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Degenerations of nilpotent Lie algebras

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Abstract. In this paper we study degenerations of nilpotent Lie algebras. If λ, μ are two points in the variety of nilpotent Lie algebras, then λ is said to degenerate to μ, $\lambda \rightarrow_{\deg} \mu$, if μ lies in the Zariski closure of the orbit of λ. It is known that all degenerations of nilpotent Lie algebras of dimension $n < 7$ can be realized via a one-parameter subgroup. We construct degenerations between characteristically nilpotent filiform Lie algebras. As an application it follows that for any dimension $n \geq 7$ there exist examples of degenerations of nilpotent Lie algebras which cannot be realized via a one–parameter subgroup.

1. Introduction

Let V be a vector space of dimension n over a field K. An n–dimensional Lie algebra $\mathfrak{g}$ may be considered as an element λ of the affine variety $\operatorname{Hom}(\Lambda^2 V, V)$ via the bilinear skew–symmetric mapping $\lambda : \mathfrak{g} \otimes \mathfrak{g} \mapsto \mathfrak{g}$ defining the Lie bracket on $\mathfrak{g}$. The set of Lie algebra structures is an algebraic subset $\mathcal{L}_n$ of the variety $\operatorname{Hom}(\Lambda^2 V, V)$ and the linear reductive group $\mathbf{GL}_n(K)$ acts on $\mathcal{L}_n$ by $(g * \mu)(x, y) = g(\mu(g^{-1}(x), g^{-1}(y)))$. The orbits under this action are the isomorphism classes. We say that λ *degenerates* to μ, if μ is in $\overline{O(\lambda)}$, the Zariski closure of the orbit of λ. We denote this by $\lambda \rightarrow_{\deg} \mu$. The degeneration is nontrivial if μ lies in the boundary of $O(\lambda)$. In this paper we consider the $\mathbf{GL}_n(K)$–stable subvariety $\mathcal{N}_n$ of $\mathcal{L}_n$ consisting of nilpotent Lie algebras. In particular, we deal with the open subset $\mathcal{F}_n$ of $\mathcal{N}_n$ consisting of filiform nilpotent Lie algebras. If not otherwise declared we will assume $K = \mathbb{C}$. The determination and classification of orbit closures is a difficult problem. An important tool is the study of degenerations via a one–parameter subgroup, in short 1–PSG. Here there are many results known for the action of a linear reductive group on an algebraic variety (e.g., the theorems of Hilbert, Kraft and Mumford). In [6] there is discussed whether every degeneration of A–modules can be obtained via a 1–PSG. (See the remark following the theorem of Hilbert–Mumford–Birkes, p. 232. Here A is a finitely generated associative $\mathbb{C}$–algebra.) It is shown that for two A–modules $M, N \in \operatorname{mod}_{A,V}$ there is a 1–PSG $g : \mathbb{C}^* \rightarrow \mathbf{GL}(V)$ with

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$\lim_{t\to 0} g_t \cdot M = N$ if and only if there is a filtration on M such that the associated graded module is isomorphic to N. In fact, there exist degenerations of A–modules which cannot be realized via a 1–PSG. This question is more difficult for the variety of nilpotent Lie algebras. It is known that all degenerations of nilpotent Lie algebras of dimension $n < 7$ can be realized via a one–parameter subgroup. In higher dimensions this was not known. In [4] a criterion is proved deciding whether a nilpotent Lie algebra is a degeneration of some other nilpotent Lie algebra via a one–parameter subgroup. It is the analog of Kraft's criterion:
If μ is a degeneration of λ via a one–parameter subgroup g_t, then μ is the associated graded Lie algebra given by the filtration on λ induced by g_t.
Also the converse holds. We will establish nontrivial degenerations between characteristically nilpotent Lie algebras. By the above criterion such a degeneration cannot be realized via a one–parameter subgroup: If λ, μ are characteristically nilpotent Lie algebras, then they do not admit such a gradation. It is natural to look for nilpotent Lie algebras of maximal nilpotence index, i.e., filiform Lie algebras. We construct *column–degenerations* of filiform nilpotent Lie algebras. These are the main degenerations $\lambda \to_{\text{deg}} \mu$ which are possible such that λ and μ are both filiform nilpotent.

2. Preliminaries

A Lie algebra $\mathfrak{g}$ over a field K determines a multiplication table relative to each basis of $\mathfrak{g}$. If $[e_i, e_j] = \sum_{k=1}^n \gamma_{ij}^k e_k$, then $(\gamma_{ij}^k) \in K^{n^3}$ is called a *structure* for $\mathfrak{g}$. A point in $\mathcal{L}_n$ is a Lie algebra structure which can be identified with the bilinear skew–symmetric mapping $\lambda : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ defining the Lie bracket on $\mathfrak{g}$. Since the Jacobi identity and the antisymmetry are defined by polynomial conditions, i.e., by $(n^3 - n)/6$ algebraic equations, $\mathcal{L}_n$ is an affine algebraic subvariety of $\operatorname{Hom}(\Lambda^2 V, V)$. $\mathbf{GL}_n(K)$ acts on $\mathcal{L}_n$ via change of basis, i.e. by $(g * \mu)(x, y) = g(\mu(g^{-1}(x), g^{-1}(y)))$. An orbit $O(\mu)$ under this action consists of all structures in a single isomorphism class.

Definition 1. A Lie algebra λ is said to *degenerate* to another Lie algebra μ, if μ is represented by a structure which lies in the Zariski closure of the $\mathbf{GL}_n(K)$–orbit of a structure which represents λ. In this case the entire orbit $O(\mu)$ lies in the closure of $O(\lambda)$. We denote this by $\lambda \to_{\text{deg}} \mu$.

Degeneration (for isomorphism classes) is transitive: If $\lambda \to_{\text{deg}} \mu$ and $\mu \to_{\text{deg}} \nu$, then $\lambda \to_{\text{deg}} \nu$. Let $K = \mathbb{C}$. It is known that the usual analytic topology on $\mathbb{C}^{n^3}$ leads to the same degenerations as does the Zariski topology. Therefore the following condition will imply that $\lambda \to_{\text{deg}} \mu$:

$$\exists\, g_t \in \mathbf{GL}_n(\mathbb{C}(t)) \quad \text{such that} \quad \lim_{t\to 0} g_t * \lambda = \mu$$

Here $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.

Example 1. Any n–dimensional Lie algebra λ degenerates to the abelian Lie algebra K^n which corresponds to $\mathbf{0} \in \operatorname{Hom}(\Lambda^2 V, V)$: Let $g_t = t^{-1} E_n$, where

E_n is the identity matrix. Then we have $(g_t * \lambda)(x,y) = t^{-1}\lambda(tx,ty) = t\lambda(x,y)$, hence $\lim_{t\to 0} g_t * \lambda = \mathbf{0}$. ∎

Let $Z(\lambda)$ denote the center of the Lie algebra λ and $[\lambda,\lambda]$ the commutator subalgebra. To construct degenerations we have to respect the following necessary conditions:

Lemma 1. *Let $\lambda \to_{\deg} \mu$ be a nontrivial degeneration. Then it follows that:*

(1) $\dim \operatorname{Der}(\lambda) < \dim \operatorname{Der}(\mu)$
(2) $\dim O(\lambda) > \dim O(\mu)$
(3) $\dim[\lambda,\lambda] \geq \dim[\mu,\mu]$
(4) $\dim Z(\lambda) \leq \dim Z(\mu)$
(5) $\operatorname{rank}(\lambda) \leq \operatorname{rank}(\mu)$

If λ is solvable of step k, then μ is solvable of step $\leq k$. The same holds for nilpotent. In that case, $\dim \lambda^{(i)} \geq \dim \mu^{(i)}$ where $\lambda^{(1)} = \lambda, \lambda^{(i+1)} = [\lambda, \lambda^{(i)}]$. ∎

Note that $\dim O(\lambda) = (\dim \lambda)^2 - \dim \operatorname{Der}(\lambda)$. The proof can be found more or less in the literature, see [1], [7]. The main argument relies on the following fact: Let B be a Borel subgroup of $\mathbf{GL}_n(K)$ and $\lambda, \mu \in \mathcal{N}_n$. If $\lambda \to_{\deg} \mu$ and λ lies in a B–stable closed subset $\mathcal{R} \subset \mathcal{N}_n$, then μ must also be represented by a structure in $\mathcal{R}$.
We remark that a Borel subgroup plays an important role for degenerations. We will use also Borel subgroups in Proposition 3. The following fact is proved in [4]: Let G be a complex reductive algebraic group acting rationally on some algebraic set X. Let B be a Borel subgroup of G. Then $\overline{G * x} = G * \overline{(B * x)}$ for all $x \in X$.

Definition 2. A degeneration $\lambda \to_{\deg} \mu$ is called a *one–parameter subgroup degeneration* (1–PSG) if it can be realized by a group homomorphism $g : K^* \to \mathbf{GL}_n(K)$ such that $\mu = \lim_{t\to 0} g_t * \lambda$.

The notion of a 1–PSG degeneration is independent of the choice of a basis. We have the following criterion [4]:

Proposition 1. *If $\lambda \to_{\deg} \mu$ via a 1–PSG then μ is the associated $\mathbb{Z}$–graded Lie algebra given by the filtration on λ induced by g_t. Conversely, if μ is the associated graded Lie algebra given by some filtration of λ then $\lambda \to_{\deg} \mu$ via a 1–PSG.* ∎

Example 2. Let $K = \mathbb{C}$. Every degeneration $\lambda \to_{\deg} \mu$ of nilpotent Lie algebras of dimension $n < 7$ can be obtained via one–parameter subgroups [4], [7]. Every degeneration in $\mathcal{L}_3$ can be realized via a 1–PSG, but not in $\mathcal{L}_4$ [1]. ∎

Definition 3. A Lie algebra $\mathfrak{g}$ is called *characteristically nilpotent* if all its derivations are nilpotent.

Lemma 2. *Let $\mathfrak{g}$ be a characteristically nilpotent Lie algebra. Then $\mathfrak{g}$ is nilpotent and admits no nontrivial $\mathbb{Z}$–gradation.*

Proof. The first part follows by Engel's theorem since all inner derivations $\mathrm{ad}(x)$ are nilpotent. Suppose that $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$ is a gradation. Then we can construct a derivation by $D(x_i) = ix_i$ for all $x_i \in \mathfrak{g}_i$ which clearly is not nilpotent. This is a contradiction. ■

Characteristically nilpotent algebras form a relatively large subclass of nilpotent Lie algebras. In [3] the following result is proved:

Proposition 2. *For $n \geq 8$ any irreducible component of the variety $\mathcal{F}_n$ contains a nonempty Zariski–open subset of characteristically nilpotent Lie algebras.* ■

Finally, let us mention the connection between degenerations and deformations. Gerstenhaber's definition of a deformation is as follows: Let V be a vector space over K and $\lambda \in \mathrm{Hom}(\Lambda^2 V, V)$ be a Lie product. If $\lambda_t := \lambda + t\phi_1 + t^2\phi_2 + t^3\phi_3 + \ldots$ is a Lie product on $V \otimes_K K((t))$ where $K((t))$ is the formal power series field and $\phi_i \in \mathrm{Hom}(\Lambda^2 V, V)$, then λ_t is called a *deformation* of λ.

A more general definition is given in [2]. This paper deals with the connection of degenerations and deformations. It is proved that every nontrivial degeneration $\lambda \to_{\mathrm{deg}} \mu$ defines a nontrivial deformation of μ. The converse is not true. However, there are special classes of deformations which do define degenerations. In the following we will consider infinitesimal deformations of filiform Lie algebras which define, under certain conditions, degenerations between filiform Lie algebras.

3. Degenerations of filiform Lie algebras

We will construct degenerations between filiform Lie algebras in any dimension $n \geq 7$ which cannot be realized via 1–PSGs.

Let $L = L(n)$ be the standard graded filiform Lie algebra generated by $e_1, \ldots, e_n$ with Lie brackets $[e_1, e_i] = e_{i+1}$ for $i = 2, \ldots, n-1$. Denote by $H^2(L, L)$ the second Lie algebra cohomology of L with adjoint coefficients and by $\{F_k H^2(L, L)\}_{k\in\mathbb{Z}}$ the canonical filtration of this space [5]. Let ψ be an integrable 2–cocycle and define the Lie algebra L_ψ by $[x, y]_\psi = [x, y]_L + \psi(x, y)$. There is a canonical basis for $H^2(L, L)$. Define the index set $\mathcal{I}_n^0 := \{(k, s) \in \mathbb{N}^2 \mid 2 \leq k \leq [n/2],\ 2k + 1 \leq s \leq n\}$ and let

$$\mathcal{I}_n = \begin{cases} \mathcal{I}_n^0, & n \equiv 1(2) \\ \mathcal{I}_n^0 \cup \{\left(\frac{n}{2}, n\right)\}, & n \equiv 0(2) \end{cases}$$

The space $F_1 H^2(L, L)$ has a canonical basis consisting of the cohomology classes of the 2–cocycles $\psi_{k,s}$ for $(k, s) \in \mathcal{I}_n$. The 2–cocycles are defined by $\psi_{k,s}(e_i, e_{i+1}) = \delta_{ik} e_s$. We have

$$\dim F_1 H^2(L, L) = \begin{cases} \dfrac{(n-3)^2}{4}, & n \equiv 1(2) \\ \dfrac{(n-4)(n-2)+4}{4}, & n \equiv 0(2) \end{cases}$$

The following result is known [5]:

Lemma 3. *Every filiform nilpotent Lie algebra of dimension $n \geq 3$ is isomorphic to an infinitesimal deformation L_ψ of L where ψ is an integrable 2–cocycle whose cohomology class lies in $F_1H^2(L, L)$.* ∎

It follows that we can obtain a special form for the structure constants γ_{ij}^k of a filiform Lie algebra:

Lemma 4. *Let $\mathfrak{g}$ be a complex filiform nilpotent Lie algebra of dimension n. Then there exists a basis $\{e_1, \ldots, e_n\}$ such that*

(a) $[e_1, e_i] = e_{i+1}$ *for* $i = 2, \ldots, n-1$

(b) *The structure constants in* $[e_i, e_j] = \sum_k \gamma_{ij}^k e_k$, $2 \leq i < j$ *can be written as*

$$\gamma_{ij}^k = \sum_{\ell=0}^{[(j-i-1)/2]} (-1)^\ell \binom{j-i-\ell-1}{\ell} \alpha_{i+\ell}^{k-j+i+2\ell+1}$$

where the constants α_i^j are zero for all pairs (i, j) not in $\mathcal{I}_n$. ∎

There are $(n-3)^2/4$ structure constants α_i^j if n is odd, and $\frac{1}{4}(n-2)(n-4)+1$ otherwise. The formula above yields a convenient way to describe filiform Lie algebras. The Jacobi identity is not satisfied automatically, unless $n < 8$. However, the polynomial conditions are much simpler with respect to the above basis.

We may represent such a filiform Lie algebra by a diagram of the structure constants: Let $N = [\frac{n}{2}]$:

$$\begin{array}{cccccc}
\boxed{\alpha_2^5} & \boxed{\alpha_2^6} & \boxed{\cdots} & \boxed{\cdots} & \boxed{\alpha_2^{n-1}} & \boxed{\alpha_2^n} \\
\boxed{\alpha_3^7} & \boxed{\alpha_3^8} & \boxed{\cdots} & \boxed{\alpha_3^n} & & \\
\boxed{\cdots} & \boxed{\cdots} & & & & \\
\boxed{\alpha_N^n} & & & & &
\end{array}$$

Denote the columns of that diagram from the left by $A_{n-4}, A_{n-3}, \ldots, A_2, A_1$.

Example 3. Let $\mathfrak{g}$ be a complex filiform Lie algebra of dimension 9. Then there is a basis $\{e_1, \dots, e_9\}$ such that

$$\begin{aligned}
[e_1, e_i] &= e_{i+1},\ i \geq 2 \\
[e_2, e_3] &= \alpha_1 e_5 + \alpha_2 e_6 + \alpha_3 e_7 + \alpha_4 e_8 + \alpha_5 e_9 \\
[e_2, e_4] &= \alpha_1 e_6 + \alpha_2 e_7 + \alpha_3 e_8 + \alpha_4 e_9 \\
[e_2, e_5] &= (\alpha_1 - \alpha_6) e_7 + (\alpha_2 - \alpha_7) e_8 + (\alpha_3 - \alpha_8) e_9 \\
[e_2, e_6] &= (\alpha_1 - 2\alpha_6) e_8 + (\alpha_2 - 2\alpha_7) e_9 \\
[e_2, e_7] &= (\alpha_1 - 3\alpha_6 + \alpha_9) e_9 \\
[e_3, e_4] &= \alpha_6 e_7 + \alpha_7 e_8 + \alpha_8 e_9 \\
[e_3, e_5] &= \alpha_6 e_8 + \alpha_7 e_9 \\
[e_3, e_6] &= (\alpha_6 - \alpha_9) e_9 \\
[e_4, e_5] &= \alpha_9 e_9
\end{aligned}$$

where $\mathfrak{g}$ depends on 9 parameters $\{\alpha_k^s \mid (k,s) \in \mathcal{I}_9\} = \{\alpha_1, \dots, \alpha_9\}$. The Jacobi identity for $\mathfrak{g}$ is equivalent to $\alpha_9(2\alpha_1 + \alpha_6) - 3\alpha_6^2 = 0$. The diagram is given by

α_1	α_2	α_3	α_4	α_5
α_6	α_7	α_8		
α_9				

■

Let λ be a complex filiform Lie algebra of dimension n with Lie brackets given in Lemma 4, i.e., λ is an infinitesimal deformation of L, the standard graded filiform. The question is whether there exists a degeneration $\lambda \to_{\deg} L$. In general, the answer is no – see the Lie algebra of example 1.5. in [2], which is the filiform Lie algebra λ with $n = 7$. The bracket $[e_2, e_4]$ given there should be equal to $\beta e_6 + \gamma e_7$.
However, under certain conditions one can degenerate "by columns", and iterating this process will yield a degeneration to L.

Definition 4. Let λ be as above and $k \in \{2, \dots, n-5\}$. Suppose that the columns $A_{n-4}, \dots A_{k+2}$ of the diagram of λ have only zero entries, but A_{k+1} has a nonzero entry. Let $\mathbf{T}_n(\mathbb{C}(t))$ denote the Borel subgroup of $\mathbf{GL}_n(\mathbb{C}(t))$ consisting of lower–triangular matrices and define a matrix $g_{t,k}^{-1} \in \mathbf{T}_n(\mathbb{C}(t))$ by

$$\begin{aligned}
g_{t,k}^{-1}(e_1) &= t e_1 \\
g_{t,k}^{-1}(e_i) &= t^{n+i-4-k} \Big(e_i + \sum_{j=1}^{k-1} f_j(s) e_{i+j+1} \Big),\ i \geq 2
\end{aligned}$$

with polynomial functions $f_i(s) \in \mathbb{C}(\alpha_k^l)[s]$ in the variable $s = 1 - t$ and coefficients in $\mathbb{C}(\alpha_k^l)$. The inverse matrix $g_{t,k}$ is called the *column degeneration matrix of level k.*

This matrix will realize a degeneration $\lambda \to_{\deg} \mu_{0,k}$ to a filiform Lie algebra $\mu_{0,k}$ as follows: Under certain conditions on the polynomials f_i the Lie algebra $\mu_{t,k} := g_{t,k} * \lambda$ will have the *same diagram* as λ except for the entries of the first column A_{k+1} multiplied by t. In that case $\mu_{0,k} := \lim_{t\to 0}\mu_{t,k}$ (which is just $\mu_{t,k}$ for $t = 0$) is a degeneration of λ. Then the columns $A_{n-4},\dots,A_{k+1}$ of the diagram of $\mu_{0,k}$ are zero. There are of course necessary conditions on such polynomials f_i, e.g., the properties of Lemma 1. In particular, $\dim \mathrm{Der}(\lambda) < \dim \mathrm{Der}(\mu_{0,k})$ must be satisfied. If $k = 2$ then these conditions are obvious and we obtain nontrivial degenerations between filiform Lie algebras of any dimension $n \geq 7$:

Proposition 3. *Let λ be a complex filiform Lie algebra of dimension $n \geq 7$ with basis $\{e_1,\dots,e_n\}$, $\alpha = \alpha_2^{n-2}, \beta = \alpha_2^{n-1}, \gamma = \alpha_2^n, \delta = \alpha_3^n$ and defining brackets*

$$\begin{aligned}
\lambda(e_1,e_i) &= e_{i+1},\ i\geq 2\\
\lambda(e_2,e_3) &= \alpha e_{n-2}+\beta e_{n-1}+\gamma e_n\\
\lambda(e_2,e_4) &= \alpha e_{n-1}+\beta e_n\\
\lambda(e_2,e_5) &= (\alpha-\delta)e_n\\
\lambda(e_3,e_4) &= \delta e_n
\end{aligned}$$

and let $s = 1-t$ and $f(s) = \frac{\gamma}{2\delta}s$ with $\delta \neq 0$. Define $g_t \in \mathbf{T}_n(\mathbb{C}(t))$ by

$$\begin{aligned}
g_t(e_1) &= \frac{1}{t}e_1\\
g_t(e_i) &= \sum_{j=0}^{[(n-i)/2]} (-1)^j \frac{f(s)^j}{t^{n+i+2j-6}} e_{i+2j},\ i\geq 2
\end{aligned}$$

Then $\mu_t := g_t * \lambda$ is a filiform Lie algebra with brackets

$$\begin{aligned}
\mu_t(e_1,e_i) &= e_{i+1},\ i\geq 2\\
\mu_t(e_2,e_3) &= t\alpha e_{n-2}+\beta e_{n-1}+\gamma e_n\\
\mu_t(e_2,e_4) &= t\alpha e_{n-1}+\beta e_n\\
\mu_t(e_2,e_5) &= t(\alpha-\delta)e_n\\
\mu_t(e_3,e_4) &= t\delta e_n
\end{aligned}$$

Therefore λ degenerates to $\mu := \lim_{t\to 0}\mu_t$: $\begin{array}{|c|c|c|}\hline \alpha & \beta & \gamma \\ \hline \delta & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \longrightarrow_{\deg} \begin{array}{|c|c|c|}\hline 0 & \beta & \gamma \\ \hline 0 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}$

Proof. Note that the Jacobi identity for λ is satisfied. The inverse of g_t is given by

$$\begin{aligned}
g_t^{-1}(e_1) &= te_1\\
g_t^{-1}(e_i) &= t^{n+i-6}(e_i+f(s)e_{i+2}),\ i\geq 2
\end{aligned}$$

We have to compute the Lie brackets of μ_t. The crucial ones are the following:

$$\begin{aligned}
\mu_t(e_1,e_i) &= g_t(\lambda(g_t^{-1}(e_1),g_t^{-1}(e_i))) = g_t(\lambda(te_1,t^{n+i-6}e_i+f(s)t^{n+i-6}e_{i+2}))\\
&= t^{n+i-5}\Big(\sum_{j=0}^{[(n-i-1)/2]}(-1)^j\frac{f(s)^j}{t^{n+i+2j-5}}e_{i+1+2j}\\
&\quad + \sum_{j=0}^{[(n-i-3)/2]}(-1)^j\frac{f(s)^{j+1}}{t^{n+i+2(j+1)-5}}e_{i+1+2(j+1)}\Big) = e_{i+1}
\end{aligned}$$

$$\begin{aligned}\mu_t(e_2,e_3)&=g_t(\lambda(g_t^{-1}(e_2),g_t^{-1}(e_3)))\\&=g_t(\lambda(t^{n-4}e_2+f(s)t^{n-4}e_4,t^{n-3}e_3+f(s)t^{n-3}e_5))\\&=t^{2n-7}g_t(\alpha e_{n-2}+\beta e_{n-1}+(\gamma+(\alpha-2\delta)f(s))e_n)\\&=t^{2n-7}\Big(\frac{\alpha}{t^{2n-8}}e_{n-2}-\frac{\alpha f(s)}{t^{2n-6}}e_n+\frac{\beta}{t^{2n-7}}e_{n-1}+\frac{\gamma+(\alpha-2\delta)f(s)}{t^{2n-6}}e_n\Big)\\&=t\alpha e_{n-2}+\beta e_{n-1}+\frac{\gamma-\gamma(1-t)}{t}e_n\\&=t\alpha e_{n-2}+\beta e_{n-1}+\gamma e_n\end{aligned}$$

$$\begin{aligned}\mu_t(e_3,e_4)&=g_t(\lambda(g_t^{-1}(e_3),g_t^{-1}(e_4)))\\&=g_t(\lambda(t^{n-3}e_3+f(s)t^{n-3}e_5,t^{n-2}e_4+f(s)t^{n-2}e_6))\\&=t^{2n-5}g_t(\delta e_n)=t\delta e_n\end{aligned}$$

■

Corollary 1. *Let $\alpha,\beta,\gamma,\delta\neq 0$. Then the above degeneration $\lambda\to_{\deg}\mu$ cannot be realized via a 1–PSG.*

Proof. The degeneration is nontrivial since λ is not isomorphic to μ: The commutator subalgebra $[\mu,\mu]$ is abelian, whereas $[\lambda,\lambda]$ is not. It is easy to see, that μ is characteristically nilpotent: For $\beta,\gamma\neq 0$ the derivations are stricly lower triangular matrices relative to the given basis and hence nilpotent. By Lemma 2, μ does not admit any nontrivial $\mathbb{Z}$–gradation. Hence the corollary follows by Proposition 1. ■

To show that Proposition 3 can be generalized for $k=3$ we state the following result:

Proposition 4. *Let λ be a complex filiform Lie algebra of dimension $n\geq 8$ with basis $\{e_1,\dots,e_n\}$ and defining brackets*

$$\begin{aligned}\lambda(e_1,e_i)&=e_{i+1},\ i\geq 2\\\lambda(e_2,e_3)&=\alpha e_{n-3}+\beta e_{n-2}+\gamma e_{n-1}+\delta e_n\\\lambda(e_2,e_4)&=\alpha e_{n-2}+\beta e_{n-1}+\gamma e_n\\\lambda(e_2,e_5)&=(\alpha-\varepsilon)e_{n-1}+(\beta-\kappa)e_n\\\lambda(e_2,e_6)&=(\alpha-2\varepsilon)e_n\\\lambda(e_3,e_4)&=\varepsilon e_{n-1}+\kappa e_n\\\lambda(e_3,e_5)&=\varepsilon e_n\end{aligned}$$

Let $s=1-t$, $\varepsilon\neq 0$ and define $g_t\in\mathbf{T}_n(\mathbb{C}(t))$ by

$$\begin{aligned}g_t^{-1}(e_1)&=te_1\\g_t^{-1}(e_i)&=t^{n+i-7}\big(e_i+f_1(s)e_{i+2}+f_2(s)e_{i+3}\big),\ i\geq 2\end{aligned}$$

where

$$\begin{aligned}f_1(s)&=\frac{\gamma}{2\varepsilon}s\\f_2(s)&=\frac{-\delta}{3\varepsilon}s^2+\left(\frac{2\delta\varepsilon-\gamma\kappa}{3\varepsilon^2}\right)s.\end{aligned}$$

Then the Lie algebra $\mu_t := g_t * \lambda$ has the same brackets as λ except for α, ε replaced by $t\alpha, t\varepsilon$. Hence λ degenerates to the limit algebra μ:

$$\begin{array}{|c|c|c|c|}\hline \alpha & \beta & \gamma & \delta \\ \hline \varepsilon & \kappa & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} \xrightarrow{}_{\text{deg}} \begin{array}{|c|c|c|c|}\hline 0 & \beta & \gamma & \delta \\ \hline 0 & \kappa & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

Proof. The proof consists of the computation of the Lie brackets. ∎

For $k \geq 4$ it is no longer true that the conditions of Lemma 1 do not imply conditions on the structure constants of $\mathfrak{g}$. Let $\mathfrak{g}$ be as in example 3. The Jacobi identity is equivalent to $\alpha_9(2\alpha_1 + \alpha_6) - 3\alpha_6^2 = 0$. In order to obtain a column degeneration of level 4, the following conditions (given by Lemma 1) have to be satisfied:

$$\alpha_6, 2\alpha_6 - \alpha_9 \neq 0, \ \alpha_3\alpha_9 = \alpha_6\alpha_8$$

Then the degeneration

$$\begin{array}{|c|c|c|c|c|}\hline \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \hline \alpha_6 & \alpha_7 & \alpha_8 & \multicolumn{2}{c}{} \\ \cline{1-3} \alpha_9 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array} \xrightarrow{}_{\text{deg}} \begin{array}{|c|c|c|c|c|}\hline 0 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \\ \hline 0 & \alpha_7 & \alpha_8 & \multicolumn{2}{c}{} \\ \cline{1-3} 0 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

is given by the following matrix:

$$g_{t,4}^{-1} = \begin{pmatrix} t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t^4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_1(s)t^3 & 0 & t^5 & 0 & 0 & 0 & 0 & 0 \\ 0 & f_2(s)t^3 & f_1(s)t^4 & 0 & t^6 & 0 & 0 & 0 & 0 \\ 0 & f_3(s)t^3 & f_2(s)t^4 & f_1(s)t^5 & 0 & t^7 & 0 & 0 & 0 \\ 0 & 0 & f_3(s)t^4 & f_2(s)t^5 & f_1(s)t^6 & 0 & t^8 & 0 & 0 \\ 0 & 0 & 0 & f_3(s)t^5 & f_2(s)t^6 & f_1(s)t^7 & 0 & t^9 & 0 \\ 0 & 0 & 0 & 0 & f_3(s)t^6 & f_2(s)t^7 & f_1(s)t^8 & 0 & t^{10} \end{pmatrix}$$

Here $s = 1 - t$ and f_i are polynomials in $\mathbb{C}[s]$ as follows:

$$f_1(s) = \frac{\alpha_3}{2\alpha_6} s,$$

$$f_2(s) = \frac{-\alpha_4}{3\alpha_6} s^2 + \frac{2\alpha_4\alpha_6 - \alpha_3\alpha_7}{3\alpha_6^2} s,$$

$$f_3(s) = \frac{\alpha_5}{2(2\alpha_6 - \alpha_9)} s^3 + \frac{2\alpha_3^2\alpha_6 + \alpha_3^2\alpha_9 - 12\alpha_6^2\alpha_5 + 4\alpha_6\alpha_4\alpha_7}{8\alpha_6^2(2\alpha_6 - \alpha_9)} s^2 +$$

$$\frac{\alpha_3\alpha_7^2 - \alpha_3^2\alpha_9 + 3\alpha_6^2\alpha_5 - 2\alpha_6\alpha_4\alpha_7}{2\alpha_6^2(2\alpha_6 - \alpha_9)} s$$

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