ be its Lie algebra and let $\exp : \mathfrak{t} \to T$ be its exponential map. Put $e = \exp\ \times \mathrm{id} : \mathfrak{t} \times T\mathbb{F} \to T \times T\mathbb{F}$.

Let Θ be the subgroup of $\mathfrak{t} \times T\mathbb{F}$ generated by $\{e^{-1}(\gamma)\,,\, \gamma \in S_1\}$.

Test (ii) $\operatorname{rank}_{\mathbb{Q}} \Theta = \dim \mathbb{F} + \dim T$.

The group Γ acts on the Lie algebra $\mathfrak{o}$ of $O(V)$ via $Ad \circ \lambda$. Also, Γ acts on $T\mathbb{F}$ by $\gamma \cdot t = \lambda(\gamma)t = \gamma t \gamma^{-1}$, hence on the product $\mathfrak{o} \times T\mathbb{F}$. So the following condition makes sense.

Test (iii) *Every* $\gamma \in \Gamma$ *maps* $\Theta_{\mathbb{Q}}$ *to itself.*

Now comes the last test which is the hardest to perform. Put $B_1 = \langle S_1 \rangle$. Note that $r(B_1)$ is a subgroup of Δ of finite index, by 4.5 a) and b). Let $\mathbb{E}_m$ be as in 3.2, i.e. $m \in M_S$ and $T\mathbb{E}_m = V^{\lambda(\Gamma)}$. Let $x_0 \in \mathbb{E}_m \cap \mathbb{F}$. We have $r(B_1)_{\mathbb{Q}} = \Delta_{\mathbb{Q}} = D(x_0)_{\mathbb{Q}}$ by 3.2. So we can pick an element $\beta = \beta_\gamma \in e(\Theta_{\mathbb{Q}})$ for every $\gamma \in S$ such that

$$\beta(x_0) = \gamma x_0$$

Here is our last test:

Test (iv) *The group $\langle \gamma^{-1}\beta_\gamma\,;\,\gamma \in S\rangle$ is finite.*

As always, let S be a finite subset of G, $\Gamma = \langle S\rangle$, $\mathbb{F}$ a minimal Γ–invariant affine subspace of $\mathbb{E}$.

Theorem 4.7. *Γ is discrete iff $r(\Gamma)$ is crystallographic and one (any) S_1 as in* 4.5 *passes the Tests* (i)–(iv).

The rest of this chapter is devoted to questions of the type how to find the relevant objects ($S_1, C_0(U), \beta_\gamma, \dots$), how to perform the tests, possible variations and, of course, a proof of 4.7.

4.8. The first question we deal with is how to find for an element U in $O(V)$ an integer $m \neq 0$ such that $C(U^m) = C_0(U)$, where $C(U) = \langle U\rangle^-$ and $C_0(U)$ is the connected component of $C(U)$. Note that $m \neq 0$ has this properties iff m is a multiple of the order r of the finite cyclic group $C(U)/C_0(U)$.

To actually compute r one can proceed as follows. Choose a torus T in the special orthogonal group of V which contains U^2, for instance by decomposing V into mutually orthogonal one and two dimensional U–invariant subspaces and taking for T the corresponding product of SO_2's, one for every 2–dimensional U–invariant subspace.

Let $\mathfrak{t}$ be the Lie algebra of T and let $\exp : \mathfrak{t} \to T$ be its exponential map. The lattice $\Lambda := \ker \exp$ gives $\mathfrak{t}$ a $\mathbb{Z}$–structure. We use this structure to define the following notion: A linear form ℓ on $\mathfrak{t}$ is said to be defined over $\mathbb{Z}$ or $\mathbb{Q}$, respectively, if $\ell(\Lambda) \subset \mathbb{Z}$ or $\mathbb{Q}$, respectively. Let $\mathfrak{t}^*_\mathbb{Z}$ and $\mathfrak{t}^*_\mathbb{Q}$ be the set of linear forms defined over $\mathbb{Z}$ and $\mathbb{Q}$, respectively. Here is a description of $C(U)$ and $C_0(U)$. Choose an element $X \in \mathfrak{t}$ with $\exp X = U$. Put $D(X) = \{\ell \in \mathfrak{t}^*_\mathbb{Q} \,|\, \ell(X) \in \mathbb{Q}\}$.

4.9. *Then*

a) $\exp^{-1} C(U) = \{Y \in \mathfrak{t} \,|\, \ell(Y) \in \mathbb{Z}\cdot\ell(X) + \mathbb{Z}\ell(\Lambda) \mathit{for\ every}\ \ell \in D(X)\}$

b) *For the Lie algebra $\mathcal{L}C(U) = \mathcal{L}C_0(U)$ we have*

$$\mathcal{L}C(U) = \bigcap_{\ell \in D(X)} \ker \ell$$

c) *If we write $X = \sum_{i=1}^{n} \xi_i e_i$ with respect to a basis $e_1, \dots, e_n$ of Λ, then $1 + \dim C_0(U)$ is equal to the dimension of the $\mathbb{Q}$–vector subspace of $\mathbb{R}$ spanned by 1, $\xi_1, \dots, \xi_n$.*

d) *The order r of the cyclic group $C(U)/C_0(U)$ is the maximal denominator of the numbers $\ell(X)$, $\ell \in D(X) \cap \mathfrak{t}^*_\mathbb{Z}$. Hence if $\ell_1, \dots, \ell_d$ is a basis of the lattice $D(X) \cap \mathfrak{t}^*_\mathbb{Z}$ in $D(X)$, then r is the least common multiple of the denominators of the numbers $\ell_1(X), \dots, \ell_d(X)$.*

Proof. These claims are clear for the torus $\mathbb{R}/\mathbb{Z}$ with exponential map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and follow in general from Pontryagin duality: Every closed subgroup of a torus T is the intersection of kernels of homomorphisms $T \to \mathbb{R}/\mathbb{Z}$. These homomorphisms correspond on the level of Lie algebras to linear maps in $\mathfrak{t}^*_\mathbb{Z}$. Concerning c) look at the $\mathbb{Q}$–linear map $F : \mathbb{Q}^{n+1} \to \mathbb{R}$, $F(a_0, \dots, a_n) =$

$-a_0 + a_1\xi_1 + \cdots + a_n\xi_n$. For $(a_1, \ldots, a_n) \in \mathbb{Q}^n$ we have $\sum_{i=1}^n a_i e_i^* \in D(X)$ iff $(a_0 = \sum_{i=1}^n a_1\xi_i, a_1, \ldots, a_n) \in \mathbb{Q}^{n-1} \cap \ker F$ where $e_1^*, \ldots, e_n^*$ is a dual basis of the basis $e_1, \ldots, e_n$ in c) of the $\mathbb{Q}$–vector space $\mathfrak{t}_{\mathbb{Q}}$. Hence $\dim D(X) = \dim \ker F$ and thus $\dim C(U) = n - \dim D(X) = n - (n + 1 - \mathrm{rank}_{\mathbb{Q}}\{1, \xi_1, \ldots, \xi_n\})$. ∎

4.10. *Necessity of* (i)–(iv) *in* 4.6. If Γ is discrete then $(\lambda(\Gamma)^-)_0$ is a torus, hence the tori $C(\lambda(\gamma))$, $\gamma \in S_1$, commute. Furthermore, the restriction homomorphism $r : \Gamma \to \mathrm{Iso}(\mathbb{F})$ has finite kernel, hence so does $r|B_1$, which is equivalent to (ii) by 4.4. Furthermore, B_1 is of finite index in Γ, since r has finite kernel and $r(B_1)$ is a lattice in $T\mathbb{F}$ contained in $r(\Gamma) \cap T\mathbb{F}$, hence of finite index in $r(\Gamma)$. It follows that there is a normal subgroup B_2 of Γ contained in B_1 and of finite index in B_1. Then $\Theta_2 := e^{-1}(B_2)$ is a subgroup of Θ of finite index, hence $\Theta_{2,\mathbb{Q}} = \Theta_{\mathbb{Q}}$ and thus every $\gamma \in \Gamma$ maps $\Theta_{\mathbb{Q}}$ to itself.

Concerning (iv), there is a non–zero integer m such that for $\Theta_3 := \frac{1}{m}\Theta$ we have $\Theta_3 \supset \Theta$ and $e(\Theta_3)$ contains all the elements β_γ we have picked, one for every $\gamma \in S$. So the group $\Gamma_3 = \Gamma \cdot e(\Theta_3)$ contains Γ of finite index and thus is discrete. It follows that the group generated by $\{\gamma^{-1}\beta_\gamma\,;\, \gamma \in S\}$ which fixes x_0 is finite.

4.11. *Sufficiency of the conditions* (i)–(iv) *in* 4.6. The torus $C(\lambda(\gamma))$ acts on the lattice $r(B_1)$ in $T\mathbb{F}$, so the action of $C(\lambda(\gamma))$ on $T\mathbb{F}$ is trivial. Thus B_1 is abelian, by (i), and $r|B_1$ has finite kernel by (ii), in view of 4.4. So B_1 is discrete. The group $\Theta = e^{-1}(B_1)$ is finitely generated, since B_1 and the kernel of $\exp : \mathfrak{t} \to T$ are so. It follows from (ii) that Θ is a lattice in $\mathfrak{t} \times T\mathbb{F}$.

Let O_1 be the subgroup of those elements of the orthogonal group of V which fix $T\mathbb{F}$. Embed $O_1 \times T\mathbb{F}$ into G by $i(U,t) = A(x,t,U)$ where $x \in \mathbb{F}$, see 1.2. This embedding does not depend on the chosen point x of $\mathbb{F}$, by 1.5. Hence if $\gamma \in G$ maps $\mathbb{F}$ to itself we have

$$\gamma i(t,U)\gamma^{-1} = i(\lambda(\gamma)U\lambda(\gamma)^{-1}\,,\, \lambda(\gamma)t)\ .$$

It follows from (iii) that the differential of the conjugation by $\gamma \in \Gamma$ maps the Lie algebra $\Theta_{\mathbb{R}}$ of $T \times T\mathbb{F}$ to itself and thus Γ normalizes $D := T \times T\mathbb{F}$. The group $D \cdot \Gamma$ acts on D by conjugation and hence on its Lie algebra $\Theta_{\mathbb{R}}$. Here D acts trivially. For the corresponding homomorphism $\Gamma \cdot D \to \mathrm{Aut}(\Theta_{\mathbb{R}})$ the image is finite, by (iv), since $\beta_\gamma \in D$. Notice that the homomorphism takes values in $\mathrm{Aut}\,(\Theta_{\mathbb{Q}})$ by (iii). It follows that there is a lattice Θ_2 in $\Theta_{\mathbb{Q}}$ with the following properties: $e(\Theta_2)$ contains β_γ for every $\gamma \in S$, Θ_2 contains Θ of finite index and Θ_2 is Γ-invariant. So $B_2 = e(\Theta_2)$ contains β_γ for every $\gamma \in S$, contains B_1 of finite index and is normalized by Γ. So the group $\Gamma_2 = \Gamma \cdot B_2$ contains Γ of finite index. On the other hand, Γ_2/B_2 is a quotient of the group in (iv), hence finite. So Γ is commensurable to B_1 and hence discrete. ∎

Remark 4.12. Note the following variation of condition (iv). Pick for every $\gamma \in S$ a pair of elements $\beta_i(\gamma) \in e(\Theta_{\mathbb{Q}})$ such that $\beta_2(\gamma) \circ \gamma \circ \beta_1(\gamma)$ fixes the point $x_0 \in \mathbb{F}_m$. Then theorem 4.7 holds with (iv) replaced by

Test (iv′) *The group generated by* $\{\beta_2(\gamma) \circ \gamma \circ \beta_1(\gamma)\,;\, \gamma \in S\}$ *is finite.*

For the proof note that $\gamma \circ \beta_1(\gamma) \circ \gamma^{-1} \in e(\Theta_{\mathbb{Q}})$ by (iii), hence (iv) is equivalent to (iv$'$).

4.13. It remains to describe a procedure how to find a subset of Γ with the properties of 4.5. The first step is to

4.13a. *Find an element $\gamma \in \Gamma$ such that $r(\gamma)$ is a translation.*
Form a list of elements of Γ by taking successive generations of S, as follows. Let $l_S(\gamma) = \min\{q; \gamma = \gamma_1^{\varepsilon_1} \cdot \ldots \cdot \gamma_q^{\varepsilon_q}, \gamma_i \in S, \varepsilon_i \in \{\pm 1\}\}$ be the length of $\gamma \in \Gamma$ with respect to the generating set S. Put $B_m = \{\gamma \in \Gamma; l_S(\gamma) \leq m\}$ and $S_m = \{\gamma \in \Gamma; l_S(\gamma) = m\}$. The sets $B_m = \{e\} \dot\cup S_1 \dot\cup \cdots \dot\cup S_m$ can be computed inductively. Recall that there is an upper bound on the orders of the finite subgroups of $SL(d,\mathbb{Z}), d = \dim \mathbb{F}$, by a theorem of Minkowski, see 3.12. Hence, if $\# B_m$ is larger than this bound there are two different elements g and h in B_m such that $\lambda(g)|T\mathbb{F} = \lambda(h)|T\mathbb{F}$ and hence $g^{-1}h$ restricts to a translation on $\mathbb{F}$.

4.13b. *Take elements $\gamma_1, \ldots, \gamma_t \in \Gamma$ such that $r(\gamma_1), \ldots, r(\gamma_t)$ are linearly independent elements of $T\mathbb{F}$ and span a $\lambda(\Gamma)$–module.*
This can be done starting from elements $\alpha \in \Gamma$ with $r(\alpha) \in T\mathbb{F}$ by taking conjugates $\gamma\alpha\gamma^{-1}$, $\gamma \in \Gamma$. Linear independence and spanning can be taken over $\mathbb{Q}$ or $\mathbb{R}$, this leads to the same conditions, since $r(\Gamma)$ is crystallographic on $\mathbb{F}$ and hence $r(\Gamma) \cap T\mathbb{F}$ a lattice in $T\mathbb{F}$. At this point one can perform easy finiteness checks: If when looking for linearly independent elements in $T\mathbb{F}$ one finds elements $\gamma_1, \ldots, \gamma_t$ in Γ such that $r(\gamma_1), \ldots, r(\gamma_t)$ are linearly dependent, say $\sum n_i r(\gamma_i) = 0$, $n_i \in \mathbb{Z}$, then the products of $\gamma_1^{n_1}, \ldots, \gamma_t^{n_t}$ in any order must be contained in a finite group, in particular of finite exponent.

4.13c. *If we have a situation as in 4.13b we can use induction on* $\dim \mathbb{F}$ *to find more elements $\gamma \in \Gamma$ with $\tau(\gamma) \neq 0$, as follows.*
More precisely, let $\gamma_1, \ldots, \gamma_t \in \Gamma$ be such that $r(\gamma_1), \ldots, r(\gamma_t)$ are in $T\mathbb{F}$ and their $\mathbb{R}$–span $W := \langle r(\gamma_1), \ldots, r(\gamma_t)\rangle_{\mathbb{R}}$ is an $\mathbb{R}[\lambda(\Gamma)]$–module. Then Γ acts on $\mathbb{E}/W$ and $\mathbb{F}/W$ by affine isometries and the image of Γ in the affine group of $\mathbb{F}/W$ is a crystallographic group on $\mathbb{F}/W$, since $r(\Gamma)$ is crystallographic on $\mathbb{F}$. So we can apply the procedure described in 4.13a on the space $\mathbb{F}/W$ of smaller dimension than $\dim \mathbb{F}$ and we find an element $\gamma \in \Gamma$ which induces a translation on $\mathbb{F}/W$, so $\tau(\gamma) \notin W$, and hence $\gamma^q \mid \mathbb{F}$ is a translation in $T\mathbb{F}$ not in W if q is a multiple of the order of $\lambda(\gamma)|W$.

Note that the image of Γ on $\mathbb{E}/W$ may not be discrete even if Γ is discrete on $\mathbb{E}$, e.g. if $W = T\mathbb{F}$.

4.14. It would be nice to have an easier procedure to produce an element $\gamma \in \Gamma$ with $\tau(\gamma) \neq 0$, easier than 4.13a which uses Minkowski's theorem. If $\tau(\gamma) = 0$ for every $\gamma \in S$ and $\bigcap_{\gamma \in S} \mathrm{Fix}\,(\gamma) = \emptyset$ then $\Gamma = \langle S \rangle$ is infinite, but the example 2.18 of an infinite group Γ such that $\mathrm{Fix}\,(\gamma) \neq 0$ for every $\gamma \in \Gamma$ shows that one has to use the fact that the relevant group is crystallographic or at least has a finite group of linear parts. So there is no general procedure, i.e. one which does not use the arithmeticity or at least finiteness of the relevant linear parts of Γ, to find a non–commutative analogue of $\mathrm{perp}(\mathbb{E}_g, \mathbb{E}_h)$,i.e. an element $f \in \langle g, h\rangle$ with non–zero $\tau(f) \in \mathbb{Q} \cdot \mathrm{perp}(\mathbb{E}_g, \mathbb{E}_h)$.

5. Examples

In this chapter we give various examples and counterexamples. We also present a complete answer to the question when a finite set S of affine isometries of a Euclidean affine space $\mathbb{E}$ generates a discrete group for dimensions of $\mathbb{E}$ up to 3.

We use the following notation. If $\mathbb{F}$ is an affine subspace of our Euclidean affine space then the *reflection* $\sigma_{\mathbb{F}}$ in $\mathbb{F}$ is the affine map defined by

$$\sigma_{\mathbb{F}}(x+v) = x - v$$

where x is a point of $\mathbb{F}$ and v is a vector orthogonal to $T\mathbb{F}$. Every involution σ, i.e. element of order ≤ 2, in G is the reflection in the affine subspace $\operatorname{Fix}(\sigma) = \{x \in \mathbb{E} \mid \sigma(x) = x\}$.

5.1. *Every finite non–abelian simple group Γ when faithfully represented is generated by linear involutions $\sigma_{\mathbb{F}_i}$ where all the $\mathbb{F}_i$ have the same dimension.*

By the theorem of Feit–Thompson Γ has even order, so contains an involution σ by the Sylow theorems. Then Γ is generated by the set of involutions $\gamma\sigma\gamma^{-1}$, $\gamma \in \Gamma$. Note that $\gamma\sigma\gamma^{-1}$ is the reflection in the *linear* subspace $\gamma\mathbb{F}$, where $\mathbb{F}$ is the linear subspace of fixed points of σ. ■

5.2. *Let $\sigma_1 = \sigma_{\mathbb{E}_1}$ and $\sigma_2 = \sigma_{\mathbb{E}_2}$ be two involutions in G. Then $\Gamma = \langle \sigma_1, \sigma_2 \rangle$ is discrete iff*
a) $\mathbb{E}_1 \cap \mathbb{E}_2 = \emptyset$
or
b) *$\sigma_1 \cdot \sigma_2$ is of finite order.*

Proof. Necessity: If $\mathbb{E}_1$ and $\mathbb{E}_2$ intersect, say $x \in \mathbb{E}_1 \cap \mathbb{E}_2$, then Γ fixes x hence must be finite if Γ is discrete. Sufficiency: The subgroup $\Gamma_1 = \langle \sigma_1 \cdot \sigma_2 \rangle$ is of index 2 in Γ. We thus may assume that $\mathbb{E}_1 \cap \mathbb{E}_2 = \emptyset$. An affine subspace $\mathbb{F}$ of $\mathbb{E}$ is minimal Γ–invariant iff $\mathbb{F}$ is a line perpendicular to and intersecting both $\mathbb{E}_1$ and $\mathbb{E}_2$. Then $\sigma_1\sigma_2$ induces the translation by $\operatorname{perp}(\mathbb{E}_2, \mathbb{E}_2)$ on $\mathbb{F}$, which implies that Γ is discrete. ■

Here is the counterexample promised in the paragraph following Corollary 3.15.

5.3. *There is a discrete group Γ for which the set of axes $\mathbb{E}_\gamma$, $\gamma \in \Gamma$, is not locally finite.*

Let $V = V_1 \oplus V_2$, $\dim V_1 = \dim V_2 = 2$ and embed $O(V_2)$ into $O(V)$ as $\{1_{V_1}\} \times O(V_2)$. Let $\gamma_1 = A(t_i, U_i)$, $i = 1, 2$ with $t_i \in V_1$, $U_1 \in O(V_2)$ such that $\det U_1 = -1$, $U_2 \in SO(V_2)$ of finite order and $\{t_1, t_2\}$ is a basis of V_1. Then the minimal Γ–invariant subspace is $\mathbb{F} = x_0 + V_1$ and for $\gamma = \gamma_2^n \cdot \gamma_1 \cdot \gamma_2^{-n}$ we have $\mathbb{E}_\gamma = \mathbb{F} + U_2^n \operatorname{Fix}(U_1)$, an infinite set of different subspaces of $\mathbb{E}$, all containing $\mathbb{F}$. ■

5.4. *For a finite set of points $\mathbb{E}_1, \ldots, \mathbb{E}_m$ in $\mathbb{E}$ the subgroup Γ generated by $\sigma_i := \sigma_{\mathbb{E}_i}$ is discrete iff for the family of difference vectors $t_{ij} = \mathbb{E}_i - \mathbb{E}_j$,*

$1 \leq i,j \leq m$ *we have*

$$\mathrm{rank}_{\mathbb{Q}}\{t_{ij}; 1 \leq i,j \leq m\} = \mathrm{rank}_{\mathbb{R}}\{t_{ij}; 1 \leq i,j \leq m\} .$$

The group Γ generated by the σ_i's contains the group Δ of translations generated by $2t_{ij}$, $1 \leq i,j \leq m$, as a subgroup of index 2. ∎

5.5. We proceed to give a complete discussion of our problem for affine spaces $\mathbb{E}$ of small dimension, i.e. $\dim \mathbb{E} \leq 3$.

Let S be a finite subset of the affine group G of isometries of the affine Euclidean space $\mathbb{E}$. Let $\Gamma = \langle S \rangle$ and let $\mathbb{F}$ be a minimal Γ–invariant affine subspace of $\mathbb{E}$.

5.6. dim $\mathbb{E} = 1$. The group G has two connected components, one consisting of the translations by elements $t \in V = T\mathbb{E}$ and one consisting of the reflections $\sigma_{\mathbb{P}}$ in points $\mathbb{P} \in \mathbb{E}$.

The group $\Gamma = \langle S \rangle$ *for* $S = \langle t_1, \ldots, t_m\, ;\ \sigma_{\mathbb{P}_1}, \ldots, \sigma_{\mathbb{P}_r} \rangle$ *is discrete iff* $\mathrm{rank}_{\mathbb{Q}}\{t_1, \ldots, t_m\, ,\ \mathbb{P}_i - \mathbb{P}_j\, ,\ 1 \leq i,j \leq r\} \leq 1$.

The subgroup Δ of translations in Γ is generated by $\{t_1, \ldots, t_m\, ;\ 2t_{\mathbb{P}_i - \mathbb{P}_j}\, ,$ $1 \leq i,j \leq r\}$ and $[\Gamma : \Delta] \leq 2$. ∎

5.7. dim $\mathbb{E} = 2$. The group $G = \mathrm{Iso}(\mathbb{E})$ has three types of elements:

a) $\lambda(g) = 1$. Then g is a *translation* by a vector $t \in T\mathbb{E}$.
b) $\lambda(g) \in SO(V)$, $\lambda(g) \neq 1$. Then g has a unique fixed point, $\mathbb{P}$ say, and is a *rotation* by a unique angle $\varphi \in \mathbb{R}/2\pi\mathbb{Z}, \varphi \notin 2\pi\mathbb{Z}$, around $\mathbb{P}$. We write $g = D(\mathbb{P}, \varphi)$. Here we suppose that $\mathbb{E}$ is oriented.
c) $\lambda(g) \notin SO(V)$. Then g has a one–dimensional axis, $\mathbb{L}$ say, and g is of the form $t \circ \sigma_{\mathbb{L}} = \sigma_{\mathbb{L}} \circ t$ where $t \in T\mathbb{L}$ and $\sigma_{\mathbb{L}}$ is the orthogonal reflection in $\mathbb{L}$. Then g is called a *glide reflection.* Both t and $\mathbb{L}$ are uniquely determined by g.

We distinguish cases according to the dimension of invariant subspaces.

Fixed point.

Clearly, all the elements of S, and hence of Γ, have the point $\mathbb{F}$ as a common fixed point iff S contains no non–trivial translation, for all the rotations $D(\mathbb{P}, \varphi)$ in S we have $\mathbb{P} = \mathbb{F}$ and for all the glide reflections $t \circ \sigma_{\mathbb{L}}$ in S we have $\mathbb{F} \in \mathbb{L}$ and the translational part t is equal to zero.

If this is the case, then Γ *is discrete iff* Γ *is finite iff the following two conditions hold:*

a) *For every rotation* $D(\mathbb{P}, \varphi)$ *in* S *the rotation angle* φ *is commensurable with* π, *i.e.* $\varphi \in \mathbb{Q} \cdot \pi$.
b) *For any two reflections* $\sigma_{\mathbb{L}_i}$, $\sigma_{\mathbb{L}_j}$ *in* S *the angle* $\measuredangle(\mathbb{L}_i, \mathbb{L}_j)$ *is commensurable with* π.

The subgroup $\Gamma \cap \lambda^{-1}(SO)$ is of index ≤ 2 in Γ and is generated by the rotations $D(\mathbb{P}, \varphi)$ in S and the products $\sigma_{\mathbb{L}_1} \circ \sigma_{\mathbb{L}_j} = D(\mathbb{P}, 2\alpha_{i,j})$ where $\sigma_{\mathbb{L}_i}, \sigma_{\mathbb{L}_j}$ are in S and $\alpha_{i,j} = \measuredangle(\mathbb{L}_i, \mathbb{L}_j)$. ∎

Invariant line.

All the elements of S, and hence of Γ, leave the line $\mathbb{L}$ invariant iff

a) all the translations of S are in $T\mathbb{L}$
b) for every rotation $D(\mathbb{P},\varphi)$ in S we have $\mathbb{P}\in\mathbb{L}$ and $\varphi=\pi \mod 2\pi\mathbb{Z}$
c) for every glide reflection $t\circ\sigma_{\mathbb{L}'}$ in S we have
c1) $\mathbb{L}=\mathbb{L}'$ or
c2) $\mathbb{L}\perp\mathbb{L}'$ and $t=0$.

Suppose this is the case. We then have for the restriction homomorphism $r:\Gamma\to \mathrm{Iso}(T\mathbb{L})$ in the various cases:

a) $r(t)=t$ for the translation t in $T\mathbb{L}$.
b) $r(D(\mathbb{P},\pi))=\sigma_{\mathbb{P}}$ for $\mathbb{P}\in\mathbb{L}$.
c1) $r(t\circ\sigma_{\mathbb{L}})=t$ for the glide reflection $t\circ\sigma_{\mathbb{L}}$.
c2) $r(\sigma_{\mathbb{L}'})=\sigma_{\mathbb{L}\cap\mathbb{L}'}$ for the reflection in the line $\mathbb{L}'$ orthogonal to $\mathbb{L}$.

Let

$$\begin{aligned}T_a &= \{r(t); t\in S\}\\ T_{c1} &= \{r(t\circ\sigma_{\mathbb{L}}); t\circ\sigma_{\mathbb{L}}\in S\}\\ P_b &= \{\mathbb{P}; D(\mathbb{P},\pi)\in S\}\\ P_{c2} &= \{\mathbb{L}\cap\mathbb{L}'; \mathbb{L}\perp\mathbb{L}' \text{ and } \sigma_{\mathbb{L}'}\in S\}\end{aligned}$$

So T_a and T_{c1} are subsets of $T\mathbb{L}$ and P_b and P_{c2} are subsets of $\mathbb{L}$.

Then Γ is discrete iff the $\mathbb{Q}$–rank of the following subset of $T\mathbb{L}$ is ≤ 1:

$$T_a\cup T_{c1}\cup\{\mathbb{P}-\mathbb{P}';\ \textit{where } \mathbb{P} \textit{ and } \mathbb{P}' \textit{ are in } P_b\cup P_{c2}\}\ .$$

The proof follows from the case $\dim\mathbb{E}=1$ in view of the following facts: $r(\Gamma)$ is generated by $T_a\cup T_{c1}\cup\{\sigma_{\mathbb{P}};\mathbb{P}\in P_b\cup P_{c2}\}$. An element γ of the kernel of r is uniquely determined by $\lambda(\gamma)\mid(T\mathbb{L})^{\perp}$, so $\ker(r)$ has at most two elements.

Note that although the line $\mathbb{L}$ is invariant in the case under consideration it may not be a minimal Γ–invariant subspace in certain very special cases.

Γ crystallographic

The whole plane $\mathbb{E}$ is the only Γ–invariant subspace iff none of the cases described above occurs, i.e. iff there is no fixed point and no invariant line. Then Γ is crystallographic if Γ is discrete. The following well known fact is crucial for $\dim\mathbb{E}=2$:

5.8. *If Γ is crystallographic then every element of $\lambda(\Gamma)$ has order* $1,2,3,4$ *or* 6.

Proof. Every element of $O(V)$, not in $SO(V)$, has order 2. If $\lambda(\gamma)$ is in $SO(V)$, say $\lambda(\gamma)$ is rotation by the angle φ, then $\mathrm{trace}\,\lambda(\gamma)=2\cos\varphi$.

On the other hand $\lambda(\gamma)$ is represented by a matrix in $SL(2;\mathbb{Z})$ with respect to a basis of $\Delta=\Gamma\cap V$, hence $\mathrm{trace}\,\lambda(\gamma)\in\mathbb{Z}$, which immediately implies that $\varphi\equiv 0,\ \pm\pi/3,\ \pm\pi/2,\ \pm 2\pi/3,\ \pi \ \mathit{mod}\ 2\pi$. ∎

5.9. dim $\mathbb{F}$ = 2, *S finite $\subset G=\mathrm{Iso}(\mathbb{E})$. Then $\Gamma=\langle S\rangle$ is discrete if the following two conditions A and B hold. Conversely, if Γ is crystallographic then A and B hold.*

To formulate the two conditions we need the following set of notations:

$$
\begin{aligned}
&T_a = \{t \in T\mathbb{E};\ t \in S\}\\
&\mathbb{E}_b = \{\mathbb{P};\ D(\mathbb{P},\varphi) \in S \text{ and } \varphi \not\equiv 0\}\\
&\lambda_b = \{\varphi \in \mathbb{R}/2\pi\mathbb{Z};\ D(\mathbb{P},\varphi) \in S \text{ and } \varphi \not\equiv 0\}\\
&\mathbb{E}_c = \{\mathbb{L} \text{ line in } \mathbb{E};\ t \circ \sigma_{\mathbb{L}} \in S \text{ for some } t \in T\mathbb{L}\}\\
&\lambda_c = \{\sigma_{\mathbb{L}};\ t \circ \sigma_{\mathbb{L}} \in S \text{ for some } t \in T\mathbb{L}\}\\
&T_c = \{t \in T\mathbb{E};\ \text{there is a line } \mathbb{L} \text{ in } \mathbb{E} \text{ such that } t \in T\mathbb{L} \text{ and } t \circ \sigma_{\mathbb{L}} \in S\}
\end{aligned}
$$

A. (Angle Condition) *All the angles in the following set of angles are integer multiples of $\pi/6$ or all of them are integer multiples of $\pi/4$:*

$$\lambda_b \cup \{2 \cdot \measuredangle(\mathbb{L}_i, \mathbb{L}_j); \mathbb{L}_i \text{ and } \mathbb{L}_j \text{ in } \mathbb{E}_c\}\ .$$

Note that $\lambda(\Gamma)$ is generated by $\lambda_b \cup \lambda_c$ and hence $\lambda(\Gamma) \cup SO$ is generated by the set in condition A, for any $S \subset G$.

In order to formulate the translational condition we use the following notations. Let M and N be sets of affine subspaces of $\mathbb{E}$. Then we define

$$
\begin{aligned}
\operatorname{perp}(M,N) &:= \{\operatorname{perp}(\mathbb{E}_1,\mathbb{E}_2);\ \mathbb{E}_1 \in M, \mathbb{E}_2 \in N\}\\
\operatorname{perp}(M) &:= \operatorname{perp}(M,M)\\
\operatorname{feet}(M,N) &:= \{\mathbb{E}_{1,2};\ \mathbb{E}_1 \in M; \mathbb{E}_2 \in N\}\\
\operatorname{feet}(M) &:= \operatorname{feet}(M,M)
\end{aligned}
$$

where as usual the set $\mathbb{E}_{1,2}$ is the affine subset of $\mathbb{E}_1$ consisting of the feet of perpendiculars from $\mathbb{E}_2$ to $\mathbb{E}_1$.

B. (Translational Condition) *The smallest $\lambda(\Gamma)$–module containing the following subset of $T\mathbb{E}$ is discrete:*

$$T_a \cup T_c \cup \operatorname{perp}(\operatorname{feet}(\mathbb{E}_b \cup \mathbb{E}_c))\ .$$

Let us write more explicitly the last part $T_3 := \operatorname{perp}(\operatorname{feet}(\mathbb{E}_b \cup \mathbb{E}_c))$

$$\operatorname{feet}(\mathbb{E}_b \cup \mathbb{E}_c) = \mathbb{E}_b \cup \mathbb{E}_c \cup \mathbb{E}_{c,b} \cup \mathbb{E}_{c,c}$$

where $\mathbb{E}_{c,b} = \operatorname{feet}(\mathbb{E}_c, \mathbb{E}_b)$ and $\mathbb{E}_{c,c} = \operatorname{feet}(\mathbb{E}_c, \mathbb{E}_c)$ is the disjoint union of $\mathbb{E}_c$ and the set $\dot{\mathbb{E}}_{c,c}$ of points $\mathbb{P}$ such that $\{P\} = \mathbb{L}_i \cap \mathbb{L}_j$ for two non parallel lines $\mathbb{L}_i$ and $\mathbb{L}_j$ in $\mathbb{E}_c$. We thus have to consider 10 types of translations, using $\operatorname{perp}(M,N) = -\operatorname{perp}(N,M)$. Things simplify considerably in the following two special cases to which we can always reduce. Namely we assume that our set S of generators of Γ contains at most one glide reflection. This can be achieved by replacing the set of glide reflections $\gamma_1, \ldots, \gamma_r$ in S by the set $\gamma_1, \gamma_1^{-1}\gamma_2, \ldots \gamma_1^{-1}\gamma$ which does not change the group generated but $\gamma_1^{-1}\gamma_i$ is a translation or a rotation.

Case 0. S contains no glide reflection. Then

$$T_3 = \text{perp}(\mathbb{E}_b, \mathbb{E}_b) = \{\mathbb{P}_i - \mathbb{P}_j;\ \mathbb{P}_i \text{ and } \mathbb{P}_j \text{ in } \mathbb{E}_b\}$$

Case 1. S contains exactly one glide reflection, say $t \circ \sigma_{\mathbb{L}}$ with axis $\mathbb{L}$ and translational part $t \in T\mathbb{L}$.

Then $\mathbb{E}_c = \mathbb{E}_{c,c} = \{\mathbb{L}\}$, $T_c = \{t\}$ and $\mathbb{E}_{c,b}$ is the set of feet of perpendiculars from points $\mathbb{P} \in \mathbb{E}_b$ to the line $\mathbb{L}$. It is easy to see that the set $T_{b,c} \cup T_{(c,b),(c,b)}$ generates the same $\mathbb{Z}$–module as T_3 does, where

$$T_{b,c} = \text{perp}(\mathbb{E}_b, \mathbb{E}_c) \subset (T\mathbb{L})^{\perp}$$

and

$$T_{(c,b),(c,b)} = \text{perp}(\mathbb{E}_{c,b}, \mathbb{E}_{c,b} \subset T\mathbb{L}\ .$$

Thus in case 1 the translational condition B reduces to one–dimensional conditions for $T\mathbb{L}$ and $(T\mathbb{L})^{\perp}$ if the angle condition A holds. So suppose A holds. Note that in the case 1 at hand the second set

$$\{2 \cdot \measuredangle(\mathbb{L}_i, \mathbb{L}_j); \mathbb{L}_i \text{ and } \mathbb{L}_j \text{ in } \mathbb{E}_c\}$$

in condition A reduces to 0, since $\mathbb{E}_c = \{\mathbb{L}\}$. We distinguish the following three cases of the angle condition A.

A 2: *Every element of* λ_b *has oder* ≤ 2.
A 4: *There is an element in* λ_b *of order* 4.
A 6: *There is an element in* λ_b *of order* 3 *or* 6.

Let p_W be the orthogonal projection of $T\mathbb{E}$ onto the linear subspace W of $T\mathbb{E}$. Put

$$T_1 = p_{T\mathbb{L}}(T_a \cup T_{(c,b),(c,b)})$$
$$T_2 = p_{(T\mathbb{L})^{\perp}}(T_a \cup T_c \cup T_{b,c})$$

and let $U = D(0,\ \pm\pi/2)$ be one of the two elements of $SO(T\mathbb{E})$ of order 4. So U maps $(T\mathbb{L})^{\perp}$ to $T\mathbb{L}$. Then in case 1 under the hypothesis that the angle condition A holds the translational condition B is equivalent to

B, A 2: $rk_{\mathbb{Q}} T_1 \leq 1$ *and* $rk_{\mathbb{Q}} T_2 \leq 1$ *if* A 2 *holds.*
B, A 4: $rk_{\mathbb{Q}}(T_1 \cup UT_2) \leq 1$ *if* A 4 *holds.*
B, A 6: $rk_{\mathbb{Q}}(T_1 \cup \sqrt{3}UT_2) \leq 1$ *if* A 6 *holds.*

Remark . If the subgroup F of $O(V)$ contains a rotation of order 4, respectively 3, then every F–invariant lattice Θ in $V = T\mathbb{E}$ is equal to the lattice generated by the roots of the root system B_2 or BC_2, respectively A_2 or G_2, up to homothety. If we identify V with $\mathbb{C}$ then $\Theta = \mathbb{Z} + \xi\mathbb{Z}$ where $\xi = i$ resp. ξ is a primitive third (or sixth) root of unity, up to a conformal linear map, i.e. multiplication by a non–zero complex number.

5.10. Proof of 5.9. With notations as in 5.9 note that for an arbitrary subset S of G the group $F := \lambda(\Gamma)$ of $O(T\mathbb{E})$ is generated by $\lambda_b \cup \lambda_c$ and hence $F \cap SO(T\mathbb{E})$ is generated by the set $\lambda_b \cup \lambda_c\lambda_c^{-1}$ mentioned in condition A. Let

T be the F–module generated by the set $T_a \cup T_b \cup T_3$ of condition B. We embed F into G by taking as center of the rotations a point of $\mathbb{E}_b \cup \mathbb{E}_{c,c}$, this set is not empty, otherwise F is the trivial group. Then $\Gamma \subset F \ltimes T$, so conditions A and B are sufficient. Conversely, if Γ is crystallographic then every element of $F \cap SO$ is of order $1, 2, 3, 4$, or 6 and hence the set of rotations $\lambda_b \cup \lambda_c \lambda_c^{-1}$ generating this group cannot contain one of order 4 and of order 3 or 6. So A is necessary. That B is necessary follows from proposition 3.14.

The statements claimed under case 1 are seen as follows. That the $\mathbb{Z}$–module $T_{3,\mathbb{Z}}$ generated by T_3 is the same as the $\mathbb{Z}$–module generated by its subset $T_{b,c} \cup T_{(c,b),(c,b)}$ is immediate. Let Θ be the $\mathbb{Z}F$–module generated by $T_{3,\mathbb{Z}}$. Then Θ is commensurable with $p_{T\mathbb{L}}\Theta \oplus p_{(T\mathbb{L})^\perp}\Theta = T_{1,\mathbb{Z}} \oplus T_{2,\mathbb{Z}}$, since the reflection in $T\mathbb{L}$ is in F. If we are then in case Ai, $i = 2, 4$ or 6, the equivalence of B with B, Ai follows readily. ∎

Example 5.11. We are now ready to give the counterexample promised after 3.13, namely that the set of $\operatorname{perp}(\mathbb{E}_m, \mathbb{E}_n)$, m, n in M_Γ, need not be discrete. Let $\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3$ be three lines in the Euclidean plane $\mathbb{E}$ any two of which intersect at an angle of $\pi/3$ and $\mathbb{E}_1 \cap \mathbb{E}_2 \cap \mathbb{E}_3 = \emptyset$, thus forming an equilateral triangle. The group Γ generated by the reflections $\sigma_i := \sigma_{\mathbb{E}_i}$, $i = 1, 2, 3$, is crystallographic, by 5.9, Case 1, B, A 6 for $\sigma_1, \sigma_1\sigma_2, \sigma_1\sigma_3$. It is in fact the affine Weyl group $\widetilde{A}_2$. Let now $\mathbb{E}_{m_k}$ be the following sequence of iterated feet of perpendiculars, where m_k is defined inductively by $m_0 = (2, 3)$, $m_{2k-1} = (1, m_{2k-2})$, $m_{2,k} = (3, m_{2k-1})$ for $k \geq 1$. Then the sequence of points $\mathbb{E}_{m_k}$ converges to the point $\mathbb{E}_{(1,3)}$.

5.12. $\dim \mathbb{E} = 3$.

Let us fix notations. We choose an orientation on $V = T\mathbb{E}$. Every element g of $SO(V)$ fixes some line $\mathbb{L}$ and induces a rotation on the plane $\mathbb{L}^\perp$. The line $\mathbb{L}$ is unique unless $g = 1$. The sign of the rotation angle on $\mathbb{L}^\perp$ depends on the orientation of $\mathbb{L}^\perp$. Every element of $O(V)$ of determinant -1 has a line $\mathbb{L}$ as $-1-$eigenspace and induces a rotation of $\mathbb{L}^\perp$. The line $\mathbb{L}$ is unique unless $g = -1$. Again the sign of the rotation angle depends on the orientation of $\mathbb{L}^\perp$.

We have the following classification of elements g of G according to the dimension of the $+1-$eigenspace of $\lambda(g)$, i.e. the dimension of the axis of g.

a) **dim (axis) = 3**: $\lambda(g) = 1$. Then g is a *translation* by a vector $t \in V$.

c2) **dim (axis) = 2**: $\lambda(g)$ has eigenvalues $\{+1, +1, -1\}$. Then g induces a translation t on a unique plane $\mathbb{F}$ in $\mathbb{E}$. We have $t \in T\mathbb{F}$ and $g = \sigma_{\mathbb{F}} \circ t = t \circ \sigma_{\mathbb{F}}$, where $\sigma_{\mathbb{F}}$ is the orthogonal reflection in the plane $\mathbb{F}$. So g is a *glide reflection.*

b) **dim (axis) = 1**: $\lambda(g) \in SO(V)$, $\lambda(g) \neq 1$. Then g induces a translation t on a unique line $\mathbb{L}$. Then $t \in T\mathbb{L}$ and

$$g(x + v) = x + t + \lambda(g)v$$

for $x \in \mathbb{L}$ and $v \in T\mathbb{L}^{\perp}$. We denote $g = \mathrm{Screw}(\mathbb{L}, t, \varphi)$ if φ is the rotation angle where we orient $\mathbb{L}^{\perp}$ so that (t, e_1, e_2) is an oriented basis of V if (e_1, e_2) is an oriented basis of $\mathbb{L}^{\perp}$. This makes sense unless $t = 0$ in which case we write $g = \mathrm{Screw}(\mathbb{L}, 0, \pm\varphi)$. This ambiguity of φ causes no problems since we are only interested in groups containing g. So in case b) g is a *screw motion* along the line $\mathbb{L}$ for $t \neq 0$ or a *rotation* around the line $\mathbb{L}$.

c0) **dim (axis)=0**: the number $+1$ is not an eigenvalue of $\lambda(g)$. Then $\det \lambda(g) = -1$, g has a unique fixed point, say $\mathbb{P}$, and

$$g(\mathbb{P} + v) = \mathbb{P} + Uv \qquad \text{for } v \in V$$

where $U = \lambda(g) \in O(V)$. We denote $g = D(\mathbb{P}, U)$. So we can think of g as a *linear orthogonal map* U if we take $\mathbb{P}$ as the origin of a linear coordinate system.

Let now S be a finite subset of G and let $\Gamma = \langle S \rangle$. Again, we discuss the cases according to the dimension of invariant subspaces.

Fixed point

The point $\mathbb{P}$ of $\mathbb{E}$ is a common fixed point for Γ iff S contains only elements with translational part zero and axis containing $\mathbb{P}$. If this is the case we can think of $\mathbb{P}$ as the origin of the vector space $V = \mathbb{E}$ and $\Gamma \subset O(V)$. And Γ is discrete iff Γ is finite.

Invariant line

Let $\mathbb{L}$ be a line in $\mathbb{E}$. The subgroup $G_{\mathbb{L}} = \{g \in G; g\mathbb{L} = \mathbb{L}\}$ is isomorphic to $\mathrm{Iso}(\mathbb{L}) \times O((T\mathbb{L})^{\perp})$ via $g \mapsto (g \mid \mathbb{L}\ ,\ \lambda(g) \mid (T\mathbb{L})^{\perp})$. Choosing coordinates we can thus identify $G_{\mathbb{L}}$ with $G_1 \times O(2)$, where G_1 is the group of (affine) isometries of the Euclidean line $\mathbb{R}$. The group of connected components of $G_1 \times O(2)$ can be identified with $\{\pm 1\} \times \{\pm 1\}$ via Det: $G_1 \times O(2) \to \{\pm 1\} \times \{\pm 1\}$, $\mathrm{Det}(g, A) = (\lambda(g), \det A)$.

Now suppose S is a finite subset of $G_1 \times O(2)$ then it is easy to find a set S_0 of generators for the intersection Γ_0 of $\Gamma = \langle S \rangle$ with the connected component $\mathbb{R} \times SO(2)$ of $G_1 \times O(2)$. A wasteful but short way to write down such set is

$$S_0 = \{\gamma_1 \gamma_2 \gamma_3^{-1}; \gamma_i \in S_{\delta_i, \varepsilon_i}\ ,\ \delta_3 = \delta_1 \delta_2\ ,\ \varepsilon_3 = \varepsilon_1 \varepsilon_2\}$$

where $S_{\delta, \varepsilon} = \{\gamma \in S \cup \{e\};\ \mathrm{Det}(\gamma) = (\delta, \varepsilon)\}$.

Finally, let $e: \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \times SO(2)$, $e(t,\varphi) = (t, e^{i\varphi})$, be the exponential map of $\mathbb{R} \times SO(2)$. Then
Γ *is discrete iff* $\operatorname{rank}_{\mathbb{Q}} e^{-1}(S_0) \leq 2$
Γ *is finite iff* $\operatorname{rank}_{\mathbb{Q}} e^{-1}(S_0) \leq 1$.

The proof follows immediately from proposition 4.4.

Let us make explicit what it means for an element $g \in G$ in the classification above to leave the line $\mathbb{L}$ invariant.

a) A translation t leaves $\mathbb{L}$ invariant iff $t \in T\mathbb{L}$.
b) A screw motion or rotation g leaves $\mathbb{L}$ invariant iff
   b1) g has axis $\mathbb{L}$, or
   b2) g is a rotation by the angle π around an axis $\mathbb{L}'$, which intersects $\mathbb{L}$ orthogonally.
c0) An orthogonal transformation $g = D(\mathbb{P}, U)$ with unique fixed point $\mathbb{P}$ leaves $\mathbb{L}$ invariant iff $\mathbb{P} \in \mathbb{L}$ and $T\mathbb{L}$ is contained in the $-1-$eigenspace of U.
c2) A glide reflection $g = t \circ \sigma_{\mathbb{F}}$ leaves $\mathbb{L}$ invariant iff
   c2.1 $\mathbb{L} \subset \mathbb{F}$ and $t \in T\mathbb{L}$, or
   c2.2 $t = 0$ and $\mathbb{F}$ intersects $\mathbb{L}$ orthogonally

We summarize the information in the following table.

	$\lambda(g) \mid T\mathbb{L}$	$\det U$	eigenvalues of $U = \lambda(g) \mid (T\mathbb{L})^{\perp}$	translational part $t \in T\mathbb{L}$	dim(axis)	axis $\mathbb{A}$	type
a	1	1	$\{1,1\}$		3	$\mathbb{E}$	translation
b1	1	1	$e^{\pm i\varphi} \neq 1$		1	$\mathbb{L}$	screw motion or rotation around $\mathbb{L}$
b2	-1	-1	$\{+1,-1\}$	0	1	*)	reflection inline $\mathbb{A}$
c0	-1	1	$e^{\pm i\varphi} \neq 1$	0	0	$\in \mathbb{L}$	"linear" orthogonal map
c2.1	1	-1	$\{+1,-1\}$		2	$\supset \mathbb{L}$	glide reflection
c2.2	-1	1	$\{+1,+1\}$	0	2	*)	reflection in plane $\mathbb{A}$

*) $\mathbb{A} \cap \mathbb{L} \neq \emptyset$, $\mathbb{A} \perp \mathbb{L}$

Invariant plane

Let $\mathbb{F}$ be a plane in $\mathbb{E}$. Then the group $G_{\mathbb{F}} = \{g \in G; g\mathbb{F} = \mathbb{F}\}$ is isomorphic to $\operatorname{Iso}(\mathbb{F}) \times O(T\mathbb{F})^{\perp})$ via $g \mapsto (g \mid \mathbb{F}, \lambda(g) \mid T\mathbb{F})$. So a subgroup Γ of $G_{\mathbb{F}}$ is discrete iff $r(\Gamma)$ is discrete, where $r : G_{\mathbb{F}} \to \operatorname{Iso}(\mathbb{F})$ is the restriction homomorphism. We are thus reduced to the 2–dimensional case.

Γ crystallographic

The whole space $\mathbb{E}$ is the only Γ–invariant affine subspace if none of the cases above occurs, i.e. if there is no invariant point, line or plane. Then Γ is crystallographic if Γ is discrete. The following well known facts are crucial.

5.13. *If* Γ *is crystallographic then every element of* $\lambda(\Gamma)$ *has order* $1, 2, 3, 4$ *or* 6.

The **proof** is very similar to the 2-dimensional case. One eigenvalue of $\lambda(\gamma)$ is $\det \lambda(\gamma) \in \{\pm 1\}$,the other two are $e^{\pm i\varphi}$. We have $\operatorname{trace} \lambda(\gamma) = 2\cos\varphi \pm 1 \in \mathbb{Z}$, hence $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ has order p, where p is one of the numbers of the claim. So the order of $\lambda(\gamma)$ is p, if $\det \lambda(\gamma) = +1$, and the least common multiple of 2 and p if $\det \lambda(\gamma) = -1$. ∎

5.14. *A finite subgroup of* $SO(3)$ *is either contained in a dihedral group or an octahedral, a tetrahedral or an icosahedral group.*

Here the latter are the groups of elements of $SO(3)$ preserving a regular octahedron, tetrahedron or icosahedron, respectively. The corresponding groups are isomorphic to S_4, A_4 and A_5, respectively. These two facts viz. 5.13 and 5.14, together leave only very few cases for $\lambda(\Gamma)$ if Γ is crystallographic. In particular, the icosahedral group is impossible. One can hence discuss the various cases according to the possible groups $\lambda(\Gamma)$. We will not go into the details.

Remark 5.15. For hyperbolic groups of motions the following result is due to Jørgensen [9]. A non–elementary group Γ of motions of the hyperbolic space of dimension at most 3 is discrete if every two generator subgroup is discrete. This is not true for subgroups Γ of the group G of affine isometries of a Euclidean space $\mathbb{E}$. This is already false for subgroups Θ of the group V of translations of $\mathbb{E}$. E.g. if $\dim V = 2$ and Θ is the $\mathbb{Z}$–module in V generated by $(e_1, e_2, \alpha_1 e_1 + \alpha_2 e_2)$ where (e_1, e_2) is a basis of V and α_1, α_2 are real numbers such that $(1, \alpha_1, \alpha_2)$ are linearly independent over $\mathbb{Q}$, then Θ is not discrete but every two generator subgroup of Θ is discrete since any two elements of Θ are either linearly dependent over $\mathbb{Q}$ or linearly independent over $\mathbb{R}$.

The question when two elements of $PSL(2, \mathbb{R})$ generate a discrete group is difficult, but it has been answered, see [8].

References

[1] Abels, H. and A. Dress, *An algebraic version of a theorem of L. Bieberbach, concerning invariant subspaces of discrete isometry groups*, J. Algebra, **99** (1986), 496–517.

[2] Bieberbach, L., *Über die Bewegungsgruppen des n–dimensionalen euclidischen Raumes mit einem endlichen Fundamentalbereich*, Nachr. Königl. Ges. Wiss. Göttingen. Math. phys. Klasse (1910), 75–84.

[3] —, *Über die Bewegungsgruppen der Euklidischen Räume (Erste Abhandlung)*, Math. Ann. **70** (1911), 297–336.

[4] —, *Über die Bewegungsgruppen der Euklidischen Räume (Zweite Abhandlung). Die Gruppen mit einem endlichen Fundamentalbereich*, Math. Ann. **72** (1912), 400–412.

[5] Brown, H., R. Bülow, J. Neubüser, H. Wondratschek and H. Zassenhaus, "Crystallographic groups of four-dimensional space", Wiley, New York, 1978.

[6] Dress, A., *A combinatorial theory of Grünbaum's new regular polyhedra. Part II: complete enumeration*, Aequationes Math. **29** (1985), 222–243.

[7] Frobenius, G., *Über die unzerlegbaren diskreten Bewegungsgruppen*, Sitzber. Preuß. Akad. Wiss. Berlin (Ges. Abh. Nr. 89, Band III 507–518) (1911), 654–665.

[8] Gilman, J., "Two–generator discrete subgroups of PSL(2,$\mathbf{R}$)", Mem. Amer. Math. Soc. **117** (1995).

[9] Jørgensen, T., *A note on subgroups of* SL(2,$\mathbf{C}$), Quart. J. Math. Oxford Ser. (2) **28** (1977), 209–211.

[10] Vinberg, E. B., and O. V. Shvartsman, "Discrete groups of motions of spaces of constant curvature", Encyclopedia of Mathematical Sciences **29**, Geometry II, Springer-Verlag 1993, English translation of Itogi nauki i tekhniki, Sovremennye problemy matematiki, Fundamental'nye napravleniya, **29**, Geometriya 2, Publisher VINITI, Moscow 1988.

[11] Wolf, J. A., "Spaces of constant curvature", Publish or Perish Press, Fourth ed. 1977.

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