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The supports of simple modules over toroidal algebras

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Abstract. We present a description of possible supports for simple modules over toroidal Lie algebras associated with $\mathfrak{sl}(2,\mathbb{C})$. This description is analogous to that known for finite-dimensional simple Lie algebras and affine Lie algebras.

1. Introduction

The problem to describe the support of a simple weight module over a Lie algebra with triangular decomposition is very popular and has been studied in many cases. The final results were obtained for complex simple finite-dimensional algebras in [1], for superalgebras in [2], for the $A_1^{(1)}$ case in [4], for all affine Lie algebras by I. Penkov (work in progress), for rank two generalized Witt algebras in [6] and for Harish-Chandra modules over higher rank Virasoro algebras in [7].

In the present paper we give the complete answer on the formulated question in the case of an arbitrary toroidal Lie algebra that can be obtained from $\mathfrak{sl}(2)$ by the method described in [3]. In fact, the final result states that any simple weight module over such an algebra is either dense (i.e. for any weight λ and root β an element $\lambda+\beta$ is again a weight) or cut (i.e. its support is a subset of the support of some induced module).

The paper is organized as follows: In Section 1 we collect all necessary preliminaries and formulate the main result of this paper. In Section 3 we will prove some auxiliary lemmas that will be used in the proof of the main theorem presented in Section 4. Finally, in Section 5 we construct an example of a simple cut $\mathfrak{G}$-module without semi-primitive elements.

2. Toroidal algebras and main theorem

Let $\mathbb{C}$ denote the set of complex numbers, $\mathbb{Z}$ denote the set of integers, $\mathbb{Z}_+$ denote the set of non-negative integers and $\mathbb{N}$ denote the set of positive integers. For a Lie algebra L we will denote by $U(L)$ its universal enveloping algebra.

Fix $n\in\mathbb{N}$. Let $\mathfrak{A}=\mathfrak{sl}(2,\mathbb{C})$ be the Lie algebra of 2×2 complex matrix with zero trace and $A=\mathbb{C}[t_1^{\pm1},\dots,t_n^{\pm1}]$ be the algebra of Laurent polynomials

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with complex coefficients. Let e, f, h be the standard basis of $\mathfrak{A}$. Consider the Lie algebra $\mathfrak{G}_A = \mathfrak{A}\otimes_{\mathbb{C}} A$ with bracket $[x\otimes a, y\otimes b] = [x, y]\otimes ab$, $x, y \in \mathfrak{A}$ and $a, b \in A$. Let $\hat{\mathfrak{G}}$ be the universal covering algebra for $\mathfrak{G}_A$ ([5]). The algebra $\hat{\mathfrak{G}}$ is usually called toroidal. To get the representation theory of $\hat{\mathfrak{G}}$ reasonable we factor out the central ideal consisting of the span of all homogeneous elements of non-zero degree in Ω_A/dA and obtain the algebra $\tilde{\mathfrak{G}}$. Then for our convenience we will extend $\tilde{\mathfrak{G}}$ by commuting differentials $d_1, \dots, d_n$ such that $[d_i, x\otimes t_1^{k_1} \dots t_n^{k_n}] = k_i x\otimes t_1^{k_1} \dots t_n^{k_n}$, $x \in \mathfrak{A}$ (d_i commutes with Ω_A/dA) to form the Lie algebra $\mathfrak{G}$.

Let $H = \langle h\rangle$ be the standard Cartan subalgebra of $\mathfrak{A}$. Then

$$\mathfrak{H} = (H \otimes 1) \oplus \Omega_A/dA \oplus \langle d_1, \dots, d_n\rangle$$

is the standard Cartan subalgebra of $\mathfrak{G}$. Let $\Delta \subset \mathfrak{H}^*$ be the root system of $\mathfrak{G}$ with respect to $\mathfrak{H}$. For $\beta \in \Delta$ let $\mathfrak{G}_\beta$ denote the corresponding root space in $\mathfrak{G}$. In a standard way Δ can be decomposed into the disjoint union $\Delta = \Delta_{\Re} \cup \Delta_{\Im}$, where $\Delta_{\Re}$ is the set of roots of elements of the form $e\otimes a$ or $f\otimes a$, a is a monomial in A and $\Delta_{\Im} = \Delta \setminus \Delta_{\Re}$. We will also denote by $\Delta_{\Re}^+$ ($\Delta_{\Re}^-$) the set of roots of the elements of the form $e\otimes a$ ($f\otimes a$) and by $\mathfrak{G}_+$ ($\mathfrak{G}_-$) the corresponding subalgebras in $\mathfrak{G}$. A root $\alpha \in \Delta_{\Im}$ will be called simple if α/k is not a root for any $k \in \mathbb{N}$. It follows easily from the definition of $\mathfrak{G}$ that the sum of two elements from $\Delta_{\Re}^+$ (or $\Delta_{\Re}^-$) is never a root and the sum of any element form $\Delta_{\Re}^+$ (or $\Delta_{\Re}^-$) with any element from $\Delta_{\Im}$ is always a root.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ let $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}$ denote the weight subspace of V corresponding to a weight λ. A module V is said to be a weight module provided it can be decomposed into a direct sum of its weight subspaces. For a weight $\mathfrak{G}$-module V let $\operatorname{supp} V$ be the set of all weights of V with non-trivial weight subspaces. This set will be called the support of V.

Let P denote the abelian group spanned by Δ. One can easily see that $P \simeq \mathbb{Z}^{n+1}$. For any order $\leqslant$ on P (here and on by an order we mean an order on an abelian group, hence, we assume that this order is compatible with the group structure) we will denote $P_+^{\leqslant} = \{p \in P \mid 0 \leqslant p, p \not\leqslant 0\}$, $P_-^{\leqslant} = \{p \in P \mid p \leqslant 0, 0 \not\leqslant p\}$, $P_0^{\leqslant} = \{p \in P \mid 0 \leqslant p, p \leqslant 0\}$ and set $\Delta_i^{\leqslant} = P_i^{\leqslant} \cap \Delta$, $i = 0, +, -$. We will say that $\leqslant$ is non-trivial if $P_+^{\leqslant}$ is not empty.

Clearly, $\operatorname{supp} V \subset \lambda + P$ for any simple weight module V and any $\lambda \in \operatorname{supp} V$. A weight module V is called dense if $\operatorname{supp} V = \lambda + P$, $\lambda \in \operatorname{supp} V$ and cut if $\operatorname{supp} V \subset \lambda + P_-^{\leqslant}$ for a non-trivial order $\leqslant$ and some $\lambda \in \mathfrak{H}^*$.

For any weight module V and any $v \in V_\lambda$ ($\lambda \in \operatorname{supp} V$) we will say that a subset $S \subset \Delta$ is an annihilating v-set provided $\mathfrak{G}_\beta v = 0$ for any $\beta \in S$. A non-zero element $v \in V_\lambda$ will be called semi-primitive if there exists a non-trivial order $\leqslant$ on P such that $\Delta_+^{\leqslant}$ is an annihilating v-set. Any subset $S \subset \Delta$ can be enlarged by adding to it all the roots of the form $\alpha + \beta$, where $\alpha, \beta \in S$ and at least one of them belongs to $\Delta_{\Re}$. Starting from S we can enlarge it to the set S_1, then we can enlarge the obtained set to the set S_2 and so on. The set $\overline{S} = \cup_{i=1}^{\infty} S_i$ will be called the additive closure of S. The following lemma follows easily from the fact that $\dim \mathfrak{G}_\delta = 1$ for all $\delta \in \Delta_{\Im}$.

Lemma 2.1. *Let V be a weight $\mathfrak{G}$-module, $\lambda \in \operatorname{supp} V$, $v \in V_\lambda$ and S be an annihilating v-set. Then $\overline{S}$ is an annihilating v-set.*

Now we can formulate the main result of this paper:

Theorem 2.2. *Let V be a simple weight $\mathfrak{G}$-module. Then V is either dense or cut.*

We have to remark that in the case $n = 1$ our algebra $\mathfrak{G}$ is an affine Lie algebra. In this case theorem 2.2 was obtained in [4].

3. Preliminary lemmas

For a simple weight $\mathfrak{G}$-module V we will denote by $P(V)$ the set $\lambda + P$, $\lambda \in \operatorname{supp} V$. Clearly, $P(V)$ does not depend on the choice of λ. We also set $s(V) = P(V) \setminus \operatorname{supp} V$. During this section we fix a non-trivial non-dense simple weight $\mathfrak{G}$-module V. For a fixed $\mu \in \mathfrak{H}^*$ we will write $\mathcal{H}_\mu$ for the set $\{\mu\} \cup \mu + \Delta$. We note that $s(V)$ is not empty since V is not dense.

A non-zero element v of a weight module V will be called bounded of type $(\beta_1, \dots, \beta_n)$, where $\beta_i \in \Delta_{\mathfrak{I}}$, $1 \leqslant i \leqslant n$ are linearly independent simple roots provided there exist $\alpha^{\pm} \in \Delta_{\mathfrak{R}}^{\pm}$ such that $\{\alpha^{\pm} + k_1\beta_1 + \dots + k_n\beta_n \mid k_i \in \mathbb{Z}_+, i = 1, \dots, n\}$ is an annihilating v-set.

Lemma 3.1. *There exists $\mu \in s(V)$ and $\lambda \in \operatorname{supp} V \cap \mathcal{H}_\mu$ such that $\mu - \lambda \in \Delta_{\mathfrak{R}}$.*

Proof. Clearly, there exists $\mu_1 \in s(V)$ such that $\operatorname{supp} V \cap \mathcal{H}_{\mu_1}$ is not empty. If $\mu_1 + \Delta_{\mathfrak{R}} \subset s(V)$ then one can take $\mu \in \mu_1 + \Delta_{\mathfrak{R}}$ and $\lambda \in \operatorname{supp} V \cap \mathcal{H}_{\mu_1}$. ■

Lemma 3.2. *Suppose that $v \in V_\lambda$ is a non-zero element. Then either $\mathfrak{G}_+ v \neq 0$ ($\mathfrak{G}_- v \neq 0$) or v is semi-primitive.*

Proof. Follows from the fact that by setting $\Delta_+^{\leqslant} = \Delta_{\mathfrak{R}}^+$ one defines a non-trivial order $\leqslant$ on P. ■

Lemma 3.3. *Let $v \in V_\lambda$ and $S \subset \Delta$ be an annihilating v-set. Suppose that $g \in \mathfrak{G}_\beta$ and $S_1 \subset S$ such that $\beta + S_1 \cap \Delta \subset S$. Then S_1 is an annihilating gv-set.*

Proof. Let $\alpha \in S_1$ and $x \in \mathfrak{G}_\alpha$. One has $xgv = [x, g]v + gxv$. The right hand side of this equality will be zero as soon as $[x, g] = 0$ since $xv = 0$. Moreover, if $[x, g] \neq 0$ it follows that $[x, g] \in \mathfrak{G}_{\alpha+\beta}$ with $\alpha + \beta \in S$. Hence $xgv = 0$ and the statement follows. ■

Lemma 3.4. *Let V be a simple weight module and $0 \neq v \in V$ be a bounded element of type $(\beta_1, \dots, \beta_n)$. Then any non-zero element in V is bounded of the same type.*

Proof. Let $0 \neq w \in V$. Since V is simple there exists $u \in U(\mathfrak{G})$ such that $w = uv$. Thus it is sufficient to prove the statement of the lemma for any element of the form xv where $x \in \mathfrak{G}_\gamma$, $\gamma \in \Delta$.

Consider an r-dimensional Euclidian space X and fix linearly independent $x_1, \dots, x_r \in X$. For $y \in X$ set C_y be the cone consisting of all elements of the form $y + s_1x_1 + \dots + s_rx_r$ where s_i are non-negative for $1 \leqslant i \leqslant r$. Since X is r-dimensional it follows immediately that for any two cones C_{y_1} and C_{y_2} there exists $y \in X$ such that $C_y \subset C_{y_1} \cap C_{y_2}$. The statement now follows from Lemma 3.3. ■

Lemma 3.5. *Let V be a non-trivial simple weight $\mathfrak{G}$-module and $v \in V$ be a non-zero bounded element of type $\beta_1, \dots, \beta_n$. Suppose that V is not dense and $\mu \in s(V)$. Then $\mu + s_1\beta_1 + \dots + s_n\beta_n \in s(V)$ for $s_i \in \mathbb{N}$, $1 \leqslant i \leqslant n$.*

Proof. Suppose that $\lambda = \mu + s_1\beta_1 + \cdots + s_n\beta_n \in \operatorname{supp} V$ for some $s_i \in \mathbb{N}$, $1 \leqslant i \leqslant n$ and $0 \neq w \in V_\lambda$. Then w is a bounded element of type $\beta_1, \dots, \beta_n$ by Lemma 3.4. Let T denotes the corresponding annihilating w-set. Then the additive closure of $T \cup \{\mu - \lambda\}$ is an annihilating w-set by Lemma 3.3. Using previous lemmas one can show that this closure coincides with Δ and thus w generates a trivial $\mathfrak{G}$-submodule in V and we obtain a contradiction. ■

Lemma 3.6. *Let V be a simple weight $\mathfrak{G}$-module. Suppose that there exists $\alpha \in \Delta_{\mathfrak{R}}$, $0 \neq X_\alpha \in \mathfrak{G}_\alpha$, $k \in \mathbb{N}$ and $0 \neq v \in V$ such that $X_\alpha^k v = 0$. Then X_α acts locally nilpotent on V.*

Proof. Follows from the fact that $\mathrm{ad} X_\alpha$ is nilpotent on $\mathfrak{G}$. ■

Lemma 3.7. *Let V be a simple weight non-dense $\mathfrak{G}$-module, $\lambda \in \operatorname{supp} V$, $\mu \in s(V)$ such that $\mu - \lambda = k\alpha$ for some $\alpha \in \Delta_{\mathfrak{R}}$ and $k \in \mathbb{N}$. Then $\lambda + l\alpha \notin \operatorname{supp} V$ for all $l \geqslant k$.*

Proof. Let $0 \neq v \in V_\lambda$ and $0 \neq X_\alpha \in \mathfrak{G}_\alpha$. Then $X_\alpha^k v = 0$ and thus X_α is locally nilpotent on V by Lemma 3.6. Suppose that $\lambda + l\alpha \in \operatorname{supp} V$ for some $l \geqslant k$. Then $V_{\lambda + l\alpha} \neq 0$ and $\mathfrak{G}_{-\alpha}^{l-k} V_{\lambda + l\alpha} = 0$ and thus $X_{-\alpha}$ is also locally nilpotent on V. Since $\mathfrak{G}_{\pm\alpha}$ generate an $\mathfrak{sl}(2)$-subalgebra of $\mathfrak{G}$ it follows that $\mathfrak{sl}(2)$-module $\oplus_{m \in \mathbb{Z}} V_{\lambda + m\alpha}$ contains two finite dimensional subquotients and their supports have empty intersection. The last is impossible and thus we obtain the statement of the Lemma. ■

4. Proof of the main theorem

Suppose that V is a simple weight non-dense $\mathfrak{G}$-module and $n > 1$ (for $n = 1$ our proof reduces to that from [4]). Clearly, the existence of a semi-primitive vector in V will imply the statement of the main theorem. It is impossible to prove the existence of a semi-primitive element in the general case, so, in fact we will prove the following statement: *Let V be a non-dense simple $\mathfrak{G}$-module, then either there exists a semi-primitive element in V or V is cut.* Thus we can suppose that there were no semi-primitive elements in V. Our first goal is to prove that there exist some bounded element in V.

Consider the elements $\mu \in s(V)$ and $\lambda \in \operatorname{supp} V$ given by Lemma 3.1. Without loss of generality we can assume that $\mu - \lambda \in \Delta_{\mathfrak{R}}^+$. Let $v \in V_\lambda$ be a non-zero element. Then the set $S = \{\mu - \lambda\}$ is an annihilating v-set.

Since v is not semi-primitive, by Lemma 3.2 there exist $\hat{\alpha} \in \Delta_{\mathfrak{R}}^+$ and $g \in \mathfrak{G}_{\hat{\alpha}}$ such that $v_1 = gv \neq 0$. Let δ be a simple root such that $\mu - \lambda - \hat{\alpha} = N\delta$ for some $N \in \mathbb{N}$. One can choose $\hat{\alpha}$ such that $\mathfrak{G}_{\hat{\alpha} + k\delta} v = 0$, $0 < k < N$.

By Lemma 3.3 we have that S is an annihilating v_1-set. Moreover, we can enlarge S to an annihilating v_1-set S_1 by elements $\hat{\alpha} + k\delta$, $0 < k < N$, $N\delta$ and thus by $\mu - \lambda + kN\delta$, $k \in \mathbb{N}$. Applying $\mathfrak{G}_\delta$ to v_1 we can find an element $0 \neq v_2 \in V_{\mu + N_1\delta}$ such that $S_2 = \{\delta\} \cup \{\mu - \lambda + k\delta \,|\, k \in \mathbb{N}\}$ is an annihilating v_2-set.

Since v_2 is not semi-primitive, using Lemma 3.2 and the same procedure as above one can find an element $\alpha \in \Delta_{\mathfrak{R}}^+$ and $g_1 \in \mathfrak{G}_\alpha$ such that $v_3 = g_1 v_2 \neq 0$. Suppose that $\alpha = \mu - \lambda + k\delta$ for some $k \in \mathbb{Z}$ and it is impossible to choose α that is not of this form. Then $\Delta_{\mathfrak{R}}^+ \setminus \{\mu - \lambda + k\delta \,|\, k \in (\mathbb{Z} \setminus \mathbb{N})\}$ is an annihilating

v_3-set. Applying to v_3 any non-zero element x from $\mathfrak{G}_\beta$ where $\beta \in \Delta_{\mathfrak{I}}$ such that β and δ are linearly independent we immediately obtain that either v_3 or $xv_3 \neq 0$ is semi-primitive.

Thus we can choose α such that $\alpha \neq \mu-\lambda+k\delta$ for all $k \in \mathbb{Z}$. Let α_1 be the weight of v_3. In this case an additive closure T of $\{\mu-\lambda+k\delta \mid k \in \mathbb{N}\} \cup \{\mu-\alpha_1\}$ is an annihilating v_3-set by Lemma 2.1 and Lemma 3.3. Moreover, one can see that α can be chosen such that it would be possible to find simple $\beta_1, \beta_2 \in \Delta_{\mathfrak{I}}$ and $\gamma^\pm \in \Delta_{\mathfrak{R}}^\pm$ such that T contains all $\gamma^\pm + s_1\beta_1 + s_2\beta_2$ for $s_, s_2 \in \mathbb{Z}_+$.

Applying the same procedure to v_3 with the use of elements from $\Delta_{\mathfrak{R}}^-$ and then again from $\Delta_{\mathfrak{R}}^+$ and so on, one can construct a bounded element $0 \neq w \in V$ of type $\beta_1, \dots, \beta_n$ for some simple linearly independent $\beta_i \in \Delta_{\mathfrak{I}}$, $1 \leqslant i \leqslant n$. Thus, any element of V should be bounded of type $\beta_1, \dots, \beta_n$ by Lemma 3.4. By Lemma 3.5 we also obtain that $\mu + s_1\beta_1 + \dots + s_n\beta_n \in s(V)$ for $s_i \in \mathbb{N}$, $1 \leqslant i \leqslant n$.

Now, using Lemma 3.4 and the same arguments as in the preceding paragraph it is easy to see that there exists a non-trivial order $\leqslant_{\mathfrak{I}}$ on $\Delta_{\mathfrak{I}}$ and $\xi \in (\mu+\Delta_{\mathfrak{I}}) \cup \{\mu\}$ such that $\xi + P_{\mathfrak{I}}^+ \subset s(V)$, where $P_{\mathfrak{I}}^+ = \{\alpha \in \Delta_{\mathfrak{I}} \mid \alpha \not\leqslant_{\mathfrak{I}} 0, 0 \leqslant_{\mathfrak{I}} \alpha\}$. Indeed, fixing the support of V and assuming that this statement is false it follows with the same arguments as above that $s(V)$ can be enlarged. Clearly, we can choose ξ minimal with respect to $\leqslant$.

The last observation together with Lemma 3.4 immediately implies the following: if $\mu' \in \operatorname{supp} V$ and $\alpha \in \Delta_{\mathfrak{I}}$ such that $\mu' + \alpha \in s(V)$ then there exists minimal $\xi' \in \mu' + \Delta_{\mathfrak{I}}$ such that $\xi' + P_{\mathfrak{I}}^+ \subset s(V)$. Otherwise, one can easy find $0 \neq v \in V$ such that $\mathfrak{G}v = 0$.

Consider the subsets $\mathcal{H}_\pm$ of $\mathcal{H}_\mu$ defined as follows: $\mathcal{H}_\pm = \mu + \Delta_{\mathfrak{R}}^\pm$. Suppose that $\mathcal{H}_- \subset \operatorname{supp} V$ ($\mathcal{H}_+ \subset \operatorname{supp} V$). Then $\mathcal{H}_+ \subset s(V)$ ($\mathcal{H}_- \subset s(V)$) by Lemma 3.7 and we obtain that V contains a semi-primitive element. Hence we can fix elements $\xi' \in \mathcal{H}_-$ and ξ as above. Set $\beta = \xi - \xi' \in \Delta_{\mathfrak{R}}^+$.

Consider the non-trivial order $\leqslant$ on Δ (and thus on P) such that $\Delta_+^{\leqslant} = P_{\mathfrak{I}}^+ \cup \pm\beta + P_{\mathfrak{I}}^+$. Now Lemma 3.7 guarantees that $\xi + P_+^{\leqslant} \subset s(V)$ which completes the proof of the theorem.

5. Examples

Example of cut $\mathfrak{G}$-modules with semi-primitive elements can easily be constructed as the unique simple quotients of Verma modules using partitions of Δ as it was done for example in [4, 8]. At the same time, examples of dense modules also can be constructed by using the standard technique. Unlike the classical case of affine Kac-Moody Lie algebras we can not state that any cut $\mathfrak{G}$-module contains a semi-primitive element. The aim of this section is to construct an example of a simple cut module without semi-primitive elements.

Let $P_{\mathfrak{I}} = \mathbb{Z}\Delta_{\mathfrak{I}}$ and consider an order $\leqslant$ on $P_{\mathfrak{I}}$ that satisfies the following condition: for any $0 \leqslant x \leqslant y$, $x, y \in P_{\mathfrak{I}}$ there exists $k \in \mathbb{N}$ such that $y \leqslant kx$. Let P_+ denote the set $\{x \in P_{\mathfrak{I}} \mid 0 \leqslant x, x \not\leqslant 0\}$. Fix $\alpha \in \Delta_{\mathfrak{R}}^+$ and set $\Delta_+ = P_+ \cup \{\alpha\} \cup \alpha + P_+ \cup -\alpha + P_+$, $\Delta_- = -\Delta_+$. Let $\mathfrak{G} = \mathfrak{G}_+ \oplus \mathfrak{H} \oplus \mathfrak{G}_-$ be the corresponding decomposition of $\mathfrak{G}$. Consider $\mathbb{C}$ as the trivial $\mathfrak{G}_+ \oplus \mathfrak{H}$-module and form a module M as follows:

$$M = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{G}_+ \oplus \mathfrak{H})} \mathbb{C}.$$

Let $P(-)$ be the semigroup generated by $\Delta_- \cup \{0\}$. Clearly, M is a weight module and $\operatorname{supp} M = P(-)$. Let N be the subspace of M generated by all M_λ, $\lambda \neq 0$.

Proposition 5.1. 1. *N is a $\mathfrak{G}$-submodule of M.*

2. *N is simple.*
3. *N contains no semi-primitive elements.*

Proof. The first statement follows from the fact that N is the kernel of the canonical epimorphism of M onto trivial $\mathfrak{G}$-module. The last one follows from the second and the description of $\operatorname{supp} N$. Thus we only need to prove the second statement.

Let v denote a canonical generator of M. First we note that it is enough to show that for any $w \in N$ the module $N_w = U(\mathfrak{G})w$ contains an element of the form $X_\beta v$ for some $\beta \in \Delta_-$. Fix a non-trivial w. The order $\leqslant$ trivially induces an order on Δ_- which we will also denote by $\leqslant$. Thus by PBW theorem w can be written as a linear combination of the monomials and, moreover, each monomial is a product of X_β for $\beta \in \Delta_-$. By the length of a monomial we will mean the number of multiplicands occurring in it.

Fix the set $S(w)$ of monomials of maximal length in w and choose the smallest β_1 that occurs as a multiplicand in these monomials. Applying the elements X_{β_2} to w such that $-\beta_2$ is smaller than arbitrary multiplicand of a monomial from $S(w)$ but β_1 one can easily show that $\operatorname{supp} N_w = \operatorname{supp} N$ and, moreover, that N_w contains an element w_1 such that $|S(w_1)| = 1$. Thus we can assume that $|S(w)| = 1$.

Now we can consider the sets $S_1 = \{-\beta_1 + \mathbb{Z}\alpha\} \cap \Delta$, $S_2 = S_1 + \beta$ for some $\beta \in \Delta_+$ small enough and $S_3 = S_1 \cup S_2$. Consider the elements $X_\gamma w$, $\gamma \in S_3$. One can see that β can be chosen in such way that at least one of the elements $X_\gamma w$ is non-zero. Now it is easy to obtain that any monomial occurring in this non-zero element has length smaller than $|S(w)|$. Trivial induction completes our proof. ∎

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