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Schur duality for the Cartan type Lie algebra W_n

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Abstract. We decompose tensor products of the defining representation of a Cartan type Lie algebra $W(n)$ in the case where the number of tensoring does not exceed the rank of the Lie algebra. As a result, we get a kind of Schur duality between $W(n)$ and a finite dimensional non-semisimple algebra, which is the semi-group ring of the transformation semigroup $\mathfrak{T}_m$.

Introduction

Cartan type Lie algebras are Lie subalgebras of algebraic vector fields on a flat affine space $\mathbb{F}^n$, where $\mathbb{F}$ is a field of characteristic zero. They are $\mathbb{Z}$-graded, simple Lie algebras with polynomial growth. By the result of Kac and Mathieu, Lie algebras with such properties are known to be (1) finite dimensional simple Lie algebras; (2) their loop algebras; (3) Witt algebra; and (4) Cartan type Lie algebras (see [11]).

Irreducible representations of a Cartan type Lie algebra $\mathfrak{g}$ were studied extensively by Rudakov ([15, 16]) and Kostrikin ([9]) in 1970's. If an irreducible representation admits a weight decomposition with respect to Euler's degree operators, then it is a lowest weight module or its dual except the only one case $\mathfrak{g} = W_1$. Therefore, the description of the irreducible representations are quite easy. However, it is rather difficult to do analysis on them because representations of W_n are not semisimple.

In the previous paper [12] (see also [13], [14]), we found an interesting phenomenon on the tensor product representations of $\mathfrak{g} = W_n$, which is one of the four series of Cartan type Lie algebras. Since, by definition, W_n is a Lie algebra of all the derivations on the polynomial ring $P[z_1, \cdots, z_n]$ of n-variables (see Section 1.), W_n acts naturally on $P = P[z_1, \cdots, z_n]$. Form the m-fold tensor product $\otimes^m P$. Then the full commutant algebra of W_n in $\otimes^m P$ becomes a finite dimensional algebra. So we expect simultaneous decomposition of $\otimes^m P$ as a module of W_n and its commutant algebra.

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To be more precise, let us assume that $m \leq n$. Then the full commutant algebra of W_n in End $(\otimes^m P)$ becomes the semigroup ring of the transformation semigroup $\mathfrak{T}_m$, which is a semigroup consisting of all the maps from the finite set $[m] = \{1, 2, \cdots, m\}$ to itself (see, for example, [6, Chapter 1]). In this paper, we discuss on the decomposition of $\otimes^m P$ as a representation of $W_n \times \mathfrak{T}_m$. Unfortunately, the representations of W_n and $\mathfrak{T}_m$ are not semisimple, and even worse, $\otimes^m P$ admits a composition series of infinite length. However, we can still get a control on the irreducible quotients of $\otimes^m P$.

Let U be a representation of $W_n \times \mathfrak{T}_m$ and $\pi \otimes \Sigma$ an irreducible representation of $W_n \times \mathfrak{T}_m$. Then we say that $\pi \otimes \Sigma$ has *quotient multiplicity* k in U if

$$\dim \mathrm{Hom}_{W_n \times \mathfrak{T}_m} (U, \pi \otimes \Sigma) = k$$

holds. Further, we say that U is *quotient multiplicity free* if

$$\dim \mathrm{Hom}_{W_n \times \mathfrak{T}_m} (U, \pi \otimes \Sigma) \leq 1$$

for any irreducible representation $\pi \otimes \Sigma$. With these terminologies, we can state our main result of this paper.

Theorem 1. *[Theorem 5.1] If $m \leq n$, then $\otimes^m P$ is quotient multiplicity free. There is a one-to-one correspondence between the subsets of irreducible representations $W_n^\wedge \ni \pi \leftrightarrow \Sigma \in \mathfrak{T}_m^\wedge$, which is defined by the following property:*

$$\dim \mathrm{Hom}_{W_n \times \mathfrak{T}_m} (\otimes^m P, \pi \otimes \Sigma) = 1.$$

Note that the above correspondence $\pi \leftrightarrow \Sigma$ cannot be bijective. However, it involves all the irreducible representations of the semigroup $\mathfrak{T}_m$. We give the correspondence in terms of Young diagrams or, equivalently to say, partitions of various sizes. The correspondence is an explicit one, hence we obtain a realization of irreducible representations of W_n as irreducible quotients of $\otimes^m P$. (See Corollary 5.2.)

Let GL_n act on $V = \mathbb{C}^n$ as its defining representation and $\mathfrak{S}_m$ act on its m-fold tensor product $\otimes^m V$ by the permutations of coordinates. Then Schur duality says that there is a correspondence between the irreducible representations of GL_n and $\mathfrak{S}_m$ via Young diagrams. Weyl became the first mathematician who pointed out the machinery works for two mutually commutant algebras which are semisimple ([18]). Our theorem is a generalization of their theory to non-semisimple and infinite dimensional algebras.

On the other hand, our method is under strong influence of series of works of Howe ([3, 4, 5]). He developed correspondences of representations of various pairs of reductive groups, which are called Howe duality. Although the groups are reductive, their $(\mathfrak{g}, K)$-modules, which are infinite dimensional, are not semisimple. So he needed to consider irreducible quotients to establish one-to-one correspondence. Since our theorem is proved by using the restrictions to semisimple subalgebras (or subgroups) instead of using K-types, it is the same as Howe's in its spirit.

Recently, Benkart and Melville independently get a similar result on a certain quotient representation of $\otimes^k P$ with no restriction on the power of tensor products for Lie superalgebra $W(m,n)$. Their result is a kind of $W(m,n) \times \mathfrak{S}_k$ duality, which corresponds to one of our intermediate results (see Theorem 2.4). Our emphasis is on the use of the transformation semigroup $\mathfrak{T}_m$ which describes the filtered structure of $\otimes^m P$.

Our theory also applies to the finite dimensional Lie superalgebra $\mathcal{W}_n$ (we use the script letter $\mathcal{W}$ to avoid the confusion). In this case, the whole representation space becomes finite dimensional and we expect an explicit description of composition series (cf. [17]). We also expect that it applies to representations of Cartan type Lie algebras over a field of positive characteristic. However, these are the future subjects of ours.

Acknowledgment. This work has been inspired by the explicit calculations done by H. Wang for Cartan type Lie superalgebras. The author expresses sincere thanks to him. He is also grateful to the referee for pointing out the reference [2] to him, which enriches our main result.

1. Preliminaries; Cartan-type Lie algebra W_n

Let $V = \mathbb{F}^n$ be an n-dimensional vector space over a field $\mathbb{F}$ of characteristic 0. We fix a basis $\{z_i \mid 1 \leq i \leq n\}$ of V and denote the dual basis in V^* by $\{z_i^* \mid 1 \leq i \leq n\}$. Let W_n be the Lie algebra of vector fields on $V^* = \mathbb{F}^n$ with polynomial coefficients. The algebra W_n is considered as the algebra of all the derivations on the polynomial algebra $P(V^*) = \mathbb{F}[z_1, z_2, \cdots, z_n]$ on V^* (cf. [11]).

As derivations, W_n acts naturally on the polynomial algebra $P(V^*)$. We call $P(V^*)$ the *natural representation* (or defining representation) of W_n, and denote it by ψ. Then $(\psi, P(V^*))$ becomes a $\mathbb{Z}$-graded W_n-module with the natural grading of polynomials, where the $\mathbb{Z}$-grading of W_n is given by

$$\begin{aligned} W_n &= \bigoplus_{j=-1}^{\infty} W_n(j), \\ W_n(j) &= \left\{ \sum_{i=1}^{n} f_i(z) \frac{\partial}{\partial z_i} \,\middle|\, f_i(z) \text{ is a homogeneous polynomial of degree } j+1 \right\}. \end{aligned}$$

In this grading, the homogeneous elements of degree zero form a finite dimensional Lie algebra which is isomorphic to $\mathfrak{gl}_n(\mathbb{F})$. This realization of $\mathfrak{gl}_n(\mathbb{F})$ is obtained by differentiating the natural action of $GL(V)$ on the polynomial ring $P(V^*) = S(V)$.

We briefly review the results on irreducible representations of W_n. Let E be an irreducible graded representation of W_n ($n \geq 2$) which is of finite type in the sense of [9]. Then E or its contragredient has a vector killed by $W_n(j)$ for arbitrary $j \geq 1$ ([9]). These representations are called height 1 and studied concretely by Rudakov ([15, §13]). In particular, E is a lowest weight module or a highest weight module. In this paper, there appear only lowest weight modules. So assume that E has a lowest weight vector and put

$$E_0 = \{v \in E \mid W_n(-1)v = 0 \ (j \geq 1)\}.$$

Then E_0 becomes a non-zero irreducible $W_n(0) = \mathfrak{gl}_n$-module. Let λ be the highest weight of the $\mathfrak{gl}_n$-module E_0. Since E is completely determined by λ, we write $E = \pi_\lambda$.

Now recall the natural representation $(\psi, P(V^*))$. Let us consider the m-fold tensor product representation $(\psi^{\otimes m}, \otimes^m P(V^*))$ of W_n. We studied this representation in [12], in comparison with the classical Schur-Weyl's duality for $GL_n(\mathbb{C}) \times \mathfrak{S}_m$. In the paper, we determined the structure of the commutant algebra of W_n in End $(\otimes^m P(V^*))$ in case that the number of tensor product m does not exceed n, which is rank of the Lie algebra W_n ([12, Theorem 2.3]). The commutant algebra is a certain semigroup ring. Let us explain it briefly.

Denote the set of positive integers less than or equal to m by $[m] = \{1, 2, \cdots, m\}$. Then the set of all the maps from $[m]$ to itself, $\mathfrak{T}_m = \{\varphi : [m] \to [m]\}$, becomes a semigroup with unit by composition of maps (see, for example, [6]). Note that the group elements (i.e., the elements which have inverses in $\mathfrak{T}_m$) are the permutations in the symmetric group $\mathfrak{S}_m$. We denote the semigroup ring of $\mathfrak{T}_m$ by $\mathbb{F}[\mathfrak{T}_m]$.

We identify the m-fold tensor product with the polynomial ring in $n \times m$-variables:

$$\otimes^m P(V^*) \simeq \mathbb{F}[z_{i,j} \mid 1 \le i \le n, 1 \le j \le m].$$

An element $\varphi \in \mathfrak{T}_m$ acts on $\otimes^m P(V^*)$ as an endomorphism of algebra $\mathbb{F}[z_{i,j}] \simeq \otimes^m P(V^*)$ and it acts on generators $\{z_{i,j}\}_{i,j}$ as follows:

$$\varphi(z_{i,j}) = z_{i,\varphi(j)}.$$

We extend it to the semigroup ring $\mathbb{F}[\mathfrak{T}_m]$ by linearity. By easy calculations, one can convince that $\mathbb{F}[\mathfrak{T}_m]$ is a subalgebra in the commutant algebra. In fact, they coincide if $m \le n$.

2. Duality on the top level

Put $R = \mathbb{F}[\mathfrak{T}_m]$, and consider the following filtration by two-sided ideals of R:

$$R_k = \langle \varphi \in \mathfrak{T}_m \mid \# \operatorname{Im} \varphi \le k \rangle \text{ (generated as a vector space)},$$

where $\# N$ means the cardinality of a finite set N. Clearly R becomes a filtered algebra by this filtration.

Put $\mathcal{V}_k = R_k \cdot (\otimes^m P(V^*))$ for $k \ge 1$ and $\mathcal{V}_0 = \mathbb{F}$. In other words, $\mathcal{V}_k$ is spanned by those polynomials in the matrix $z = (z_{i,j})$ that actually depends on at most k columns of z. Then $(\mathcal{V}_k)_{0 \le k \le m}$ becomes a natural filtration of the $W_n \times \mathbb{F}[\mathfrak{T}_m]$-module $\mathcal{V}_m = \otimes^m P(V^*)$. Consider the graded module

$$\operatorname{gr} \mathcal{V} = \bigoplus_{0 \le k} \mathcal{V}_k / \mathcal{V}_{k-1} = \bigoplus_{0 \le k} \mathcal{V}(k) \quad (\mathcal{V}_{-1} = (0)).$$

Our aim in the present note is to decompose this module as a $W_n \times \mathfrak{T}_m$-module.

Let us begin with decomposition of $\mathcal{V}(m) = \mathcal{V}_m / \mathcal{V}_{m-1}$. If we put $P(V^*)^+ = P(V^*)/\mathbb{F}$ ($\mathbb{F}$ being the constant functions), then clearly

$$\mathcal{V}(m) \simeq \otimes^m P(V^*)^+$$

holds. However, we prefer the realization of $\mathcal{V}(m)$ as the quotient space of $\mathbb{F}[z_{i,j} | \ 1 \le i \le n, 1 \le j \le m] = \otimes^m P(V^*)$.

Lemma 2.1. *Assume that $m \leq n$. Then, the commutant algebra of W_n in* End $\mathcal{V}(m)$ *is isomorphic to $\mathbb{F}[\mathfrak{S}_m]$, the group ring of the symmetric group of degree m.*

Proof. The proof of the lemma is essentially the same as that of Theorem 2.3 (1) in [12]. Let us recall it.

Take $v = z_{1,1}z_{2,2}\cdots z_{m,m} \in \mathcal{V}(m)$. Then v is a cyclic vector of $\mathcal{V}(m)$ for the action of W_n. So, if E belongs to the commutant of W_n, then E is completely determined by the image of v under E. On the other hand, since W_n contains Euler's degree operators, the image Ev is contained in the subspace of some fixed homogeneous degree.

Let us explain it more precisely. Define the degree of the monomial

$$\prod_{1\leq i\leq n, 1\leq j\leq m} z_{i,j}^{d_{i,j}} \qquad (d_{i,j} \in \mathbb{Z})$$

with respect to $(z_{i,1}, \cdots, z_{i,m})$ by $\sum_{j=1}^{m} d_{i,j}$. Then Ev has the homogeneous degree 1 with respect to $(z_{i,1}, \cdots, z_{i,m})$ for each $i \leq m$, and 0 for $m < i \leq n$.

Since the subspace of the homogeneous polynomials in $\mathcal{V}(m)$ of the degree described above has a basis

$$\{\prod_{i=1}^{m} z_{i,\sigma(i)} \mid \sigma \in \mathfrak{S}_m\},$$

the dimension of the commutant algebra cannot exceed $m!$. On the other hand, it clearly contains $\mathbb{F}[\mathfrak{S}_m]$ which is $m!$-dimensional, hence the lemma is proved. ■

Note that $R = \mathbb{F}[\mathfrak{T}_m]$ acts on $\mathcal{V}(m)$ and, by definition, the action of the two-sided ideal R_{m-1} is trivial. So, the quotient algebra $\mathbb{F}[\mathfrak{T}_m]/R_{m-1} \simeq \mathbb{F}[\mathfrak{S}_m]$ naturally acts on $\mathcal{V}(m)$. Therefore, the above lemma says that the image of $\mathbb{F}[\mathfrak{T}_m]$ on $\mathcal{V}(m)$ is the full commutant of W_n.

Since $\mathbb{F}[\mathfrak{S}_m]$ is semisimple, we decompose $\mathcal{V}(m)$ as a $\mathbb{F}[\mathfrak{S}_m]$-module:

$$\mathcal{V}(m) \simeq \sum_{D\in\mathcal{Y}_m}^{\oplus} \operatorname{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m)) \otimes \sigma_D,$$

where $\mathcal{Y}_m$ is the set of Young diagrams of size m and σ_D is the corresponding irreducible representation of $\mathfrak{S}_m$ (see, for example, [8], [7], ...); we prefer the terminology *size* to weight here, since 'weight' is used in the different meaning (cf. [10]). The fact that $\mathbb{F}[\mathfrak{S}_m]$ is the full commutant algebra of W_n assures that

$$\operatorname{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$$

becomes a W_n-module and, moreover, it is indecomposable. We define a grading on $\operatorname{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$ by defining that $f \in \operatorname{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$ is of homogeneous degree d if and only if $f(\sigma_D)$ is contained in the homogeneous space of degree d. Then, by this grading, $\operatorname{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$ becomes a graded W_n-module.

Lemma 2.2. *Let $D \in \mathcal{Y}_m$ be a Young diagram of size m and $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ the corresponding partition of m. Then the restriction of the W_n-module $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$ to the subalgebra $W_n(0) \simeq \mathfrak{gl}_n$ contains the irreducible representation ρ_D of $\mathfrak{gl}_n$ with highest weight $(\lambda_1, \lambda_2, \cdots, \lambda_m, 0, \cdots, 0)$ with multiplicity one:*

$$\dim \mathrm{Hom}_{\mathfrak{gl}_n}(\rho_D, \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))) = 1.$$

Moreover, the homogeneous component of $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$ of the lowest possible degree m coincides with the embedded ρ_D.

Proof. Let $P(V^*)_+$ be the space of the polynomials without constant term. Then $\otimes^m P(V^*)_+ \simeq \mathcal{V}(m)$ as a vector space, and we take representatives of the elements in $\mathcal{V}(m)$ from $\otimes^m P(V^*)_+ \subset \otimes^m P(V^*)$. This identification preserves degrees and $\mathfrak{gl}_n \times \mathfrak{S}_m$-module structures.

Let U be the subspace of the elements of degree m in $\mathcal{V}(m)$. Note that m is the lowest possible degree in $\mathcal{V}(m)$. Since $\mathfrak{gl}_n \times \mathfrak{S}_m$ preserves degree, U is a $\mathfrak{gl}_n \times \mathfrak{S}_m$-module. It holds that

$$U \simeq \otimes^m V \simeq \sum_{D \in \mathcal{Y}_m}^{\oplus} \rho_D \otimes \sigma_D$$

as a $\mathfrak{gl}_n \times \mathfrak{S}_m$-module, where ρ_D is the irreducible representation of $GL(V)$ with highest weight $(\lambda_1, \cdots, \lambda_m, 0, \cdots, 0)$. The first part of the above formula is obvious by definition and the second part follows from Schur duality.

By the above formula, we get

$$U \cap (\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m)) \otimes \sigma_D) \simeq \rho_D \otimes \sigma_D \neq 0.$$

Denote the above non-zero space by U_D. Note that U_D does not have $GL(V)$-components in common with $\mathcal{V}(m)_D = \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m)) \otimes \sigma_D$ other than itself. To see this, we note that $\otimes^m P(V^*) \simeq P(V^* \otimes \mathbb{F}^m)$ is decomposed as $GL_n \times GL_m$-module in the following multiplicity-free manner:

$$P(V \otimes \mathbb{F}^m) = \sum_{E}^{\oplus} \rho_E^{(n)} \otimes \rho_E^{(m)},$$

where E ranges over all the Young diagrams with length less than or equal to $\min\{n, m\}$, and $\rho_E^{(n)}$ (respectively $\rho_E^{(m)}$) is the irreducible representation of GL_n (respectively GL_m) with highest weight $\mu(E) = (\mu_1, \mu_2, \cdots)$ which is the partition associated with E. This decomposition is well-known and we refer to, for example, [3].

Note that, in this decomposition, the subspace

$$\sum_{|E|=d}^{\oplus} \rho_E^{(n)} \otimes \rho_E^{(m)}$$

gives the full homogeneous component of $\otimes^m P(V^*)$ of degree d. Therefore, in our case, the representation ρ_D does not appear in the degree greater than m. On the other hand, m is the smallest positive degree in $\mathcal{V}(m)$. Hence ρ_D appears only in U_D.

The rest of the statements are clear from the above arguments. ∎

Proposition 2.3. *Let $D \in \mathcal{Y}_m$ be a Young diagram of size m and let $(\lambda_1, \lambda_2, \cdots, \lambda_m)$ be the corresponding partition of m. Then the W_n-module $\mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m))$ has the unique irreducible quotient which has the lowest weight $(0, \cdots, 0, \lambda_m, \lambda_{m-1}, \cdots, \lambda_1)$.*

Proof. We use the notations in the proof of Lemma 2.2.

Note that the space U generates $\mathcal{V}(m)$ as W_n-module. In fact, the vector $v = z_{1,1}z_{2,2}\cdots z_{m,m} \in U$ is a cyclic vector for $\mathcal{V}(m)$ (cf. the proof of Lemma 2.1). Since W_n commutes the action of $\mathfrak{S}_m$, U_D also generates $\mathcal{V}(m)_D = \mathrm{Hom}_{\mathfrak{S}_m}(\sigma_D, \mathcal{V}(m)) \otimes \sigma_D$. By the above lemma, U_D does not have the $GL(V)$-components in common with $\mathcal{V}(m)_D$ other than itself and is irreducible under the action of $\mathfrak{gl}_n \times \mathfrak{S}_m$.

In the following in this proof, we consider $W_n \times \mathfrak{S}_m$-module structure only unless otherwise stated. Note that $\mathfrak{S}_m$ acts just according to σ_D on $\mathcal{V}(m)_D$.

Let $\mathcal{V}(m)_D \supset A, B$ be two submodules such that both $\mathcal{V}(m)_D/A$ and $\mathcal{V}(m)_D/B$ are non-zero and irreducible. Since U_D is cyclic for $\mathcal{V}(m)_D$, U_D and A has no $GL(V)$-component in common. The same statement is applicable to B. So we know $A+B$ has no $GL(V)$-component in common with U_D. Since $\mathcal{V}(m)_D/A$ is irreducible, this means $A+B \subset A$, which concludes $B \subset A$. Similarly, we have $A \subset B$ and $A = B$ follows.

By the above argument, we know that $\mathcal{V}(m)_D$ has the largest submodule which does not contain U_D. Since U_D contains the lowest weight vector for W_n with weight $(0, \cdots, 0, \lambda_m, \lambda_{m-1}, \cdots, \lambda_1)$, the irreducible quotient has also the same lowest weight vector. ∎

In the following, for a Young diagram D of size m, $\lambda = (\lambda_1, \lambda_2, \cdots)$ denotes the corresponding partition of D. If the length of λ is less than or equal to n, then we also denote by π_D an irreducible representation of W_n with lowest weight $(\lambda_n, \lambda_{n-1}, \cdots, \lambda_1)$.

Theorem 2.4. *Let $P(V^*)^+ = P(V^*)/\mathbb{F}$ be an irreducible representation of W_n with lowest weight $(0, \cdots, 0, 1)$. Then $\mathcal{V}(m) \simeq \otimes^m P(V^*)^+$ is a $W_n \times \mathfrak{S}_m$-module and, under the assumption that $m \leq n$, its irreducible quotients are of the form*

$$\pi_D \otimes \sigma_D \quad (D \in \mathcal{Y}_m),$$

where $\mathcal{Y}_m$ is the set of Young diagrams of size m. Moreover, it is quotient multiplicity-free, i.e.,

$$\dim \mathrm{Hom}_{W_n \times \mathfrak{S}_m}(\mathcal{V}(m), \pi_D \otimes \sigma_D) = 1 \quad (\forall D \in \mathcal{Y}_m).$$

3. Irreducible representation of the semigroup $\mathfrak{T}_m$

The irreducible representations of the transformation semigroup $\mathfrak{T}_m$ were studied by Hewitt and Zuckerman [2] and they got a complete classification. However, their description of the classification is combinatorial and too complicated to fit with our present situation. In this section, we study irreducible representations of the semigroup $\mathfrak{T}_m$ which are obtained by induction from maximal parabolic sub-semigroups. Thanks to the above mentioned result of [2], we eventually find the representations constructed here using parabolic induction exhaust all the irreducible ones.

The arguments of Hewitt and Zuckerman essentially use restriction of the representations to the symmetric group, while we use the induced representations. Though our arguments here do not provide a new proof, we believe that they shed a new light on their classification.

Put

$$\mathcal{P}_{k,m-k} = \{\varphi \in \mathfrak{T}_m \mid \varphi([k]) \subset [k]\} \subset \mathfrak{T}_m.$$

Clearly $\mathcal{P}_{k,m-k}$ is a sub-semigroup of $\mathfrak{T}_m$. We may call it a maximal parabolic sub-semigroup. It projects naturally onto $\mathfrak{T}_k$ when we consider the first k letters $[k] = \{1, 2, \cdots, k\}$. The projection is a semigroup morphism. Let (σ, U_σ) be a representation of $\mathfrak{T}_k$. Then the above projection proj naturally induces a representation $\mathrm{proj}^*\sigma$ of $\mathcal{P}_{k,m-k}$ by

$$\mathrm{proj}^*\sigma(\varphi) = \sigma(\mathrm{proj}(\varphi)). \tag{1}$$

We denote this representation $(\mathrm{proj}^*\sigma, U_\sigma)$ of $\mathcal{P}_{k,m-k}$ by the same letter σ by abuse of notation. Then we define the parabolically induced representation $\mathrm{ind}\,\sigma = \mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m}\sigma$ of $\mathfrak{T}_m$ as

$$\mathrm{ind}\,\sigma = \mathrm{ind}\,(\sigma, U_\sigma) = \mathbb{F}[\mathfrak{T}_m] \otimes_{\mathbb{F}[\mathcal{P}_{k,m-k}]} U_\sigma. \tag{2}$$

Proposition 3.1. *Let (σ, U_σ) be an irreducible representation of $\mathfrak{T}_k$ on which R_{k-1} acts trivially. Then the $\mathbb{F}[\mathfrak{T}_m]$-module $\mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m}\,\sigma$ has the unique irreducible quotient.*

Proof. Since R_{k-1} acts trivially, the action of σ is completely determined by $\mathfrak{S}_k \subset \mathfrak{T}_k$. Hence, as a representation of $\mathfrak{S}_k$, σ is isomorphic to σ_D for some Young diagram $D \in \mathcal{Y}_k$. In the following, we identify $(\sigma, U_\sigma) = (\sigma_D, U_{\sigma_D})$.

Lemma 3.2. *As $\mathfrak{S}_m$-modules, we have*

$$\mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m}\sigma_D \simeq \mathrm{ind}_{\mathfrak{S}_k\times\mathfrak{S}_{m-k}}^{\mathfrak{S}_m}(\sigma_D \otimes 1).$$

Proof. We realize the representation (σ_D, U_{σ_D}) in the polynomial ring $\otimes^k V \subset \otimes^k P(V^*)^+ = \otimes^k P(V^*)/\mathbb{F}$. As mentioned before, Weyl-Schur duality tells us that

$$\otimes^k V \simeq \sum_{E\in\mathcal{Y}_k}^{\oplus} \rho_E \otimes \sigma_E,$$

and we take one copy of (σ_D, U_{σ_D}) in $\otimes^k V$. Note that $\otimes^k V$, and hence $U_{\sigma_D} \subset \otimes^k V$, is naturally embedded into $\otimes^m P(V^*)/R_{k-1}\,(\otimes^m P(V^*))$, which is a $W_n \times \mathfrak{T}_m$-module:

$$\otimes^k V \ni v_1 \otimes \cdots \otimes v_k \mapsto v_1 \otimes \cdots \otimes v_k \otimes 1 \otimes \cdots \otimes 1 \in \otimes^m P(V^*)/R_{k-1}\,(\otimes^m P(V^*)).$$

Then, on U_{σ_D}, the maximal parabolic sub-semigroup $\mathcal{P}_{k,m-k}$ acts as described in (1) and the action of R_{k-1} vanishes. By the universality of tensor products, we get a natural map:

$$\mathbb{F}[\mathfrak{T}_m] \otimes_{\mathbb{F}[\mathcal{P}_{k,m-k}]} U_{\sigma_D} \to \mathbb{F}[\mathfrak{T}_m] \cdot U_{\sigma_D} \subset \otimes^m P(V^*)/R_{k-1}\left(\otimes^m P(V^*)\right). \tag{3}$$

We shall prove the above natural map is actually an isomorphism.

Note that the right cosets of $\mathfrak{T}_m$ with respect to $\mathcal{P}_{k,m-k}$ have representatives in $\mathfrak{S}_m$:

$$\mathfrak{T}_m = \bigcup_{s \in \mathfrak{S}_m/\mathfrak{S}_k \times \mathfrak{S}_{m-k}} s\mathcal{P}_{k,m-k},$$

where the union is *not* disjoint. Using these representatives, we have

$$\operatorname{ind} \sigma = \mathbb{F}[\mathfrak{T}_m] \otimes_{\mathbb{F}[\mathcal{P}_{k,m-k}]} U_\sigma = \sum_{s \in \mathfrak{S}_m/\mathfrak{S}_k \times \mathfrak{S}_{m-k}} s \otimes U_\sigma.$$

On the other hand, as a subrepresentation of the polynomial ring, we can prove easily that

$$\mathbb{F}[\mathfrak{T}_m] \cdot U_\sigma = \sum_{s \in \mathfrak{S}_m/\mathfrak{S}_k \times \mathfrak{S}_{m-k}}^{\oplus} sU_\sigma \subset \otimes^m P(V^*)/R_{k-1}\left(\otimes^m P(V^*)\right),$$

and, as a $\mathfrak{S}_m$-module, this means that

$$\mathbb{F}[\mathfrak{T}_m] \cdot U_{\sigma_D} \simeq \operatorname{ind}_{\mathfrak{S}_k \times \mathfrak{S}_{m-k}}^{\mathfrak{S}_m} (\sigma_D \otimes 1).$$

Comparing the above two formulas, we can conclude that the map (3) must be bijective and the lemma follows. ■

Let us return to the proof of Proposition 3.1. Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ be a partition of k corresponding to the Young diagram D. We write $\lambda^\# = (\lambda_1 + m - k, \lambda_2, \cdots, \lambda_k)$, a partition of m, and denote by $D^\# \in \mathcal{Y}_m$ the corresponding Young diagram.

By Young's rule (see, for example, [8, §2.8.2]), the induced representation

$$\operatorname{ind}_{\mathfrak{S}_k \times \mathfrak{S}_{m-k}}^{\mathfrak{S}_m} (\sigma_D \otimes 1)$$

is multiplicity free and contains $\sigma_{D^\#}$. We shall show that a $\mathbb{F}[\mathfrak{T}_m]$-submodule of $\operatorname{ind} \sigma_D$ which contains $\sigma_{D^\#}$ must coincide with the whole module. Then, if we take all the summation of submodules which *does not* contain $\sigma_{D^\#}$, it gives the unique maximal proper submodule and we have done.

Take a standard tableau B of shape D such that the first row of B consists of $\{1, 2, \cdots, \lambda_1\}$, and that the second row consists of $\{\lambda_1 + 1, \lambda_1 + 2, \cdots, \lambda_1 + \lambda_2\}$ and so on. We also take a standard tableau $B^\#$ of shape $D^\#$ which is exactly the same as B except the first row. The first row of $B^\#$ consists of $\{1, 2, \cdots, \lambda_1\} \cup \{k+1, k+2, \cdots, m\}$.

Consider the Young symmetrizers c_B and $c_{B^\#}$. Here we follow the notation in [1, §4.1] (see also [8, Theorem 3.1.10]). Put

$$\begin{aligned} \mathfrak{S}_{\lambda_1} &= \mathfrak{S}(\{1, 2, \cdots, \lambda_1\}) \\ \mathfrak{S}_{m-k} &= \mathfrak{S}(\{k+1, k+2, \cdots, m\}) \\ \mathfrak{S}_{\lambda_1 + (m-k)} &= \mathfrak{S}(\{1, 2, \cdots, \lambda_1\} \cup \{k+1, k+2, \cdots, m\}), \end{aligned}$$

where $\mathfrak{S}(\{\cdots\})$ means the permutation group of a finite set $\{\cdots\}$ (this is an abuse of notation, but it would not lead to misunderstandings). Then we have

$$c_{B^\#} = \left(\sum_{s \in \mathfrak{S}_{\lambda_1+(m-k)}/\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{m-k}} s \right) \left(\sum_{t \in \mathfrak{S}_{m-k}} t \right) c_B.$$

We can take $u \in U_{\sigma_D}$ such that $c_B u \neq 0$. For this u, we have

$$c_{B^\#} \otimes u = \sum_{s \in \mathfrak{S}_{\lambda_1+(m-k)}/\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{m-k}} s \otimes \left(\sum_{t \in \mathfrak{S}_{m-k}} t \right) c_B u = (m-k)! \sum_s s \otimes c_B u.$$

Take $\varphi \in R_k$ as follows:

$$\varphi(i) = i \ (1 \leq i \leq k); \quad \varphi(i) = 1 \ (k+1 \leq i \leq m).$$

Then it is easy to see that the following three properties hold: for $s \in \mathfrak{S}_{\lambda_1+(m-k)}$,

$$\begin{array}{lcl} \varphi s & \in & \mathcal{P}_{k,m-k}, \\ \varphi s & \in & R_{k-1} \quad \text{if } s \notin (\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{m-k}) \cup (1,m)(\mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{m-k}), \\ \varphi e & = & \varphi(1,m) = \varphi, \end{array}$$

where e is the unit element and $(1,m)$ is the transposition of 1 and m. Put $u' = (m-k)!\, c_B u$. Using the above properties of φ, we calculate

$$\varphi\,(c_{B^\#} \otimes u) = \sum_s \varphi s \otimes u' = \sum_s e \otimes (\varphi s) u' = 2e \otimes \varphi u' = 2e \otimes u' \neq 0.$$

Since $c_{B^\#} \otimes u$ belongs to $U_{\sigma_{D^\#}}$, we get

$$\mathbb{F}[\mathfrak{T}_m] U_{\sigma_{D^\#}} \ni e \otimes u' \neq 0 \ (\exists u' \in U_{\sigma_D}),$$

hence

$$\mathbb{F}[\mathfrak{T}_m] U_{\sigma_{D^\#}} \supset \mathbb{F}[\mathfrak{T}_m] \otimes U_{\sigma_D} = \mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \sigma_D. \qquad \blacksquare$$

Definition 3.3. We denote by Σ_D ($D \in \mathcal{Y}_k$) the unique irreducible quotient of $\mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \sigma_D$, and call it a *standard representation* of $\mathfrak{T}_m$ corresponding to D.

Example 3.4. We shall give an example, in which $\mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \sigma_D$ is not irreducible. So it is necessary to take the irreducible quotient to obtain Σ_D.

Take $k = m-1$ and consider the digram D which corresponds to the partition $(1, \cdots, 1) = (1^{m-1})$. Then σ_D is the sign representation of $\mathfrak{S}_{m-1}$. It is easy to see that

$$\mathrm{ind}_{\mathfrak{S}_{m-1} \times \mathfrak{S}_1}^{\mathfrak{S}_m} \sigma_{(1^{m-1})} \simeq \sigma_{(2 \cdot 1^{m-2})} \oplus \sigma_{(1^m)}.$$

The space $\sigma_{(1^m)}$ is killed by R_{m-1}. In fact, this is a general property of sign representation. Let U be a $\mathfrak{T}_m$-module and U_{sgn} the sign-isotypic component of U as an $\mathfrak{S}_m$-module. Then U_{sgn} is killed by R_{m-1}, hence invariant under $\mathfrak{T}_m$.

Note that $\Sigma_{(1^{m-1})} \not\simeq \Sigma_{(2 \cdot 1^{m-2})}$ as proved in Theorem 3.5 below, while $\Sigma_{(1^{m-1})}\big|_{\mathfrak{S}_m} \simeq \Sigma_{(2 \cdot 1^{m-2})}\big|_{\mathfrak{S}_m} \simeq \sigma_{(2 \cdot 1^{m-2})}$ as $\mathfrak{S}_m$-modules.

Theorem 3.5. *The standard representations*

$$\{\Sigma_D \mid D \in \mathcal{Y}_k \ (1 \le k \le m)\}$$

of $\mathfrak{T}_m$ are mutually inequivalent, and they give a complete set of representatives of equivalence classes of irreducible representations of $\mathfrak{T}_m$.

Proof. Note that

$$R_{k-1} = \mathfrak{S}_m \left(\mathcal{P}_{k,m-k} \cap R_{k-1}\right).$$

Therefore it is easy to see that, if $D \in \mathcal{Y}_k$, $R_{k-1}\Sigma_D = 0$ holds. On the other hand, if we denote by $\varphi \in R_k$ as in the proof of Proposition 3.1, we have shown $\varphi \cdot \Sigma_D \neq 0$. This implies that

$$\Sigma_{D_1} \not\simeq \Sigma_{D_2} \quad \text{if} \quad |D_1| \neq |D_2|.$$

For $D_1, D_2 \in \mathcal{Y}_k$, the proof of Proposition 3.1 tells us that Σ_{D_i} $(i = 1, 2)$ contains $\sigma_{D_i^\#}$ as a representation of $\mathfrak{S}_m$. Introducing suitable lexicographic order on the set of Young diagram $\mathcal{Y}_m$, we conclude that $D_i^\#$ is the largest one among $\left\{D \in \mathcal{Y}_m \mid \sigma_D \subset \Sigma_{D_i}|_{\mathfrak{S}_m}\right\}$. For this, see Lemma 3.2 and consult with Young's rule. Therefore we get

$$\Sigma_{D_1}|_{\mathfrak{S}_m} \not\simeq \Sigma_{D_2}|_{\mathfrak{S}_m} \quad \text{if} \quad D_1 \neq D_2.$$

On the other hand, by [2, Th. 3.7 & Th. 7.4], the irreducible representations of $\mathfrak{T}_m$ are completely classified. Their classification shows that $\{\Sigma_D\}_D$ exhausts the irreducible ones. ■

4. Duality for general level

Let $R_k \subset \mathbb{F}[\mathfrak{T}_m]$ as in §2, and put $\mathcal{V}_k = R_k(\otimes^m P(V^*))$. In Section 2., we clarified the structure of $\mathcal{V}(m) = \mathcal{V}_m/\mathcal{V}_{m-1} = \otimes^m P(V^*)/\mathcal{V}_{m-1}$ in terms of the quotient representations. In this section, we study the structure of $\mathcal{V}(k) = \mathcal{V}_k/\mathcal{V}_{k-1}$ in general.

Let us consider $\mathcal{V}(k) = \mathcal{V}_k/\mathcal{V}_{k-1}$ as a $W_n \times \mathfrak{T}_m$-module. We note that $\otimes^k P(V^*)^+$ is contained in $\mathcal{V}(k)$ as the quotient image of the natural inclusion map $\otimes^k P(V^*) \hookrightarrow \otimes^k P(V^*) \otimes 1 \otimes \cdots \otimes 1 \subset \mathcal{V}_k$. Moreover we recall that, as a $W_n \times \mathfrak{T}_k$-module,

$$\otimes^k P(V^*)^+ \simeq \sum_{D \in \mathcal{Y}_k}^{\oplus} \mathrm{Hom}_{\mathfrak{S}_k}(\sigma_D, \otimes^k P(V^*)^+) \otimes \sigma_D,$$

where $\mathrm{Hom}_{\mathfrak{S}_k}(\sigma_D, \otimes^k P(V^*)^+)$ has the unique irreducible W_n-quotient π_D (Proposition 2.3).

Lemma 4.1.

$$\mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \otimes^k P(V^*)^+ \simeq \mathcal{V}(k)$$

Proof. By the universality of tensor products, we have a natural map

$$\begin{aligned}
&\mathbb{F}[\mathfrak{T}_m] \otimes_{\mathbb{F}[\mathcal{P}_{k,m-k}]} (\otimes^k P(V^*)^+) \\
&\simeq \ \mathbb{F}[\mathfrak{T}_m] \otimes_{\mathbb{F}[\mathcal{P}_{k,m-k}]} (\otimes^k P(V^*)^+ \ \otimes 1 \otimes \cdots \otimes 1) \\
&\quad \to \mathbb{F}[\mathfrak{T}_m] \cdot \left(\otimes^k P(V^*)^+ \otimes 1 \otimes \cdots \otimes 1\right) = \mathcal{V}(k).
\end{aligned}$$

By Lemma 3.2 this map is bijective, hence gives an isomorphism. ■

From this lemma, we calculate as

$$\begin{aligned}
\mathcal{V}(k) &\simeq \operatorname{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \ \otimes^k P(V^*)^+ \\
&\simeq \operatorname{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \ \sum_{D\in\mathcal{Y}_k}^{\oplus} \operatorname{Hom}_{\mathfrak{S}_k}(\sigma_D, \otimes^k P(V^*)^+) \otimes \sigma_D \\
&\simeq \sum_{D\in\mathcal{Y}_k}^{\oplus} \operatorname{Hom}_{\mathfrak{S}_k}(\sigma_D, \otimes^k P(V^*)^+) \otimes \left(\operatorname{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \ \sigma_D\right).
\end{aligned}$$

Therefore it is easy to see that the irreducible representation $\pi_D \otimes \Sigma_D$ is an irreducible quotient of $\mathcal{V}(k)$.

Theorem 4.2. *Let the notations be as above. An irreducible quotient of $W_n \times \mathfrak{T}_m$-module $\mathcal{V}(k)$ is of the form $\pi_D \otimes \Sigma_D$ $(D \in \mathcal{Y}_k)$. Moreover, it appears with quotient multiplicity one, i.e.,*

$$\dim \operatorname{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}(k), \pi_D \otimes \Sigma_D) = 1. \tag{4}$$

Proof. In the beginning, we shall prove the multiplicity freeness (4). Let us consider $\mathcal{V}(k)$ as a $GL(V) \times \mathfrak{S}_m$-module. Then we have proved that the multiplicity of $\rho_D \otimes \sigma_{D^\#}$ in $\mathcal{V}(k)$ is one. Since $\pi_D \otimes \Sigma_D$ contains $\rho_D \otimes \sigma_{D^\#}$ exactly once, we get

$$\dim \operatorname{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}(k), \pi_D \otimes \Sigma_D) \leq 1.$$

As mentioned before the theorem, the left hand side is positive, so the equality must hold.

Let U be an irreducible quotient of $\mathcal{V}(k)$. Then there is a proper invariant subspace $\mathcal{V}' \subset \mathcal{V}(k)$ such that $U \simeq \mathcal{V}(k)/\mathcal{V}'$. Let $f : \mathcal{V}(k) \to U$ be a projection map.

We take $\varphi \in R_k$ as before:

$$\varphi(i) = i \ (1 \leq i \leq k); \quad \varphi(i) = 1 \ (k+1 \leq i \leq m).$$

Since f is a $\mathfrak{T}_m$-morphism, we have $f(\varphi\mathcal{V}(k)) = \varphi f(\mathcal{V}(k)) = \varphi U$. Note that $\varphi\mathcal{V}(k)$ is a $W_n \times \mathcal{P}_{k,m-k}$-module, and is isomorphic to $\otimes^k P(V^*)^+$. Therefore φU is a $W_n \times \mathcal{P}_{k,m-k}$-module, which is a quotient of $\otimes^k P(V^*)^+$. Let us prove that the module φU is non-zero irreducible. In fact, it is easy to see that the space

$$\varphi\mathcal{V}(k) = \left(\otimes^k P(v) \otimes 1 \otimes \cdots \otimes 1\right) / \left(\mathcal{V}_k \cap \left(\otimes^k P(V^*) \otimes 1 \otimes \cdots \otimes 1\right)\right)$$

generates $\mathcal{V}(k)$ as a $\mathfrak{T}_m$-module, which means φU is cyclic for U. So, if $\varphi U = 0$, we have $U = 0$, which is not the case. To prove irreducibility, we assume that

φU has a proper $W_n \times \mathcal{P}_{k,m-k}$-submodule U'. Then $\mathbb{F}[\mathfrak{T}_m]U'$ becomes a proper $W_n \times \mathfrak{T}_m$-submodule of U, hence zero. Let us show it. Take $\varphi v \notin U'$. If $\varphi v \in \mathbb{F}[\mathfrak{T}_m]U'$, then we have

$$\varphi v = \varphi^2 v \in \varphi\mathbb{F}[\mathfrak{T}_m]U' = \mathbb{F}[\mathcal{P}_{k,m-k}]U' = U',$$

and this is a contradiction.

By Theorem 2.4, φU is isomorphic to $\pi_D \otimes \sigma_D$ for some $D \in \mathcal{Y}_k$. Therefore we have

$$\mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \varphi U \simeq \mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} (\pi_D \otimes \sigma_D) \simeq \pi_D \otimes \mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \sigma_D.$$

On the other hand, the universality of tensor products tells us that there is a surjective morphism

$$\pi_D \otimes \mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \sigma_D \simeq \mathrm{ind}_{\mathcal{P}_{k,m-k}}^{\mathfrak{T}_m} \varphi U \longrightarrow \mathbb{F}[\mathfrak{T}_m]\varphi U = U.$$

Since U is irreducible, it is isomorphic to $\pi_D \otimes \Sigma_D$ by Proposition 3.1. ∎

5. Howe's correspondence for $W_n \times \mathfrak{T}_m$

Recall that we say that an irreducible representation $\pi\otimes\Sigma$ has quotient multiplicity k in a representation U if

$$\dim \mathrm{Hom}_{W_n\times\mathfrak{T}_m}(U, \pi \otimes \Sigma) = k$$

holds.

Theorem 5.1. (1) *The tensor product* $\otimes^m P(V^*)$ *is quotient multiplicity free. This means that if* $\pi \otimes \Sigma$ *is an irreducible representation of* $W_n \times \mathfrak{T}_m$*, then*

$$\dim \mathrm{Hom}_{W_n\times\mathfrak{T}_m}(\otimes^m P(V^*), \pi \otimes \Sigma) \leq 1.$$

(2) *The quotient multiplicity of* $\pi \otimes \Sigma$ *is positive* (*hence it is one by* (1)), *if and only if* $\pi\otimes\Sigma$ *is isomorphic to* $\pi_D\otimes\Sigma_D$ *for some diagram* D *of size* k $(1 \leq k \leq m)$.

Proof. Consider the following exact sequence:

$$0 \to \mathcal{V}_{k-1} \to \mathcal{V}_k \to \mathcal{V}(k) \to 0.$$

It produces a long exact sequence:

$$\begin{aligned} 0 \to \mathrm{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}(k), \pi \otimes \Sigma) &\to \mathrm{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}_k, \pi \otimes \Sigma) \\ &\to \mathrm{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}_{k-1}, \pi \otimes \Sigma) \to \cdots . \end{aligned} \tag{5}$$

From the first three terms of the sequence, we get the following inequality

$$\mathrm{qmult}(\mathcal{V}_k : \pi \otimes \Sigma) \leq \mathrm{qmult}(\mathcal{V}(k) : \pi \otimes \Sigma) + \mathrm{qmult}(\mathcal{V}_{k-1} : \pi \otimes \Sigma),$$

where we denote $\dim \mathrm{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}, \pi\otimes\Sigma)$ by $\mathrm{qmult}(\mathcal{V}:\pi\otimes\Sigma)$. Using induction on k, we get

$$\mathrm{qmult}(\mathcal{V}_m:\pi\otimes\Sigma)\le\sum_{k=0}^{m}\mathrm{qmult}(\mathcal{V}(k):\pi\otimes\Sigma).$$

Assume that $k>0$. By Theorem 4.2, we know $\mathrm{qmult}(\mathcal{V}(k):\pi\otimes\Sigma)\le 1$ and the equality holds if and only if $\pi\otimes\Sigma\simeq\pi_D\otimes\Sigma_D$ for some D of size k. Therefore we get

$$\mathrm{qmult}(\mathcal{V}_m:\pi\otimes\Sigma)\le 1$$

and the equality holds only if $\pi\otimes\Sigma\simeq\pi_D\otimes\Sigma_D$ for some D or $\pi\otimes\Sigma$ is the trivial representation.

Let us show that the quotient multiplicity of the trivial representation is zero. Take $f\in\mathrm{Hom}_{W_n\times\mathfrak{T}_m}(\mathcal{V}_m,\mathbb{F})$. We identified $\mathcal{V}_m\simeq\mathbb{F}[z_{i,j}]$. Note that $z_{i,j}-f(z_{i,j})$ is contained in the kernel of f. Since $\ker f$ is graded (because W_n contains Euler's degree operators), $f(z_{i,j})$ is a member of $\ker f$, hence $f(z_{i,j})=0$. However, since $z_{i,j}$ produces a non-zero constant by differentiation, this means $f(\mathbb{F})=0$. By the similar argument as above, we conclude that $f=0$.

Now we are to prove "if"-part of the statement (2). Take $\pi_D\otimes\Sigma_D$ with $|D|=k$. By Theorem 2.4, we have

$$\dim\mathrm{Hom}_{W_n\times\mathfrak{T}_k}(\otimes^k P(V^*),\pi_D\otimes\Sigma_D^{(k)})=1,$$

where $\Sigma_D^{(k)}$ is the standard representation of $\mathfrak{T}_k$. Take $\varphi\in\mathfrak{T}_m$ such that

$$\varphi(i)=i\quad(1\le i\le k)\ ;\qquad\varphi(j)=k\quad(k<j\le m).$$

Then φ maps $\otimes^m P(V^*)$ naturally onto $\otimes^k P(V^*)$, and φ is W_n-equivariant. Hence we have a non-zero W_n-equivariant map

$$\otimes^m P(V^*)\xrightarrow{\varphi}\otimes^k P(V^*)\longrightarrow\pi_D\otimes\Sigma_D^{(k)},$$

composing the above two surjective maps. This proves

$$\mathrm{Hom}_{W_n}(\otimes^m P(V^*),\pi_D)\ne 0.$$

Let us put

$$U_{\pi_D}=\cap\{\ker f\mid f\in\mathrm{Hom}_{W_n}(\otimes^m P(V^*),\pi_D)\}.$$

Then it is easy to see $\otimes^m P(V^*)/U_{\pi_D}$ is non-zero and a finite direct sum of π_D (cf. formula (2.5) in [5]);

$$\otimes^m P(V^*)/U_{\pi_D}\simeq\pi_D\otimes\omega,$$

where ω is a certain finite dimensional vector space. Since the actions of W_n and $\mathfrak{T}_m$ are commutative, ω inherits an action of $\mathfrak{T}_m$. Choose $\omega'\subset\omega$ such that ω/ω' is irreducible. Then we get a non-zero $W_n\times\mathfrak{T}_m$-homomorphism

$$\otimes^m P(V^*)\xrightarrow{\text{proj.}}\otimes^m P(V^*)/U_{\pi_D}\simeq\pi_D\otimes\omega\xrightarrow{\text{proj.}}\pi_D\otimes(\omega/\omega').$$

Note that the irreducible quotients of $\otimes^m P(V^*)$ are $\pi_D\otimes\Sigma_D$ for certain D by the "only if"-part of the theorem (2). Therefore it must hold that $\pi_D\otimes(\omega/\omega')\simeq\pi_D\otimes\Sigma_D$, and we have done. ∎

For an algebra A and an A-module M, we denote by $\mathcal{R}_A(M)$ the set of equivalence classes of irreducible quotients of M.

Corollary 5.2. *There is a bijective correspondence between $\mathcal{R}_{W_n}(\otimes^m P(V^*))$ and $\mathcal{R}_{\mathfrak{T}_m}(\otimes^m P(V^*))$ defined by the following property. Irreducible representations $\pi \in W_n^\wedge$ and $\Sigma \in \mathfrak{T}_m^\wedge$ are in correspondence if and only if*

$$\dim \operatorname{Hom}_{W_n \times \mathfrak{T}_m}(\otimes^m P(V^*), \pi \otimes \Sigma) = 1.$$

The correspondence can be given explicitly as follows:

$$\begin{aligned} &\mathcal{R}_{W_n}(\otimes^m P(V^*)) = \{\pi_D \mid D \in \mathcal{Y}_k \ (1 \le k \le m)\}, \\ &\mathcal{R}_{\mathfrak{T}_m}(\otimes^m P(V^*)) = \{\Sigma_D \mid D \in \mathcal{Y}_k \ (1 \le k \le m)\} = \mathfrak{T}_m^\wedge, \\ &\textit{and } \pi_D \leftrightarrow \Sigma_D. \end{aligned}$$

We call the correspondence in Corollary 5.2 *Howe correspondence* (cf. [4, 5]).

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