Journal of Lie Theory
Volume **9** (1999) 39–38

On the image of a generalized d-plane transform on $\mathbb{R}^n$

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Communicated by J. Hilgert

Abstract. The generalized d-plane transform of a function f on $\mathbb{R}^n$ is defined on a set $\mathcal{E}$ of d-dimensional affine subspaces ("d-planes") of $\mathbb{R}^n$ by integration of f over each subspace in $\mathcal{E}$. In general, it renders less information about the unknown function f than in the special case of the well known Radon d-plane transform, where $\mathcal{E}$ contains every d-plane in $\mathbb{R}^n$. We study the case where $\mathcal{E}$ appears as an orbit of a matrix group and characterize the range of the spaces of Schwartz functions and of smooth ones with compact support.

Introduction

The Radon d-plane transform of a function f on $\mathbb{R}^n$ is a function on the set of d-dimensional affine subspaces ("d-planes") of $\mathbb{R}^n$, obtained by integration of f over each such subspace. For $d = n-1$, the case of hyperplanes, it was first introduced and studied by Radon in his famous work of 1917 [12]. For $d < n-1$, the idea goes back to John, who considered line integrals of point functions in $\mathbb{R}^3$ in connection with the ultrahyperbolic differential equation [10]. However, the first one who systematically studied this transform was Helgason. Beside inversion formulas, he investigated the range of certain function spaces, such as $C_c^\infty(\mathbb{R}^n)$, the smooth functions with compact support, or $\mathcal{S}(\mathbb{R}^n)$, the Schwartz functions (see [9]). The range of $C_c^\infty(\mathbb{R}^n)$ under the Radon d-plane transform has been described by Helgason in terms of the so called "moment conditions". For $d = n-1$ (the Radon transform), these characterize the range of $\mathcal{S}(\mathbb{R}^n)$, too [9]. However, this is no longer true for $d < n-1$, as Gonzalez later showed [6].

In the already mentioned article of 1938 John showed that the line functions in $\mathbb{R}^3$, which are obtained by integration of point functions over straight lines, satisfy certain differential equations, which characterize this class of line functions [10]. The idea was further developed by Gel'fand, Gindikin and Graev in 1980, who gave a characterization of the range of $\mathcal{S}(\mathbb{R}^n)$ in terms of a system of second order partial differential equations but without complete proof [3]. In 1984 Grinberg

ISSN 0949–5932 / \$2.50

used both moment conditions and the differential operators of Gel'fand et al. to determine the range [7]. The proof of the assertion of Gel'fand et al. was given in detail by Richter in 1986 [13]. It was his great achievement to observe that the differential operators arise from infinitesimal operators of the group of euclidean motions in $\mathbb{R}^n$ [14]. Later, Gonzalez and Kurusa gave other proofs, the former in the group theoretic frame of Richter, the latter following John's method of parametrizing the d-planes [11], [6].

In this article we study the generalized situation, where the d-plane transform is only defined on certain *subsets* $\mathcal{E}$ of the set $G(d,n)$ of all d-planes in $\mathbb{R}^n$. In practice, particularly in computer tomography, where the theory applies decisively, measurements can only be made for a finite set of d-planes, so it is of natural interest to consider the whole problem in the restriction to subclasses of d-planes. It is of great practical importance to see how the main facts such as inversion formulae and range theorems depend on the geometry of $\mathcal{E}$. Gel'fand, Graev and others have studied the complex case, where $\mathcal{E}$ is an n-dimensional analytic submanifold of the set of all (complex) d-planes in $\mathbb{C}^n$, and have termed $\mathcal{E}$ "admissible", if it is possible to recapture a function from its integrals over the planes in $\mathcal{E}$ (see for instance [4]). Felix has treated some more general settings [2]. In this work we show that the group theoretic approach, as it has been introduced by Helgason and applied by Richter and Gonzalez with great success to the range problem, bears its fruits in a greater generality of cases, provided that $\mathcal{E}$ is an orbit of a matrix group. So we restrict ourselves to such cases. Moreover, the group theoretic treatment of the *generalized* d-plane transform reveals the meaning of the otherwise mysterious construction of the differential operators which characterize the range in [14] (here: remark 2 to lemma 5.2).

In the sequel, by "classical case" we shall mean the case $\mathcal{E} = G(d,n)$. The Radon d-plane transform, which refers to it, has been extensively studied through the past decades by Helgason, Richter and Gonzalez, as already mentioned.

The organization of the paper is as follows. The set $\mathcal{E}$ is introduced in section 1. In section 2 the notion of rapidly decreasing functions on $\mathcal{E}$ and the Fourier transform are defined. Section 3 deals with the group theoretic nature of the differential operators which serve to characterize the range. Here we follow Richter [14] and correct the way in which he defines these operators. This part can be skipped during the first reading. The d-plane transform, the moment conditions and the differential operators in connection to it are introduced in section 4. In 5, the range of $\mathcal{S}(V)$, the set of Schwartz functions on the vector space V, is characterized by both moment conditions and differential operators. The main work is done here. Section 6 covers a lot of special cases (the classical case included), where the differential operators *alone* characterize the range of $\mathcal{S}(V)$. In section 7 it turns out that for $d = n-1$ the only example that fits into our framework is the classical case itself. The range of $C_c^\infty(V)$ is determined with the help of a "polar coordinate" version of the classical Paley-Wiener theorem, due to Helgason [8]. This is done in section 8. Finally, the theory is applied to some concrete situations (section 9). The precise characterization of those cases, where we can dispense with the moment conditions, is still an open problem.

The author would like to stress that the main facts concerning the range of $C_c^\infty(\mathbb{R}^n)$ and $\mathcal{S}(\mathbb{R}^n)$ in the classical case are here recovered as special cases. In this sense, the classical results are truly generalized.

1. Fundamental concepts

Let H be a Lie group, ρ a (C^∞-) representation of H on a finite-dimensional vector space V. The representation ρ gives rise to a semi-direct product $H \times_s V$, where multiplication is given by

$$(h_1, v_1) \cdot (h_2, v_2) := (h_1 h_2, v_1 + \rho(h_1) v_2)$$

(from now on we write $h \cdot v$ instead of $\rho(h)v$).

The Lie group $H \times_s V$ operates transitively on V:

$$(h, v) \cdot v_0 := v + h \cdot v_0 \,.$$

It also operates transitively on $G(d, V)$, the set of d-dimensional affine subspaces of V $(1 \leq d \leq n-1,\ n = \dim V)$:

$$(h, v) \cdot E := v + h \cdot E \quad (h \cdot E = \{h \cdot z | z \in E\})\,.$$

We fix a d-dimensional linear subspace E of V and denote by H_E the isotropy subgroup of E in H. Then $H_E \times_s E$ is the isotropy subgroup of E in $H \times_s V$. The orbit of E will be identified with ${}^{H\times_s V}/_{H_E \times_s E}$ and will be given the differentiable structure of this homogeneous space. ${}^{H\times_s V}/_{H_E \times_s E}$ is in fact a vector bundle over ${}^{H}/_{H_E}$: If $\sigma : W \to H$ is a local cross section of an open set $W \subseteq {}^{H}/_{H_E}$ into H and U a complement of E in V, we obtain a local trivialization by the mapping $(\dot{h}, u) \mapsto (\sigma(\dot{h}), \sigma(\dot{h}) \cdot u)(H_E \times_s E)$ $(\dot{h} = hH_E)$ of $W \times U$ into ${}^{H\times_s V}/_{H_E \times_s E}$. We put $\mathcal{E} := {}^{H\times_s V}/_{H_E \times_s E}$ and $(\dot{h}, x)$ for $(h, x)(H_E \times_s E)$.

The contragredient representation ρ^*,

$$[\rho^*(h)\xi](v) := \xi[\rho(h)^{-1} v] \quad (v \in V,\ \xi \in V^*)\,,$$

leads analogously to a semi-direct product $H \times_s V^*$. We write $h^* \cdot \xi$ instead of $\rho^*(h)\xi$. If $E^\perp \subseteq V^*$ denotes the orthogonal subspace to E, then the set

$$\mathcal{E}^* := \{(\dot{h}, \xi) | \xi \in h^* \cdot E^\perp\}$$

can be given the structure of a vector bundle over ${}^{H}/_{H_E}$ via the mappings $(\dot{h}, w) \mapsto (\dot{h}, \sigma(\dot{h})^* \cdot w)$ of $W \times E^\perp$ into $\mathcal{E}^*$ (notation as above). (Note that $h^* \cdot E^\perp$ does not depend on $h \in hH_E$.) The set $\mathcal{E}^*$ will be called "dual bundle" (its fibres can be naturally identified with the dual spaces of the fibres of $\mathcal{E}$).

2. Rapidly decreasing functions on $\mathcal{E}$ and the Fourier transform

Gonzalez introduced a class of functions on $G(d, \mathbb{R}^n)$, the so called rapidly decreasing functions [6]. They can be defined on $\mathcal{E}$ in a similar way. By a local cross section of a compact set $M \subseteq {}^{H}/_{H_E}$ into H we shall always mean a local C^∞-cross section of an open neighborhood of M into H. Let U be a complement of E in V and introduce linear coordinates $(u_1, \dots, u_{n-d})$ on U. The vector bundle $\mathcal{E}$ is of fibre type U, as already shown.

Definition 2.1. A C^∞-function φ on $\mathcal{E}$ is called rapidly decreasing if for all multiindices $\alpha, \beta \in \mathbb{Z}_+^{n-d}$, all differential operators T on ${}^H/_{H_E}$ and all compact sets $M \subseteq {}^H/_{H_E}$ admitting a local cross section σ into H

$$\sup_{\dot{h} \in M,\, u \in U} |u^\alpha T_{\dot{h}} \frac{\partial^\beta}{\partial u^\beta} \varphi(\dot{h}, \sigma(\dot{h}) \cdot u)| < \infty .$$

(We have put $u^\alpha := \prod_{i=1}^{n-d} u_i^{\alpha_i}$, $\frac{\partial^\beta}{\partial u^\beta} := \prod_{i=1}^{n-d} \frac{\partial^{\beta_i}}{\partial u_i^{\beta_i}}$.)

The above definition does not depend on the choice of U or of its basis. Rapidly decreasing functions on $\mathcal{E}^*$ can be defined similarly ($E^\perp$ replaces U). We write $\mathcal{S}(\mathcal{E})$, $\mathcal{S}(\mathcal{E}^*)$ for these classes of functions.

Let $f \in C^\infty(V^*)$ be a function of the Schwartz class. We define a function φ_f on $\mathcal{E}^*$ by

$$\varphi_f(\dot{h}, \xi) := f(\xi) .$$

Lemma 2.2. $\varphi_f \in \mathcal{S}(\mathcal{E}^*)$.

Proof. $\varphi_f \in C^\infty(\mathcal{E}^*)$ by the local trivializations of the vector bundle $\mathcal{E}^*$. We keep the notation of definition 2.1 and consider a covering of M by open, relatively compact subsets $O_1, \ldots, O_k$ having the additional property that each $\overline{O_i}$ is contained in a chart of ${}^H/_{H_E}$ and that there exists a local cross section on the union of all $\overline{O_i}$. Under these assumptions it suffices to verify the property of definition 2.1 for each O_i separately. In what follows, $\overline{O_i}$ will be regarded as a compact set in an $\mathbb{R}^m$.

Let $\|\cdot\|_{V^*}$ be a norm on V^*, $\|\cdot\|$ the corresponding norm of linear operators on the space of endomorphisms of V^*. If T is a differential operator of order r on ${}^H/_{H_E}$, let

$$C := \max\{ \sup_{0 \le |s| \le r} \sup_{\dot{h} \in O_i} \|\partial^s \rho^*(\sigma(\dot{h}))\|, \sup_{\dot{h} \in O_i} \|\rho^*(\sigma(\dot{h}))^{-1}\|\} < \infty$$

($s = (s_1, \ldots, s_m) \in \mathbb{Z}_+^m$ is a multiindex, $|s| := \sum_{i=1}^m s_i$, and ∂^s is defined as in definition 2.1 ($O_i \subseteq \mathbb{R}^m$)). Let $(w_1, \ldots, w_{n-d})$ denote the vector of coordinates with respect to a basis in $E^\perp$. The expression

$$|w^\alpha T_{\dot{h}} (\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_{n-d}})^\beta \varphi_f(\dot{h}, \sigma(\dot{h})^* \cdot w)|$$

$$= |(\,(\sigma(\dot{h})^{-1})^* \cdot (\sigma(\dot{h})^* \cdot w)\,)^\alpha (\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_{n-d}})^\beta T_{\dot{h}} f(\sigma(\dot{h})^* \cdot w)|$$

can then be estimated by a linear combination of terms of the form

$$C^l \|\sigma(\dot{h})^* \cdot w\|_{V^*}^{l'} \cdot |Df(\sigma(\dot{h})^* \cdot w)|$$

(D a constant coefficient differential operator, $l, l' \in \mathbb{Z}_+$). However, since $f \in \mathcal{S}(V^*)$, the latter are bounded for $\dot{h} \in O_i$, $w \in E^\perp$. ∎

Let p_U denote the projection of V onto U, according to the decomposition $V = U \oplus E$. In order to define a Fourier transform on $\mathcal{E}$, we have to assume that a translation-invariant integral du on U is left invariant by the mappings

$$u \longmapsto p_U(s \cdot u)\,, \quad s \in H_E\,.$$

Definition 2.3. The Fourier transform $\tilde{\varphi}$ of a rapidly decreasing function φ on $\mathcal{E}$ is a function on the dual bundle $\mathcal{E}^*$, defined by

$$\tilde{\varphi}(\dot{h}, \xi) := \int_U \varphi(\dot{h}, h \cdot u) e^{-i\langle h \cdot u, \xi \rangle} du\,, \quad \xi \in h^* \cdot E^\perp, \ \langle h \cdot u, \xi \rangle := \xi(h \cdot u)\,.$$

(The right hand side does not depend on $h \in hH_E$ because of the assumption above.)

In fact, the Fourier transform on $\mathcal{E}$ is the classical Fourier transform on its fibres. It therefore can be extended to functions which are integrable on the fibres of $\mathcal{E}$.

Proposition 2.4. $\varphi \in \mathcal{S}(\mathcal{E}) \Rightarrow \tilde{\varphi} \in \mathcal{S}(\mathcal{E}^*)$.

The proof is straightforward and is left to the reader.

3. The left regular representation of $H \times_s V$ on $C^\infty(V)$ and the kernel of its differential

As we have seen in the first section, the Lie group $H \times_s V$ operates transitively on V. The subgroup H (identified with $H \times_s \{0\}$) is the isotropy subgroup of $0 \in V$, so V can be identified (set-theoretically and as a manifold) with ${}^{H \times_s V}/_{H}$. The left regular representation λ of $H \times_s V$ on $C^\infty(V)$ is defined by

$$[\lambda(h,v)f](v') = f[(h,v)^{-1} \cdot v'] = f(h^{-1} \cdot (v' - v)) \quad (f \in C^\infty(V))\,.$$

It follows that the subspace of Schwartz functions $\mathcal{S}(V)$ is λ-invariant. The representation λ can be differentiated on $C^\infty(V)$. For $Y \in \mathfrak{h} \times_s V$, the Lie algebra of $H \times_s V$, and $f \in C^\infty(V)$ we have

$$d\lambda(Y) f(v) = \frac{d}{dt} f(\exp(-tY) \cdot v)|_{t=0}\,.$$

If $f \in \mathcal{S}(V)$, the right hand side remains in $\mathcal{S}(V)$. This shows that $\mathcal{S}(V)$ is $d\lambda$-invariant, so λ can be differentiated as a representation on $\mathcal{S}(V)$. Since $d\lambda$ is a representation of $\mathfrak{h} \times_s V$, it can be extended to a representation of the universal enveloping algebra $U(\mathfrak{h} \times_s V)$.

Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{h}$, $Z_1, \ldots, Z_n$ one of V (being identified with its Lie algebra). By the Poincaré-Birkhoff-Witt theorem the elements $X_1^{i_1} \ldots X_m^{i_m} Z_1^{j_1} \ldots Z_n^{j_n}$, $i_k, j_l \in \mathbb{Z}_+$, form a basis of $U(\mathfrak{h} \times_s V)$.

Definition 3.1. $P \in U(\mathfrak{h} \times_s V)$ is said to have X-degree $M \geq 0$ if P can be written as a linear combination of basis vectors $X_1^{i_1} \ldots X_m^{i_m} Z_1^{j_1} \ldots Z_n^{j_n}$ with $i_1 + \ldots + i_m \leq M$, where at least one vector appears with $i_1 + \ldots + i_m = M$.

This definition does not depend on the (ordered) basis $X_1, \ldots, X_m$ of $\mathfrak{h}$. It comes from the canonical filtration of $U(\mathfrak{h})$. An X-degree is assigned to every nonzero element $P \in U(\mathfrak{h} \times_s V)$.

The adjoint representation Ad of $H \times_s V$ as well as its differential ad extend to representations on $U(\mathfrak{h} \times_s V)$.

Lemma 3.2. *If $P \in U(\mathfrak{h} \times_s V)$ has X-degree $M > 0$, then for every $Z \in V$ $adZ(P)$ has X-degree $< M$ or is equal to 0.*

The proof follows immediately from the relations $\mathrm{ad}Z(X_k) \in V$ $(1 \leq k \leq m)$ for the derivations $\mathrm{ad}Z$.

Let $\pi_V : U(\mathfrak{h} \times_s V) \to U(V)$ be the canonical projection, defined by $\pi_V(X_1^{i_1} \ldots X_m^{i_m} Z_1^{j_1} \ldots Z_n^{j_n}) = 0$ for $i_1 + \ldots + i_m > 0$, $\pi_V(Z_1^{j_1} \ldots Z_n^{j_n}) = Z_1^{j_1} \ldots Z_n^{j_n}$. We define a subspace $\mathcal{K}(\mathfrak{h} \times_s V)$ of $U(\mathfrak{h} \times_s V)$ as follows:[1]

> An element $P \in U(\mathfrak{h} \times_s V)$ of X-degree $M \geq 0$ belongs to $\mathcal{K}(\mathfrak{h} \times_s V)$ iff for all $j_1, \ldots, j_n \in \mathbb{Z}_+$, $j_1 + \ldots + j_n \leq M$, $\pi_V(\mathrm{ad}Z_1)^{j_1} \ldots (\mathrm{ad}Z_n)^{j_n}(P) = 0$. Moreover, $0 \in \mathcal{K}(\mathfrak{h} \times_s V)$.

For later purposes we write $\mathcal{K}^1(\mathfrak{h} \times_s V)$ for $\{P \in \mathcal{K}(\mathfrak{h} \times_s V) \,|\, P$ has X-degree $1\} \cup \{0\}$.

Now we are ready to state and prove the main result of this section.

Theorem 3.3. *$ker(d\lambda) = \mathcal{K}(\mathfrak{h} \times_s V)$.*

Proof. We first prove the following statement:
If an element P with $\pi_V(P) = 0$ satisfies

$$d\lambda(P) = d\lambda(\mathrm{Ad}(\exp Z)P)$$

for every $Z \in V$, then $P \in \ker(d\lambda)$.

Let $P \neq 0$ be such an element,

$$P = \sum_J c_J P_J, \quad J = (i_1, \ldots, j_n), \quad P_J = X_1^{i_1} \ldots X_m^{i_m} Z_1^{j_1} \ldots Z_n^{j_n}, \quad c_J \in \mathbb{C} \setminus \{0\}\,.$$

For $f \in C^\infty(V)$ we have

$$d\lambda(P_J)(f)(v) = \frac{\partial^{|J|}}{\partial t^J} f(v(t) + h(t) \cdot v)|_{t=0}\,, \quad \text{where}$$

$$t = (t_1, \ldots, t_m, t_{m+1}, \ldots, t_{m+n}), \quad h(t) = \exp(-t_m X_m) \ldots \exp(-t_1 X_1) \in H\,,$$

$$v(t) = \exp(-t_{m+n} Z_n) \ldots \exp(-t_{m+1} Z_1) = -\sum_{i=1}^{n} t_{m+i} Z_i \in V\,,$$

[1]Richter's definition leads to a much greater space $\mathcal{K}(n)$ (following his notation), and the relation '$\mathcal{K}(n) \subseteq \mathrm{Ker}\lambda$' fails to hold in [14, Proposition 3]. For example, if P belongs to $\mathcal{K}^M(n)$, then $P + \mathcal{U}^{M-1}(n) \subseteq \mathcal{K}(n)$, but of course $P + \mathcal{U}^{M-1}(n) \not\subseteq \mathrm{Ker}\lambda$. Furthermore, the use of the operators Z_i as $\mathrm{Ad}(\exp(Z_i) - \mathrm{Id}$ instead of $\mathrm{ad}Z_i$ in the definition of $\mathcal{K}^M(n)$ makes it much harder to deduce the last equation in the proof of Proposition 3 if Y is not just one of the $Z_1, \ldots, Z_n$, but a linear combination of them.

$$\frac{\partial^{|J|}}{\partial t^J} = \frac{\partial^{i_1+\ldots+i_m+j_1+\ldots+j_n}}{\partial t_1^{i_1}\ldots\partial t_m^{i_m}\partial t_{m+1}^{j_1}\ldots\partial t_{m+n}^{j_n}} .$$

The relation $\exp(\mathrm{Ad}(g)Y) = g\exp(Y)g^{-1}$, $Y \in \mathfrak{h}\times_s V$, $g \in H\times_s V$, implies

$$d\lambda(\mathrm{Ad}(g)P)(f)(v) = d\lambda(P)(f\circ L_g)(g^{-1}\cdot v) \quad \text{for } P \in U(\mathfrak{h}\times_s V) , \tag{1}$$

L_g denoting left translation by g. We obtain

$$d\lambda(\mathrm{Ad}(\exp v)P_J)(f)(v) = d\lambda(P_J)(f\circ L_v)(0) = \frac{\partial^{|J|}}{\partial t^J} f(v + v(t) + h(t)\cdot 0)|_{t=0} = 0 ,$$

since $\frac{\partial^{|J|}}{\partial t^J}$ contains at least one derivative with respect to a t_i, $1 \le i \le m$. The assumption on P leads to $d\lambda(P)(f)(v) = 0$, which holds for all $v \in V$, $f \in C^\infty(V)$. This proves the statement.

Let now $P \in \mathcal{K}(\mathfrak{h}\times_s V)$ having X-degree M. For all $Z \in V$ and all $j_1,\ldots,j_n \in \mathbb{Z}_+$, $j_1+\ldots+j_n = M-1$, we obtain by the use of $\mathrm{Ad}(\exp Z) = e^{adZ}$:

$$d\lambda(\mathrm{Ad}(\exp Z)(\mathrm{ad}Z_1)^{j_1}\ldots(\mathrm{ad}Z_n)^{j_n}P) = d\lambda((\mathrm{ad}Z_1)^{j_1}\ldots(\mathrm{ad}Z_n)^{j_n}P) .$$

From the above statement it follows that the right hand side is equal to 0. We repeat this argument with $j_1+\ldots+j_n = M-2$, $M-3$ etc. and obtain inductively $d\lambda(P) = 0$.

For the converse we observe that $\ker(d\lambda)$ is $\mathrm{Ad}(H\times_s V)$-invariant, as equation (1) shows. By differentiation we obtain $(\mathrm{ad}Z)(\ker(d\lambda)) \subseteq \ker(d\lambda)$ for every $Z \in V$ (in fact, $\mathrm{Ad}(g)$ and $\mathrm{ad}(Z)$ operate on the finite dimensional subspaces of elements of X-degrees $\le M$). Therefore it suffices to show that if $P \in \ker(d\lambda)$ it holds $\pi_V(P) = 0$. Let $P_0 = \pi_V(P)$. For $f \in C^\infty(V)$, $v \in V$ we obtain:

$$\begin{aligned} 0 = d\lambda(P)(f)(v) &= d\lambda[\mathrm{Ad}(\exp v)P](f)(v) = d\lambda(P)(f\circ L_{\exp v})(0) \quad \text{by } (1) \\ &= d\lambda(P_0)(f\circ L_{\exp v})(0) + d\lambda(P-P_0)(f\circ L_{\exp v})(0) = d\lambda(P_0)(f\circ L_{\exp v})(0) \\ &= d\lambda[\mathrm{Ad}(\exp v)P_0](f)(v) = d\lambda(P_0)(f)(v) , \end{aligned}$$

so $d\lambda(P_0) = 0$ and $P_0 = 0$, since $d\lambda$ is injective on $U(V)$. ■

Remark . The definition of the subspace $\mathcal{K}(\mathfrak{h}\times_s V)$ is suggested by Corollary 3.6 in [1].

4. The d-plane transform

We return to the setting of the first two sections. E is a d-dimensional subspace of V ($1 \le d \le n-1$), U a complement of E in V. The following considerations contain the Radon transform ($d = n-1$) as a special case.

Let us introduce translation invariant integrals dz and du on E and U respectively. The integral $dy = dz\,du$ is then translation invariant on V. From now on we assume that H_E acts by unimodular automorphisms on E (that is of determinant ± 1). This, combined with the condition needed for the Fourier transform on $\mathcal{E}$, leads to the assumption that H_E acts unimodularly on both E and V. Moreover, we can assume that H operates by unimodular automorphisms on V; if this is not the case, we only have to replace $\rho(h)$ by $\delta(h)^{-1/n}\rho(h)$, where $\delta(h)$ denotes the modulus (= absolute value of the determinant) of $\rho(h)$.

Definition 4.1. Let f be an integrable function on V. The d-plane transform of f is a function $\hat{f}$ on $\mathcal{E}$, defined by

$$\hat{f}(\dot{h}, x) = \int_E f(x + h \cdot z) dz \,.$$

The new function is well defined almost everywhere, due to the unimodularity assumption above. The following lemma, often called "Projection-Slice theorem", is of great importance.

Lemma 4.2. *Let $f \in \mathcal{S}(V)$, $\tilde{f} \in \mathcal{S}(V^*)$ being the (classical) Fourier transform of f. The d-plane transform $\hat{f}$ is then integrable on every fibre of $\mathcal{E}$, and we have:*

$$\tilde{\hat{f}}(\dot{h}, \xi) = \tilde{f}(\xi) \,, \quad \xi \in h^* \cdot E^{\perp} \,.$$

Proof. Clearly, $\hat{f}$ is everywhere defined and integrable on the fibres by the Fubini theorem. We have:

$$\tilde{\hat{f}}(\dot{h}, \xi) = \int_U \hat{f}(\dot{h}, h \cdot u) e^{-i\langle h \cdot u, \xi \rangle} du = \int_U \int_E f(h \cdot (u + z)) e^{-i\langle h \cdot (u+z), \xi \rangle} dz \, du$$

$(\langle h \cdot z, \xi \rangle = 0)$

$$= \int_V f(h \cdot y) e^{-i\langle h \cdot y, \xi \rangle} dy = \int_V f(y) e^{-i\langle y, \xi \rangle} dy = \tilde{f}(\xi) \,.$$

■

Lemma 4.3. *For $f \in \mathcal{S}(V)$ we have $\hat{f} \in \mathcal{S}(\mathcal{E})$.*

Proof. Clearly, $\hat{f} \in C^{\infty}(\mathcal{E})$. From lemmas 4.2 and 2.2 it follows that $\tilde{\hat{f}} \in \mathcal{S}(\mathcal{E}^*)$. The desired relation is now obtained from the analogue of proposition 2.4 for the inverse Fourier transform. ■

An important property of the d-plane transforms $\hat{f}$ is that they satisfy the so called "moment conditions".

Proposition 4.4. *Let $\varphi \in \mathcal{S}(V)^{\wedge}$. For every $m \in \mathbb{Z}_+$ there exists a homogeneous polynomial P_m of degree m on V^*, such that for every $h \in {}^{H}/_{H_E}$, $\xi \in h^* \cdot E^{\perp}$:*

$$\int_U \varphi(\dot{h}, h \cdot u) \cdot \langle h \cdot u, \xi \rangle^m du = P_m(\xi) \,. \tag{2}$$

Proof. Let $\varphi = \hat{f}$, $f \in \mathcal{S}(V)$. We have:

$$\int_U \varphi(\dot{h}, h \cdot u) \langle h \cdot u, \xi \rangle^m du = \int_U \int_E f(h \cdot (u + z)) \langle h \cdot (u + z), \xi \rangle^m dz \, du$$

$$= \int_V f(h \cdot y) \langle h \cdot y, \xi \rangle^m dy = \int_V f(y) \langle y, \xi \rangle^m dy \,.$$

■

Let $\mathcal{S}_M(\mathcal{E}) := \{\varphi \in \mathcal{S}(\mathcal{E}) | \varphi \text{ satisfies } (2)\}$. We have shown: $\mathcal{S}(V)^\wedge \subseteq \mathcal{S}_M(\mathcal{E})$.

The functions in $\mathcal{S}(V)^\wedge$ have a further common property. They are annihilated by certain differential operators.

Let ν be the left regular representation of $H \times_s V$ on $C^\infty(\mathcal{E})$. For $f \in \mathcal{S}(V)$ we compute:

$$[\nu(h,v)\hat{f}](\dot{h}', x') = \hat{f}[(h,v)^{-1} \cdot (\dot{h}', x')] = \hat{f}(h^{-1}\dot{h}', h^{-1} \cdot x' - h^{-1} \cdot v)$$

$$= \int_E f(h^{-1} \cdot x' - h^{-1} \cdot v + h^{-1}h' \cdot z)dz = \int_E f((h,v)^{-1} \cdot (x' + h' \cdot z))dz$$

$$= \int_E [\lambda(h,v)f](x' + h' \cdot z)dz = [\lambda(h,v)f]^\wedge(\dot{h}', x')$$

(λ denoting the left regular representation of $H \times_s V$ on $C^\infty(V)$), so

$$[\lambda(g)f]^\wedge = \nu(g)\hat{f}\,, \quad g \in H \times_s V\,, \tag{3}$$

which means that the operator $f \mapsto \hat{f}$ intertwines λ and ν. Differentiation of (3) leads to

$$[d\lambda(Y)f]^\wedge = d\nu(Y)\hat{f}$$

for Y in $\mathfrak{h} \times_s V$ and then in $U(\mathfrak{h} \times_s V)$. Thus, if $Y \in \ker(d\lambda)$, $d\nu(Y)\hat{f} = 0$ for all $f \in \mathcal{S}(V)^\wedge$.

Let $\mathcal{S}_D(\mathcal{E}) := \{\varphi \in \mathcal{S}(\mathcal{E}) | d\nu(Y)\varphi = 0 \text{ for } Y \in \ker(d\lambda)\}$. We have shown: $\mathcal{S}(V)^\wedge \subseteq \mathcal{S}_D(\mathcal{E})$. Thus, $\mathcal{S}(V)^\wedge \subseteq \mathcal{S}_D(\mathcal{E}) \cap \mathcal{S}_M(\mathcal{E})$.

5. The image of the d-plane transform

In this section we prove the relation

$$\mathcal{S}(V)^\wedge = \mathcal{S}_D(\mathcal{E}) \cap \mathcal{S}_M(\mathcal{E})\,. \tag{4}$$

For this we have to impose the following two conditions on the action of the Lie group H:

(i) The mapping from $H \times E^\perp$ into V^*, $(h,w) \mapsto h^* \cdot w$, is surjective and its restriction to $H \times (E^\perp \setminus \{0\})$ a submersion.

(ii) If $h^* \cdot w_1 = w_2$, $w_1, w_2 \in E^\perp$, $h \in H$, then there exists $s \in H_E$ such that $s^* \cdot w_1 = w_2$.

Both conditions are satisfied in the classical case $(H = SO(n),\ V = \mathbb{R}^n)$.[2]

[2] In the first and last example of section 9 it is shown that they are also necessary for a relation like (4).

Lemma 5.1. *The d-plane transform is an injective mapping on $\mathcal{S}(V)$.*

The proof follows from lemma 4.2 and the surjectivity in assumption (i) above.

For (4) it remains to be proven that for $\varphi \in \mathcal{S}_D(\mathcal{E}) \cap \mathcal{S}_M(\mathcal{E})$ there exists $f \in \mathcal{S}(V)$ such that $\varphi = \hat{f}$. We proceed in three steps:

1. There exists a function $F \in C^\infty(V^* \setminus \{0\})$ satisfying $\tilde{\varphi}(\dot{h}, \xi) = F(\xi)$.
2. $F \in C^\infty(V^*)$.
3. $F \in \mathcal{S}(V^*)$.

It then holds for $f \in \mathcal{S}(V)$ with $\tilde{f} = F$:

$$\tilde{\varphi}(\dot{h}, \xi) = F(\xi) = \tilde{f}(\xi) = \tilde{\hat{f}}(\dot{h}, \xi)$$

by lemma 4.2, so $\tilde{\varphi} = \tilde{\hat{f}}$, and by inverse Fourier transform, $\varphi = \hat{f}$.

First step

Let $X_1, \ldots, X_m$ be a basis of $\mathfrak{h}$. For every $w \in V^*$ we denote by H_w the isotropy subgroup of w in H and by $\mathfrak{h}_w$ its Lie algebra.

Lemma 5.2. *Let $l = \inf\{\dim H_w | w \in V^* \backslash \{0\}\}$, $O = \{w \in V^* \backslash \{0\} | \dim H_w = l\}$.*

a) O is a dense open subset of $V^* \setminus \{0\}$.

b) Let $w_0 \in O$. For every w in a dense open neighborhood of w_0 in O there exists a basis

$$Y_j(w) = \sum_{k=1}^{m} a_{jk}(w) X_k\,, \quad 1 \leq j \leq l\,,$$

of $\mathfrak{h}_w$, such that the coefficients a_{jk} are homogeneous polynomials in $w \in V^*$.

Proof. If we differentiate with respect to t the equation $\rho^*[\exp t(a_1 X_1 + \ldots + a_m X_m)]w - w = 0$, we obtain:

$$Y = \sum_{k=1}^{m} a_k X_k \in \mathfrak{h}_w \Longleftrightarrow d\rho^*(Y)w = 0 \Longleftrightarrow \sum_{k=1}^{m} a_k d\rho^*(X_k)w = 0\,. \tag{5}$$

a) $O \subseteq V^*$ is the set of vectors $w \in V^*$, where the system $(d\rho^*(X_1)w, \ldots, d\rho^*(X_m)w)$ has maximal rank, that is $m - l$.

b) Let us assume that $(d\rho^*(X_k)w)_{1 \leq k \leq m-l}$ is a basis of $\langle d\rho^*(X_k)w | 1 \leq k \leq m\rangle$ for w in a dense open neighborhood of w_0 in O. If we bring the terms with $m - l < k \leq m$ in (5) to the right hand side (and choose a basis on V^*), we obtain for every choice of $a_{m-l+1}, \ldots, a_m$ an $n \times (m-l)$-linear system, whose equations are linear in w. The statement of the lemma follows by applying Cramer's rule to an $(m-l) \times (m-l)$-part of the system and multiplying the vector $(a_1, \ldots, a_m)$ by the common denominator. ∎

Remark . 1. According to the construction given in the proof, the polynomials a_{jk} have the same degree of homogeneity.
2. If we consider the polynomials a_{jk} as elements of the symmetric (= universal enveloping) algebra of V, then the Y_j become the operators in $\mathcal{K}^1(\mathfrak{h} \times_s V)$ (see section 3):

$$P_j := \sum_{k=1}^{m} X_k a_{jk} \in \mathcal{K}^1(\mathfrak{h} \times_s V)\,.$$

This can be seen as follows ($Z_1, \ldots, Z_n$ a basis of V):

$$0 = \sum_{k=1}^{m} a_{jk}(\xi) d\rho^*(X_k)\xi$$

implies

$$0 = \langle \sum_{k=1}^{m} a_{jk}(\xi) d\rho^*(X_k)\xi, Z_l\rangle = \sum_{k=1}^{m} a_{jk}(\xi)\langle d\rho^*(X_k)\xi, Z_l\rangle$$

$$= -\sum_{k=1}^{m} a_{jk}(\xi)\langle \xi, d\rho(X_k)Z_l\rangle = \sum_{k=1}^{m} a_{jk}(\xi)\langle \xi, [Z_l, X_k]\rangle = \langle \xi, (\mathrm{ad}Z_l)P_j\rangle\,.$$

This gives a geometrical meaning to $\dim \mathcal{K}^1(\mathfrak{h} \times_s V) = l$.

We observe that for $\varphi \in \mathcal{S}(\mathcal{E})$, $T \in \mathfrak{h} \times_s V$, $d\nu(T)\varphi$ remains in $\mathcal{S}(\mathcal{E})$. Therefore we can define the Fourier transform of every $d\nu(T)$, $T \in \mathfrak{h} \times_s V$, and then for $T \in U(\mathfrak{h} \times_s V)$, by iteration:

$$\widetilde{d\nu(T)}\tilde{\varphi} := [d\nu(T)\varphi]^{\sim}\,.$$

We calculate $\widetilde{d\nu(X)}$, $\widetilde{d\nu(Z)}$ for $X \in \mathfrak{h}$, $Z \in V$:

$$\widetilde{d\nu(X)}\tilde{\varphi}(\dot{h}, \xi) = \int_U d\nu(X)\varphi(\dot{h}, h \cdot u)e^{-i\langle h\cdot u, \xi\rangle}du$$

$$= \int_U \frac{d}{dt}\varphi(\exp(-tX)\dot{h}, \exp(-tX)h \cdot u)|_{t=0} \cdot e^{-i\langle h\cdot u,\xi\rangle}du$$

$$= \frac{d}{dt}\int_U \varphi(\exp(-tX)\dot{h}, \exp(-tX)h \cdot u) \cdot e^{-i\langle \exp(-tX)h\cdot u, \exp(-tX)^*\cdot\xi\rangle}du|_{t=0}$$

$$= \frac{d}{dt}\tilde{\varphi}(\exp(-tX)\dot{h}, \exp(-tX)^* \cdot \xi)|_{t=0}\,; \tag{6}$$

$$\widetilde{d\nu(Z)}\tilde{\varphi}(\dot{h}, \xi) = \int_U d\nu(Z)\varphi(\dot{h}, h \cdot u)e^{-i\langle h\cdot u,\xi\rangle}du$$

$$= \int_U \frac{d}{dt}\varphi(\dot{h}, -tZ + h \cdot u)|_{t=0} \cdot e^{-i\langle h\cdot u,\xi\rangle}du$$

$$= \int_U \frac{d}{dt}\varphi(\dot{h}, -th \cdot p_U(h^{-1} \cdot Z) + h \cdot u)|_{t=0} \cdot e^{-i\langle h\cdot u,\xi\rangle}du$$

(p_U denoting projection on U parallel to E)

$$= \frac{d}{dt}\int_U \varphi(\dot{h}, h \cdot u)e^{-i\langle h\cdot u,\xi\rangle}e^{-i\langle tZ,\xi\rangle}du|_{t=0} \quad (\varphi \in \mathcal{S}(\mathcal{E}))$$

$$= -i\langle Z, \xi\rangle \tilde{\varphi}(\dot{h}, \xi)\,. \tag{7}$$

Following the notation in lemma 5.2 we define operators $Y_j,\ 1 \le j \le l$, on $C^\infty(\mathcal{E}^*)$:

$$Y_j \psi(\dot{h}, \xi) := \sum_{k=1}^{m} a_{jk}(\xi) \widetilde{d\nu(X_k)} \psi(\dot{h}, \xi)\,.$$

For $f \in \mathcal{S}(V)$ we compute:

$$Y_j(\tilde{\hat{f}})(\dot{h}, \xi) = \sum_{k=1}^{m} a_{jk}(\xi) \widetilde{d\nu(X_k)} (\tilde{\hat{f}})(\dot{h}, \xi)$$

$$= \sum_{k=1}^{m} a_{jk}(\xi) \frac{d}{dt} (\tilde{\hat{f}})(\exp(-tX_k)\dot{h}, \exp(-tX_k)^* \cdot \xi)|_{t=0} \quad \text{by (6)}$$

$$= \sum_{k=1}^{m} a_{jk}(\xi) \frac{d}{dt} \tilde{f}(\exp(-tX_k)^* \cdot \xi)|_{t=0} \quad \text{by lemma 4.2}$$

$$= \frac{d}{dt} \tilde{f}(\exp(-tY_j(\xi))^* \cdot \xi)|_{t=0} = 0\,, \quad 1 \le j \le l\,. \tag{8}$$

We now proceed to a closer study of the operators Y_j. A homogeneous polynomial in $\xi \in V^*$ is obtained by multiplication and addition of terms of the form $\langle Z, \xi\rangle$, $Z \in V$. Thus, because of (7) there exist elements $P_{jk} \in U(V)$ satisfying $\widetilde{d\nu(P_{jk})}\psi(\dot{h}, \xi) = a_{jk}(\xi)\psi(\dot{h}, \xi)$. Using this fact, we obtain:

$$Y_j \tilde{\varphi}(\dot{h}, \xi) = \sum_{k=1}^{m} a_{jk}(\xi) \widetilde{d\nu(X_k)} \tilde{\varphi}(\dot{h}, \xi) = \sum_{k=1}^{m} \widetilde{d\nu(P_{jk})} \widetilde{d\nu(X_k)} \tilde{\varphi}(\dot{h}, \xi)$$

$$= \sum_{k=1}^{m} [d\nu(P_{jk}) d\nu(X_k)\varphi]^{\sim}(\dot{h}, \xi)\,,$$

so

$$Y_j \tilde{\varphi} = \sum_{k=1}^{m} [d\nu(P_{jk}) d\nu(X_k)\varphi]^{\sim} = [d\nu(P_j)\varphi]^{\sim} = \widetilde{d\nu(P_j)}\tilde{\varphi}\,,$$

where we have put $P_j := \sum_{k=1}^{m} P_{jk} X_k \in U(\mathfrak{h} \times_s V)$. Equation (8) now implies:

$$0 = [d\nu(P_j)\hat{f}]^{\sim} \Longrightarrow 0 = d\nu(P_j)\hat{f} = [d\lambda(P_j)f]^{\wedge} \Longrightarrow d\lambda(P_j)f = 0$$

by (3) and lemma 5.1.

We have thus proved that there exist elements $P_j \in \ker(d\lambda)$ such that

$$Y_j = \widetilde{d\nu(P_j)}\,, \quad 1 \le j \le l\,. \tag{9}$$

Remark . It holds: $P_j = \sum_{k=1}^{m} X_k P_{jk}$. In fact, for $Q_j := \sum_{k=1}^{m} X_k P_{jk}$ we have $Q_j \in \mathcal{K}^1(\mathfrak{h} \times_s V)$ (see section 3), since $\mathrm{ad}(Z)Q_j = \mathrm{ad}(Z)P_j = 0$ for every $Z \in V$ ($P_j \in \mathcal{K}^1(\mathfrak{h} \times_s V)$). It follows: $Q_j \in \ker(d\lambda) \Rightarrow Q_j - P_j \in \ker(d\lambda) \Rightarrow Q_j - P_j = 0$, because $Q_j - P_j \in U(V)$ and $d\lambda$ is injective on $U(V)$.

We come to the main result of this subsection:

Proposition 5.3. *For every* $\varphi \in \mathcal{S}_D(\mathcal{E})$ *there exists a function* $F \in C^\infty(V^* \setminus \{0\})$ *such that* $\tilde{\varphi}(\dot{h}, \xi) = F(\xi)$ *for* $\xi \in h^* \cdot E^\perp$, $\xi \neq 0$.

Proof. The restriction of $\pi : \mathcal{E}^* \to V^*$, $(\dot{h}, \xi) \mapsto \xi$, to $\pi^{-1}(V^* \setminus \{0\})$ is by assumption a submersion. Thus it is enough to show that $\tilde{\varphi}$ is constant on every subset $\pi^{-1}(\xi)$, $\xi \neq 0$. For this we recall assumption (ii) in the beginning of this section.

Let $\xi = h_1^* \cdot w_1 = h_2^* \cdot w_2$, $w_1, w_2 \in E^\perp \setminus \{0\}$. Then there exists $s \in H_E$ such that $s^* \cdot w_1 = w_2$, and we have:

$$h_1^* \cdot w_1 = (h_2 s)^* \cdot w_1 \Leftrightarrow h_1^{-1} h_2 s \in H_{w_1} \Leftrightarrow h_2 s \in h_1 H_{w_1} = H_\xi h_1$$

and $h_2 s H_E = \dot{h}_2$, so there is $r \in H_\xi$ such that $r \cdot (\dot{h}_1, \xi) := (r\dot{h}_1, r^* \cdot \xi) = (\dot{h}_2, \xi)$. It therefore remains to show that the function

$$r \longmapsto \tilde{\varphi}(r\dot{h}, r^* \cdot \xi) \tag{10}$$

is constant on H_ξ.

Relation (9) implies $Y_j \tilde{\varphi} = [d\nu(P_j)\varphi]^\sim = 0$, $1 \leq j \leq l$. On the other hand, with the help of $\Phi : H \times E^\perp \to \mathbb{C}$, $\Phi(h, w) := \tilde{\varphi}(\dot{h}, h^* \cdot w)$, we compute:

$$Y_j \tilde{\varphi}(\dot{h}, \xi) = \sum_{k=1}^m a_{jk}(\xi) \widetilde{d\nu(X_k)} \tilde{\varphi}(\dot{h}, \xi)$$

$$= \sum_{k=1}^m a_{jk}(\xi) \frac{d}{dt} \tilde{\varphi}(\exp(-tX_k)\dot{h}, \exp(-tX_k)^* \cdot \xi)|_{t=0} \quad \text{by (6)}$$

$$= \sum_{k=1}^m a_{jk}(\xi) \frac{d}{dt} \Phi(\exp(-tX_k)h, w)|_{t=0} \quad \text{if } \xi = h^* \cdot w\,,$$

$$= \frac{d}{dt} \Phi(\exp(-tY_j(\xi))h, w)|_{t=0} = \frac{d}{dt} \tilde{\varphi}(\exp(-tY_j(\xi))\dot{h}, \xi)|_{t=0}\,.$$

Since $Y_j(\xi)$, $1 \leq j \leq l$, form a basis of $\mathfrak{h}_\xi$ for ξ in a dense open set $O' \subseteq V^* \setminus \{0\}$, (10) is constant on H_ξ, $\xi \in O'$. Thus there exists a function $F \in C^\infty(O')$ satisfying $\tilde{\varphi}(\dot{h}, \xi) = F(\xi)$, $\xi \in O'$. For $0 \neq \xi_0 = h_0^* \cdot w_0 \notin O'$ let $W_{(h_0, w_0)}$ be a submanifold of $H \times E^\perp$ through (h_0, w_0) which is mapped by $(h, w) \mapsto h^* \cdot w$ diffeomorphically onto an open neighborhood W_{ξ_0} of ξ_0 in V^*, and put $F(h^* \cdot w) := \tilde{\varphi}(\dot{h}, h^* \cdot w)$ for $(h, w) \in W_{(h_0, w_0)}$. The argument shows that F can be smoothly extended to an open neighborhood of every $\xi \notin O'$, $\xi \neq 0$. If we consider a covering of $V^* \setminus \{0\}$ by such neighborhoods and by O', there exists a locally finite subcovering and a partition of unity associated with it, whereby F can be extended to a C^∞-function on $V^* \setminus \{0\}$. The equation $\tilde{\varphi} = F \circ \pi$ then holds on the dense subset $\pi^{-1}(O')$ and hence everywhere on $\pi^{-1}(V^* \setminus \{0\})$. ∎

Remark . For the proof we have only used the fact that φ is annihilated by $d\nu(Y)$, $Y \in \mathcal{K}^1(\mathfrak{h} \times_s V)$.

Second Step

The proof of $F \in C^\infty(V^*)$ is based on an application of a theorem of Glaeser [5], which we shall now recall.

Definition (Glaeser). Let $\omega \subseteq \mathbb{R}^n$ be an open set, $o \subseteq \omega$, $\mathcal{V} \subseteq C^\infty(\omega)$. A function $\Phi \in C^\infty(\omega)$ is said to be attached to $\mathcal{V}$ on o if there is $\Psi \in \mathcal{V}$ such that $\Phi - \Psi$ and all its partial derivatives vanish on o. (In that case Ψ will be called "interpolating function" for Φ). The function Φ will be called bipunctually attached to $\mathcal{V}$ if Φ is attached to $\mathcal{V}$ on every two-element subset of ω.

Furthermore, we consider an open set $\Omega \in \mathbb{R}^p$, $p \leq n$, and a C^∞-map $\theta : \omega \to \Omega$. Let A_θ denote the set of C^∞-functions on ω of the form $F \circ \theta$, $F \in C^\infty(\Omega)$.

Theorem (Glaeser). *If $\theta : \omega \to \Omega$ satisfies the following 4 conditions and $\Phi \in C^\infty(\omega)$ is bipunctually attached to A_θ, then Φ belongs to A_θ.*

Θ_1: *$\theta : \omega \to \Omega$ is real analytic.*

Θ_2: *θ has maximal rank $(= p)$ on a dense open subset of ω.*

Θ_3: *$\theta(\omega)$ is closed in Ω.*

Θ_4: *For every compact set $K \subseteq \theta(\omega)$ there is a compact set $k \subseteq \omega$ such that $\theta(k) = K$.*

We prepare the application of this theorem.

Lemma 5.4. *There is a compact set $K \subseteq H$ such that the mapping $(h, w) \mapsto h^* \cdot w$ from $K \times E^\perp$ into V^* remains surjective.*

Proof. Let S_{V^*} denote the sphere of radius 1 in a norm $\|\cdot\|$ in V^*. The map $\sigma : H \times (E^\perp \setminus \{0\}) \to S_{V^*}$, $(h, w) \mapsto \frac{h^* \cdot w}{\|h^* \cdot w\|}$ is continuous and open, since it is the composition of the submersions $(h, w) \mapsto h^* \cdot w$ and $v \mapsto \frac{v}{\|v\|}$. For every $\xi = \sigma(h, w) \in S_{V^*}$ let U_h be an open, relatively compact neighborhood of h in H, $W_h := U_h \times (E^\perp \setminus \{0\})$. Since S_{V^*} is compact, it is covered by finitely many open sets $\sigma(W_{h_1}), \ldots, \sigma(W_{h_k})$. The subset $K := \bar{A}$, $A := \cup_{i=1}^k U_{h_i}$, has the required property. ■

Let $\dot{K}$ be the projection of K on ${}^H/_{H_E}$. There is a finite open covering of $\dot{K}$ by charts (U_i, φ_i), $1 \leq i \leq q$, with the property that there exists on every U_i a cross section σ_i into H. The sets U_i can be considered as pairwise disjoint open sets in $\mathbb{R}^k$ ($k = \dim {}^H/_{H_E}$) (if we identify U_i with $\varphi_i(U_i)$ and take $\varphi_i(U_i)$ bounded, the $\varphi_i(U_i)$ can be disjointly distributed in $\mathbb{R}^p$ by the use of translations).[3]

Let now $\omega := (\cup_{i=1}^q U_i) \times E^\perp$, $\Omega := V^*$, $\theta : \omega \to V^*$, $\theta(\dot{h}, w) := \sigma(\dot{h})^* \cdot w$, $\sigma|_{U_i} := \sigma_i$.[4] The map θ is surjective and satisfies Θ_2 and Θ_3 by assumption. Our manifolds has a real analytic atlas, and the cross sections σ_i as well as the representation ρ can be taken analytic. This establishes Θ_1. The next lemma establishes Θ_4.

[3]This very technical assumption is needed, so as to construct a unique cross section σ from the various σ_i. After all, Glaeser's theorem is formulated for open sets in $\mathbb{R}^p$ and not in the context of manifolds, as is the case here.

[4]After choosing bases we can identify $E^\perp$ and V^* with $\mathbb{R}^{n-d}$ and $\mathbb{R}^n$ respectively.

Lemma 5.5. *For every compact set $K' \subseteq V^*$ there exists a compact set $k \subseteq \omega$ such that $\theta(k) = K'$.*

Proof. Similar considerations as those in the proof of lemma 5.4 lead to a compact set $\tilde{K} \subseteq \cup_{i=1}^q U_i$ with the property that the restriction of θ on $\tilde{K} \times E^\perp$ is surjective. For $\dot{h} \in \tilde{K}$ we put

$$K_{\dot{h}} := \{w \in E^\perp | \theta(\dot{h}, w) \in K'\}.$$

Obviously, $K_{\dot{h}}$ is compact. We finish the proof by verifying the compactness of $k := \cup_{\dot{h} \in \tilde{K}} \{\dot{h}\} \times K_{\dot{h}}$.

Let $((\dot{h}_n, w_n))_{n \in \mathbb{N}}$ be a sequence in k and suppose that $\dot{h}_n$ converges to $\dot{h} \in \tilde{K}$. We have:

$$w_n \in K_{\dot{h}_n} = [(\sigma(\dot{h}_n)^{-1})^* \cdot K'] \cap E^\perp.$$

Let B be a compact neighborhood of $\dot{h}$ in $\cup_{i=1}^q U_i$. Almost all w_n lie in the compact set $[(\sigma(B)^{-1})^* \cdot K'] \cap E^\perp$, so there is a subsequence of $(w_n)_{n \in \mathbb{N}}$ which converges to $w \in [(\sigma(B)^{-1})^* \cdot K'] \cap E^\perp$. Therefore there exists $\dot{h}_B \in B$ such that $\sigma(\dot{h}_B)^* \cdot w \in K'$. If we let B shrink to $\dot{h}$, we obtain a sequence $(\dot{h}_{B_n})_{n \in \mathbb{N}}$ converging to $\dot{h}$ and having the property that $\sigma(\dot{h}_{B_n})^* \cdot w = \theta(\dot{h}_{B_n}, w) \in K'$. This implies $\theta(\dot{h}, w) \in K'$, so $(\dot{h}, w) \in k$. ■

To apply Glaeser's above theorem to the function Φ on ω, defined by

$$\Phi(\dot{h}, w) := \tilde{\varphi}(\dot{h}, \theta(\dot{h}, w)), \tag{11}$$

we need to show that Φ is bipunctually attached to A_θ. Taking into account the goal of the first step, we only have to show this for the subset $\{(\dot{h}, 0) | \dot{h} \in \cup_{i=1}^q U_i\}$ of ω. We first prove an algebraic lemma.

Lemma 5.6. *Let P_k be a homogeneous polynomial of degree $k \geq 0$ on V^*. There is a unique k-linear symmetric form T_k on V^* such that $T_k(x, \dots, x) = P_k(x)$ for all $x \in V^*$.*

Proof. Let $(x^1, \dots, x^n)$ denote the vector of coordinates of $x \in V^*$ with respect to a fixed basis. The proof follows easily, if we consider the general expression of a symmetric k-linear form:

$$T_k(x_1, \dots, x_k) = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} \frac{1}{k!} c_{i_1 \dots i_k} \sum_{\sigma \in S_k} x^{i_1}_{\sigma(1)} \cdots x^{i_k}_{\sigma(k)}, \quad c_{i_1 \dots i_k} \in \mathbb{C}$$

(S_k the symmetric group of k elements). ■

Proposition 5.7. *The function Φ defined by (11) on ω is attached to A_θ on $\{(\dot{h}, 0) | \dot{h} \in \cup_{i=1}^q U_i\}$.*

Proof. Let $(u_1, \ldots, u_{n-d})$ be the vector of coordinates with respect to a basis of U, $(w_1, \ldots, w_{n-d})$ the one with respect to the dual basis of $E^\perp$. We have:

$$\frac{\partial}{\partial w'_j}\Phi(\dot{h}, w') = \frac{\partial}{\partial w'_j}\int_U \varphi(\dot{h}, \sigma(\dot{h}) \cdot u)e^{-i\langle u,w'\rangle}du$$

$$= -i \cdot \int_U \varphi(\dot{h}, \sigma(\dot{h}) \cdot u)u_j e^{-i\langle u,w'\rangle}du\,.$$

The moment conditions for φ (proposition 4.4) imply:

$$(\sum_{j=1}^{n-d} w_j \frac{\partial}{\partial w'_j})^m \Phi(\dot{h}, w')|_{w'=0} = (-i)^m P_m(\sigma(\dot{h})^* \cdot w) \tag{12}$$

(P_m a homogeneous polynomial of degree m on V^*).

For every $m \in \mathbb{Z}_+$ let T_m be the unique m-linear symmetric form on V^* such that $T_m(x, \ldots, x) = (-i)^m P_m(x)$ (according to the preceding lemma). By the generalized theorem of Emile Borel every (formal) power series is the Taylor series of a C^∞-function.[5] Therefore there exists a function $F' \in C^\infty(V^*)$ such that $D^m F'(0) = T_m$, $m \in \mathbb{Z}_+$. We shall show that $F' \circ \theta$ is an interpolating function for Φ on $\{(\dot{h}, 0) | \dot{h} \in \cup_{i=1}^q U_i\}$.

If we set

$$(\dot{h}^* T_m)(w_1, \ldots, w_m) := T_m(\sigma(\dot{h})^* \cdot w_1, \ldots, \sigma(\dot{h})^* \cdot w_m)$$

and denote by $D_2^m \Phi(\dot{h}, 0)$ the m-th derivative of $w \mapsto \Phi(\dot{h}, w)$ at 0, (12) takes the form

$$D_2^m \Phi(\dot{h}, 0)(w, \ldots, w) = (\dot{h}^* T_m)(w, \ldots, w)\,.$$

Since both sides are symmetric tensors, $D_2^m \Phi(\dot{h}, 0) = \dot{h}^* T_m$ by the preceding lemma. On the other hand, $D_2^m(F' \circ \theta)(\dot{h}, 0) = \dot{h}^* T_m$, the left hand side denoting the m-th derivative of $w \mapsto F' \circ \theta(\dot{h}, w) = F'(\sigma(\dot{h})^* \cdot w)$. Hence,

$$D_2^m \Phi(\dot{h}, 0) = D_2^m(F' \circ \theta)(\dot{h}, 0)\,, \quad m \in \mathbb{Z}_+\,.$$

Since this last equation holds for all $\dot{h} \in \cup_{i=1}^q U_i$, we can apply on both sides a differential operator of constant coefficients in $\dot{h}$ ($\cup_{i=1}^q U_i \subseteq \mathbb{R}^p$) without doing any harm to the equality. This shows that $F' \circ \theta$ is an interpolating function for Φ. ■

By Glaeser's theorem there now exists $F' \in C^\infty(V^*)$ such that Φ and $F' \circ \theta$ coincide on ω. Therefore, the function F of the first step can be extended to the C^∞-function F' on V^*. We shall continue to write F instead of F'. By continuity we have $\tilde{\varphi}(\dot{h}, \xi) = F(\xi)$ for all $\xi \in h^* \cdot E^\perp$, $\dot{h} \in {}^H/_{H_E}$.

Third step

Proposition 5.8. *The function F defined above belongs to the Schwartz class $\mathcal{S}(V^*)$.*

[5] This theorem is a special case of theorem I in [16, page 65].

Proof. It remains to show that for every $p \in \mathbb{Z}_+$ and every differential operator D with constant coefficients on V^*

$$\sup_{\xi \in V^*} \|\xi\|^p |DF(\xi)| < \infty .$$

Let $K \subseteq H$ be as in lemma 5.4, $\dot{K}$ the projection of K on ${}^H/_{H_E}$ and $(U_i)_{1 \le i \le q}$ a covering of $\dot{K}$ by open, relatively compact sets, which admit local cross sections σ_i on their topological closures. Thus we have to show that

$$\sup_{\dot{h} \in \dot{K}, w \in E^\perp} \|\sigma_i(\dot{h})^* \cdot w\|^p \cdot |DF(\sigma_i(\dot{h})^* \cdot w)| < \infty \tag{13}$$

(if $\dot{h}$ is contained in U_i). For this we shall express D in coordinates $(\dot{h}, w)$.

Let us take a basis of $E^\perp$, extend it to one of V^* and denote by $(w_1, \ldots, w_{n-d})$ and $(\xi_1, \ldots, \xi_n)$ the vectors of coordinates on $E^\perp$ and V^* respectively. The automorphisms $\rho^*(\sigma_i(\dot{h}))$ can then be regarded as $n \times n$-matrices. Putting $\theta_i : U_i \times E^\perp \to V^*$, $\theta_i(\dot{h}, w) := \sigma_i(\dot{h})^* \cdot w$, $1 \le i \le q$, we have ($X_1, \ldots, X_m$ a basis of the Lie algebra $\mathfrak{h}$ of H):

$$\frac{\partial}{\partial w_j} F \circ \theta_i(\dot{h}, w) = \sum_{l=1}^{n} \frac{\partial F}{\partial \xi_l}(\sigma_i(\dot{h})^* \cdot w) \cdot \rho^*(\sigma_i(\dot{h}))_{lj} , \quad 1 \le j \le n - d ,$$

$$X_k(F \circ \theta_i)(\dot{h}, w) := \frac{d}{dt} F \circ \theta_i(\exp(tX_k)\dot{h}, w)|_{t=0}$$

$$= \sum_{l=1}^{n} \frac{\partial F}{\partial \xi_l}(\sigma_i(\dot{h})^* \cdot w) \cdot \frac{d}{dt}[\rho^*(\sigma_i(\exp(tX_k)\dot{h}))w]_l|_{t=0} , \quad 1 \le k \le m .$$

We consider these equations as a linear system in $\frac{\partial F}{\partial \xi_1}, \ldots, \frac{\partial F}{\partial \xi_n}$. For every $(\dot{h}_0, w_0) \in \dot{K} \times (E^\perp \setminus \{0\})$ there is an index i such that $\dot{h}_0 \in U_i$, and we can choose n of the above equations such that the determinant of the coefficients of this part of the system does not vanish (θ_i is on $U_i \times (E^\perp \setminus \{0\})$ a submersion). Since this determinant is a homogeneous polynomial in w, it remains non-zero on a neighborhood of $(\dot{h}_0, w_0)$ of the form $O \times \mathbb{R}^* \cdot W$, where O is a neighborhood of $\dot{h}_0$ in U_i and W one of $\frac{w_0}{\|w_0\|}$ in the sphere $S_{E^\perp}$ of radius 1 in $E^\perp$. Using Cramer's rule, we obtain:

$$\frac{\partial F}{\partial \xi_l}(\sigma_i(\dot{h})^* \cdot w) = \sum_{j=1}^{n-d} f_j(\dot{h}, w) \frac{\partial}{\partial w_j} F \circ \theta_i(\dot{h}, w) + \sum_{k=1}^{m} g_k(\dot{h}, w) X_k(F \circ \theta_i)(\dot{h}, w) ,$$

where f_j, g_k are smooth homogeneous rational functions in w (f_j of degree 0, g_k of degree -1) (take f_j, g_k to be zero, if a term does not appear). Repeating the procedure with $\frac{\partial F}{\partial \xi_l}$ instead of F, we obtain an expression for the second derivatives of F, etc. Eventually this leads to an expression of the operator D in coordinates $(\dot{h}, w)$:

$$DF(\sigma_i(\dot{h})^* \cdot w) = \sum_j e_j(\dot{h}, w) T_j P_j(\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_{n-d}}) F \circ \theta_i(\dot{h}, w)$$

$$= \sum_j e_j(\dot{h}, w) T_j P_j(\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_{n-d}}) \tilde{\varphi}(\dot{h}, \sigma_i(\dot{h})^* \cdot w) \,. \tag{14}$$

Therein T_j is a differential operator on ${}^H/_{H_E}$, $P_j(\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_{n-d}})$ a polynomial in $\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_{n-d}}$, and e_j smooth rational functions, homogeneous in w of degree ≤ 0.

Relation (13) is a property for w in a neighborhood of infinity. For $\|w\| \leq 1$ we actually have $\|\sigma_i(\dot{h})^* \cdot w\| \leq \|\rho^*(\sigma_i(\dot{h}))\|$, and

$$A_i := \sup_{\dot{h} \in U_i} \|\rho^*(\sigma_i(\dot{h}))\| < \infty \,,$$

since U_i is relatively compact and σ_i a local cross section on $\bar{U}_i$. The region of V^* we obtain by $\sigma_i(\dot{h})^* \cdot w$ $(1 \leq i \leq q)$ from $\dot{h} \in \dot{K}$, $\|w\| \geq 1$, contains $\{\xi \in V^* | \, \|\xi\| > \max_{1 \leq i \leq q} A_i\}$. Therefore we can restrict ourselves to $\|w\| \geq 1$ in (13).

Equation (14) holds for $(\dot{h}, w) \in O \times \mathbb{R}^* \cdot W$. If we take O and W to be compact, all functions e_j are bounded on $O \times (\mathbb{R} \backslash]-1,1[) \cdot W$, and (13) holds for $(\dot{h}, w) \in O \times (\mathbb{R} \backslash]-1,1[) \cdot W$, because $\tilde{\varphi}$ is a rapidly decreasing function (cf. section 2). Due to the compactness of $\dot{K}$ and $S_{E^\perp}$, we now only need finitely many sets of the form $O \times (\mathbb{R} \backslash]-1,1[) \cdot W$ to obtain a full neighborhood of the infinity in V^*. The proof is thereby completed. ■

6. The image in the case $d < n - 1$

Richter and Kurusa have shown that in the classical case ($H = SO(n)$, $V = \mathbb{R}^n$) and for $d < n-1$ the range-characterizing partial differential equations (which are equivalent to $d\nu(T)\varphi = 0$ for $T \in \mathcal{K}^1(so(n) \times_s \mathbb{R}^n)$, see [14, page 72ff]) suffice to characterize the range $\mathcal{S}(\mathbb{R}^n)^\wedge$ (the latter author derives directly the moment conditions of Helgason) [11], [13]. Gonzalez has given a group theoretical proof [6]. The fact is also true in the general situation, provided that the set of planes in $\mathcal{E}$ (actually in $\mathcal{E}^*$) is sufficiently rich.

Definition 6.1. A collection $\mathcal{U}$ of k-dimensional linear subspaces of V^* will be called fundamental system if every homogeneous polynomial on V^* is completely determined by its restriction on the union of all elements of $\mathcal{U}$.

For instance, a set of hyperplanes ($k = n-1$) is a fundamental system iff it contains infinitely many elements.

We proceed to the following assumption (denoted in the sequel by (G)) on the action of the group H:

> There is a dense set $A \subseteq V^* \backslash \{0\}$ with the following property: For every $\xi \in A$ the planes of the form $h^* \cdot E^\perp$ through ξ form a fundamental system.

The set of $\xi \in V^* \setminus \{0\}$ for which the statement in (G) holds is $\rho^*(H)$-invariant. Therefore we only have to establish (G) for a dense set in $E^\perp \setminus \{0\}$. This makes things much easier in concrete situations.

Proposition 6.2. *Assumption (G) implies: $\mathcal{S}(V)^\wedge = \mathcal{S}_D(\mathcal{E})$.*

Proof. Let $\varphi \in \mathcal{S}_D(\mathcal{E})$, $\tilde{\varphi}$ the Fourier transform. We denote by $D_2^m\tilde{\varphi}(\dot{h},\xi_0)$ the m-th derivative of $\xi \mapsto \tilde{\varphi}(\dot{h},\xi)$ (as a function on $h^*\cdot E^\perp$) at the point $\xi_0 \in h^*\cdot E^\perp$. The proof will be given by establishing the moment conditions for φ. These are equivalent to

$$D_2^m\tilde{\varphi}(\dot{h},0)\cdot\xi^{\otimes^m} = (-i)^m P_m(\xi)\,, \quad \xi \in h^*\cdot E^\perp\,,$$

P_m homogeneous polynomials of degree $m \geq 0$ on V^* (cf. proposition 4.4).

Let $Y_j(\xi)$, $1 \leq j \leq l$, be a basis of $\mathfrak{h}_\xi$ for ξ in a dense open subset of $V^*\setminus\{0\}$ (notation as in section 5). It follows from the considerations in section 5 that for $1 \leq j \leq l$ the function

$$(t,s) \longmapsto \tilde{\varphi}(\exp(-tY_j(\xi))\dot{h}, s\xi)$$

does not depend on t, nor does the m-th partial derivative with respect to s at $s=0$:

$$D_2^m\tilde{\varphi}(\exp(-tY_j(\xi))\dot{h},0)\cdot\xi^{\otimes^m}\,.$$

Following the reasoning of section 5, there exists a function $F_m \in C^\infty(V^*\setminus\{0\})$ such that

$$D_2^m\tilde{\varphi}(\dot{h},0)\cdot\xi^{\otimes^m} = F_m(\xi) \tag{15}$$

for $\xi \in h^*\cdot E^\perp$, $\xi \neq 0$. Let $\xi_0 \in A$, A as in (G). The map

$$\xi \longmapsto D^kF_m(\xi_0)\cdot\xi^{\otimes^k}\,, \quad k > m\,,$$

(D^k: the k-th derivative) is a homogeneous polynomial, whose restriction on every plane $h^*\cdot E^\perp$ containing ξ_0 is equal to zero by (15). The fact $\xi_0 \in A$ implies $D^kF_m(\xi_0)\cdot\xi^{\otimes^k} \equiv 0$, and by lemma 5.6 $D^kF_m(\xi_0) = 0$. Since A is dense, we obtain $D^kF_m = 0$ for $k > m$. Therefore F_m equals a polynomial $(-i)^mP_m$ of degree $\leq m$ on $V^*\setminus\{0\}$ and can be extended to it on V^*. By continuity,

$$D_2^m\tilde{\varphi}(\dot{h},0)\cdot\xi^{\otimes^m} = (-i)^m P_m(\xi)\,, \quad \xi \in h^*\cdot E^\perp\,.$$

Hereby we conclude that P_m is homogeneous of degree m, and the proof is completed. ∎

However, assumption (G) is not necessary for a relation of the form $\mathcal{S}(V)^\wedge = \mathcal{S}_D(\mathcal{E})$, as the last example in section 9 shows.

7. The Radon transform ($d = n-1$)

In this section we focus our attention on the special case $d = n-1$. As it will come out, this leads to the classical Radon transform. Different situations can only appear, if we remove some of our conditions on the group action, or if we completely change the setting and allow for $\mathcal{E}$ sets of planes which do not admit a transitive group action. For the latter possibility the reader is referred to [2].

Let H be a Lie group, ρ a representation of H on a (finite-dimensional) vector space V. The contragredient representation will be denoted by ρ^*. We fix a linear subspace $E \subseteq V$ of codimension 1 and denote by $E^\perp \subseteq V^*$ its (one-dimensional) orthogonal complement. We make the following assumptions:

1. $\rho(H)$ consists of unimodular automorphisms of V, and the same holds for $\rho(H_E)$ (the isotropy subgroup of E) in the case of E.

2. The map from $H \times E^\perp$ into V^*, $(h,w) \mapsto \rho^*(h)w$, is a surjective submersion.[6]

We put $h \cdot u := \rho(h)u$, $h^* \cdot w := \rho^*(h)w$. Let $\|\cdot\|$ be a norm on V^*, S_{V^*} the sphere of radius 1. The group H acts transitively on S_{V^*}:

$$h \cdot w := \frac{h^* \cdot w}{\|h^* \cdot w\|}\,.$$

Let $w_0 \in S_{V^*} \cap E^\perp$, $H^S_{w_0}$ the isotropy subgroup of w_0. Since H_E operates by unimodular automorphisms on $E^\perp$, we obtain $\overset{o}{H}_E \subseteq H^S_{w_0} \subseteq H_E$ ($\overset{o}{H}_E$ denoting the connected component of H_E) and $H^S_{w_0} = H_{w_0}$ (the isotropy subgroup in the contragredient action). The transitivity of the action on S_{V^*} implies $H^S_\xi = H_\xi$ for every $\xi \in S_{V^*}$. Thus, ${}^H/_{H_\xi} \cong S_{V^*}$ for $\xi \in V^* \setminus \{0\}$. The orbit $\rho^*(H)\xi$ is bounded because ${}^H/_{H_\xi}$ is compact. Since this holds for every ξ, $\rho^*(H)$ itself is bounded (in the vector space $L(V^*)$ of all endomorphisms of V^*). Therefore, $\det[\rho^*(H)]$ is a bounded subgroup of $\mathbb{R}^*$, so $\det[\rho^*\overline{(H)]} \subseteq \{-1,1\}$. This relation also holds for the closure $\overline{\rho^*(H)}$ in $L(V^*)$. Thus, $\overline{\rho^*(H)}$ is a compact subgroup of $GL(V^*)$ and hence leaves a positive definite inner product invariant. With the help of it, V^* can be identified with $V^{**} \cong V$, and we have $\rho^* = \rho$. Eventually, $V \cong V^*$ is identified with $\mathbb{R}^n$ with the help of an orthonormal basis. We obtain:

> $\rho(H)$ is a group of orthogonal matrices and operates transitively on the sphere S^{n-1}. It holds $\rho = \rho^*$, and $E \subseteq \mathbb{R}^n$ is a subspace of codimension 1.

This is the situation in the classical case.

As for the range of $\mathcal{S}(V)$ under $f \mapsto \hat{f}$, it is completely characterized by the moment conditions (see [9, page 100]):

Proposition 7.1. *In the case $d = n-1$ we have: $\mathcal{S}(V)^\wedge = \mathcal{S}_M(\mathcal{E})$.*

Proof. Since $\xi \in h \cdot E^\perp$, $\xi \neq 0$, completely determines $\dot{h} = hH_E$ ($hH_E \Leftrightarrow h \cdot E = \xi^\perp$), we deduce at once the result of the first step in section 5. The differential operators are not needed for the second and third step.[7] ∎

8. The range of $C_c^\infty(V)$

The characterization of the range of $C_c^\infty(V)$, the set of C^∞-functions with compact support, under the d-plane transform will be done with the help of a modified version of the Paley-Wiener theorem, due to Helgason [8], which is recalled first. Condition (ii) in the beginning of section 5 can be dropped throughout this section.

Let $\mathcal{D}_A(\mathbb{R}^n)$ denote the class of C^∞-functions on $\mathbb{R}^n$ with support in $\{x \mid |x| \leq A\}$.

[6] Notice that condition (ii) in the beginning of section 5 here degenerates.

[7] Obviously, we have here $d\nu(Y) = 0$ for $Y \in \mathcal{K}^1(\mathfrak{h} \times_s V)$.

Theorem . *The Fourier transform* $f \mapsto \tilde{f}$ *maps* $\mathcal{D}_A(\mathbb{R}^n)$ *onto the set of functions* $\tilde{f}(\lambda\omega) = \varphi(\lambda, \omega) \in C^\infty(\mathbb{R} \times S^{n-1})$ *satisfying:*

(i) For each ω *, the function* $\lambda \mapsto \varphi(\lambda, \omega)$ *extends to a holomorphic function on* $\mathbb{C}$ *such that*

$$\sup_{\lambda,\omega} |\varphi(\lambda, \omega)\lambda^N e^{-A|Im\lambda|}| < \infty$$

for each integer $N \in \mathbb{Z}_+$ *.*

(ii) For each $k \in \mathbb{Z}_+$ *and every vector* $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ *satisfying* $\sum_{i=1}^n a_i^2 = 0$ *the function*

$$\lambda \mapsto \lambda^{-k} \int_{S^{n-1}} \varphi(\lambda, \omega)\langle a, \omega\rangle^k d\omega$$

is even and holomorphic on $\mathbb{C}$ *.*

For a proof we refer to [9, page 23ff].

Before we state the main theorem, we need some facts about spherical harmonics.

Proposition . *The eigenspaces of the Laplacian on* S^{n-1} *are of the form*

$$E_k = span\{f_{a,k} | f_{a,k}(\omega) = \langle a, \omega\rangle^k, \ \omega \in S^{n-1}\} ,$$

where $a = (a_1, \dots, a_n) \in \mathbb{C}^n$ *satisfies* $\sum_{i=1}^n a_i^2 = 0$ *, and* $k \in \mathbb{Z}_+$ *. The subspaces* E_k *are pairwise orthogonal in* $L^2(S^{n-1})$ *. If* P_k *denotes the set of restrictions on* S^{n-1} *of the homogeneous polynomials of degree* k *on* $\mathbb{R}^n$ *, it holds*

$$P_k = E_k + E_{k-2} + \ldots + E_{k-2m} , \quad m = [\frac{k}{2}] .$$

For a proof see [9, page 17ff].

Let $\langle \cdot, \cdot \rangle$ be a (positive definite) inner product on V such that E and U are orthogonal. S_V stands for the sphere of radius 1 in the induced norm. For $h \in H$ we put

$$\tau(h) := \|\rho(h^{-1})\| ,$$

ρ being the representation of H on V and $\| \cdot \|$ the operator norm. The map τ is continuous. For $u \in U$, $\|u\| \geq \tau(h)$, $h \cdot (u + E)$ has at most one point with S_V in common. If supp$f \subseteq \{x \in V | \, \|x\| \leq A\}$, we therefore have $\hat{f}(\dot{h}, h \cdot u) = 0$ for $\|u\| \geq A\tau(h)$.

Theorem 8.1. $\varphi \in \mathcal{S}_M(\mathcal{E})$ *is the* d*-plane transform of a* C^∞*-function* f *of compact support on* V *iff there exists a continuous function* c *on* H *such that for every* $h \in H$ $\varphi(\dot{h}, h \cdot u) = 0$ *for* $\|u\| \geq c(h)$ *.*

Proof. We only have to show that the condition is sufficient. We calculate:

$$\tilde{\varphi}(\dot{h}, \xi) = \int_U \varphi(\dot{h}, h \cdot u) e^{-i\langle h \cdot u, \xi\rangle} du = \int_U \varphi(\dot{h}, h \cdot u) \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \langle h \cdot u, \xi\rangle^m du$$

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_U \varphi(\dot{h}, h \cdot u) \langle h \cdot u, \xi \rangle^m du$$

(the region of integration is bounded)

$$= \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} P_m(\xi)$$

by the moment conditions. The last series converges for all $\xi \in h^* \cdot E^\perp$, and, since it is a power series, for all $\xi \in V^*$. It defines an analytic function F.

The spaces V and V^* can be identified with the help of the inner product. We choose an orthonormal basis and identify both spaces with $\mathbb{R}^n$.

$$F(\xi) = \sum_{m=0}^{\infty} \frac{(-i)^m}{m!} P_m(\xi)$$

is then an analytic function on $\mathbb{R}^n$ and can be extended to a holomorphic function on $\mathbb{C}^n$ by the above power series.

Let $F(\lambda\omega) =: \psi(\lambda, \omega) \in C^\infty(\mathbb{R} \times S^{n-1})$. We show that ψ satisfies the two conditions of the theorem in the beginning.

Ad (i): It is enough to verify the relation for even exponents N. We calculate ($\omega = h^* \cdot w \in S^{n-1}$, $w \in U$):

$$|\psi(\lambda, \omega)\lambda^{2N}| = |\tilde{\varphi}(\dot{h}, \lambda\omega)\lambda^{2N}| = |\int_U \varphi(\dot{h}, h \cdot u)\lambda^{2N} e^{-i\langle h \cdot u, \lambda h^* \cdot w\rangle} du|$$

$$= |\int_U \varphi(\dot{h}, h \cdot u)\lambda^{2N} e^{-i\lambda\langle u, w\rangle} du|$$

$$= |\int_U \varphi(\dot{h}, h \cdot u)(-\sum_{k=1}^{n-d} \frac{\partial^2}{\partial u_k^2})^N e^{-i\lambda\langle u, w\rangle} \cdot \|w\|^{-2N} du|$$

(taking such an orthonormal basis that the first $n-d$ vectors belong to U)

$$= \|w\|^{-2N} |\int_U (-\sum_{k=1}^{n-d} \frac{\partial^2}{\partial u_k^2})^N \varphi(\dot{h}, h \cdot u) e^{-i\lambda\langle u, w\rangle} du|$$

$$\leq \|w\|^{-2N} \int_U |(\sum_{k=1}^{n-d} \frac{\partial^2}{\partial u_k^2})^N \varphi(\dot{h}, h \cdot u)| du \cdot e^{c(h)\|w\||Im\lambda|} .$$

For $\tau^*(h) := \|\rho^*(h^{-1})\|$ we have: $\frac{1}{\tau^*(h^{-1})} \leq \|w\| \leq \tau^*(h)$, so

$$|\psi(\lambda, \omega)\lambda^{2N}| \leq \tau^*(h^{-1})^{2N} \int_U |(\sum_{k=1}^{n-d} \frac{\partial^2}{\partial u_k^2})^N \varphi(\dot{h}, h \cdot u)| du \cdot e^{c(h)\tau^*(h)|Im\lambda|} .$$

Let $K \subseteq H$ be a compact subset with the property of lemma 5.4. Putting

$$C_N := \sup_{h \in K} \tau^*(h^{-1})^{2N} \int_U |(\sum_{k=1}^{n-d} \frac{\partial^2}{\partial u_k^2})^N \varphi(\dot{h}, h \cdot u)| du , \quad A := \sup_{h \in K} c(h)\tau^*(h)$$

we obtain the desired relation

$$|\psi(\lambda, \omega)\lambda^{2N}| \leq C_N e^{A|Im\lambda|} .$$

Ad (ii): Since $\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}P_m(\xi)$ converges uniformly on compact sets, we have:

$$\int_{S^{n-1}}\psi(\lambda,\omega)\langle a,\omega\rangle^k d\omega = \int_{S^{n-1}}F(\lambda\omega)\langle a,\omega\rangle^k d\omega$$

$$= \int_{S^{n-1}}\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}P_m(\lambda\omega)\langle a,\omega\rangle^k d\omega$$

$$= \sum_{m=0}^{\infty}\int_{S^{n-1}}\frac{(-i)^m}{m!}P_m(\lambda\omega)\langle a,\omega\rangle^k d\omega = \sum_{m=0}^{\infty}\frac{(-i\lambda)^m}{m!}\int_{S^{n-1}}P_m(\omega)\langle a,\omega\rangle^k d\omega\,.$$

The proposition above implies that $\int_{S^{n-1}}P_m(\omega)\langle a,\omega\rangle^k d\omega$ vanishes, unless $m-k\in\mathbb{Z}_+$. This verifies (ii).

Concluding, there exists a function $f\in C_c^\infty(V)$ such that

$$\tilde{f}(\lambda\omega)=\psi(\lambda,\omega)=F(\lambda\omega)=\tilde{\varphi}(\dot{h},\lambda\omega)\,,\quad \omega\in h^*\cdot U\,.$$

Since on the other hand $\tilde{f}(\lambda\omega)=\tilde{\hat{f}}(\dot{h},\lambda\omega)$, we obtain $\varphi=\hat{f}$ by the inverse Fourier transform. ■

Remark . The application of the modified version of the Paley-Wiener theorem in the above proof shows that the support of f is contained in the ball of radius A, which depends on the function $c(h)$ and the compact set K. In the classical case the theorem reduces to theorem 2.10 in [9, page 109], since H is compact. Furthermore, for functions f of the Schwartz class, we obtain Helgason's support theorem [9, page 105] by taking $c(h)\equiv A$.

9. Examples

1. We begin by a counterexample, which shows that condition (i) in the beginning of section 5 cannot be weakened substantially.

Let

$$H=\left\{\left.\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}\right| a,b,c\in\mathbb{R}\right\}\,,$$

$V=\mathbb{R}^3$, $E=\langle(0,0,1)\rangle$, $U=\langle(1,0,0),(0,1,0)\rangle\cong E^\perp$ via the standard scalar product. The assumptions we have made in the general case are all satisfied except of (i), section 5. The vector $(0,0,1)$ is not contained in the image (which is dense in $\mathbb{R}^3$) of the mapping

$$(a,b,c,w_1,w_2)\longmapsto\begin{pmatrix}1&a&c\\0&1&b\\0&0&1\end{pmatrix}^*\cdot\begin{pmatrix}w_1\\w_2\\0\end{pmatrix}=\begin{pmatrix}w_1\\-aw_1+w_2\\(ab-c)w_1-bw_2\end{pmatrix}.$$

Let $g\in C_c^\infty(\mathbb{R})$, g nowhere zero on $[-1,1]$, and define $F:\mathbb{R}^3\to\mathbb{R}$ by

$$F(\xi_1,\xi_2,\xi_3):=g(\xi_1^2+\xi_2^2+\xi_3^2)\cdot g\Big(\frac{1}{\xi_1^2+\xi_2^2+\xi_3^2}\Big)\cdot\frac{1}{\xi_1^2+\xi_2^2}\,.$$

Then $\psi(\dot{h},\xi) := F(\xi)$, $\xi \in h^* \cdot E^\perp$, belongs to $\mathcal{S}(\mathcal{E}^*)$ and hence there exists $\varphi \in \mathcal{S}(\mathcal{E})$ such that $\tilde{\varphi} = \psi$. Due to the construction of ψ it holds $\varphi \in \mathcal{S}_D(\mathcal{E}) \cap \mathcal{S}_M(\mathcal{E})$. However, there is no $f \in \mathcal{S}(\mathbb{R}^3)$ such that $\hat{f} = \varphi$, because otherwise we would have $\tilde{f} = F$, which is a contradiction, since F does not extend to a C^∞-function on $\mathbb{R}^3$.

2. The action of $H = SO(n)$ on $V = \mathbb{R}^n$ (by matrix multiplication) leads to the classical d-plane transform. For the range of $\mathcal{S}(\mathbb{R}^n)$ the reader is referred to [13] and [6].

3. $H = SO(p, n-p)$ operates on $V = \mathbb{R}^n$ by matrix multiplication. We introduce the following basis on the Lie algebra $so(p, n-p)$ (E_{ij} denotes the matrix $(\delta_{ik}\delta_{jl})_{k,l}$):

$$\begin{array}{ll} X_{ij} := E_{ij} - E_{ji} & \text{for} \quad 1 \le i < j \le p\,,\ p+1 \le i < j \le n\,, \\ Y_{ij} := E_{ij} + E_{ji} & \text{for} \quad 1 \le i \le p\,,\ p+1 \le j \le n\,. \end{array}$$

We agree to write X_{ji} for $-X_{ij}$ and Y_{ji} for Y_{ij} (in the above range of the indices). $Z_1, \dots, Z_n$ will denote the standard basis of $\mathbb{R}^n$.

The Lie algebra $so(n) \times_s \mathbb{R}^n$ is isomorphic to the Lie algebra of all $(n+1) \times (n+1)$-matrices of the form

$$\begin{pmatrix} T & Z \\ 0 & 0 \end{pmatrix}, \quad T \in so(n)\,,\ Z \in \mathbb{R}^n\,.$$

Similarly, $so(p, n-p) \times_s \mathbb{R}^n$ is isomorphic to the Lie algebra of the $(n+1) \times (n+1)$-matrices of the form

$$\begin{pmatrix} T_1 & X & U_1 \\ X^T & T_2 & U_2 \\ 0 & 0 & 0 \end{pmatrix},$$

$T_1 \in so(p)$, $T_2 \in so(n-p)$, $U_1 \in \mathbb{R}^p$, $U_2 \in \mathbb{R}^{n-p}$ and X an arbitrary $p \times (n-p)$-matrix. The complexification of $so(n) \times_s \mathbb{R}^n$ is isomorphic to the complexification of $so(p, n-p) \times_s \mathbb{R}^n$ by the mapping

$$\begin{pmatrix} X_1 & X_2 & U_1 \\ -X_2^T & X_3 & U_2 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} X_1 & iX_2 & U_1 \\ iX_2^T & X_3 & -iU_2 \\ 0 & 0 & 0 \end{pmatrix},$$

where $X_1 \in so(p, \mathbb{C})$, $X_3 \in so(n-p, \mathbb{C})$, $U_1 \in \mathbb{C}^p$, $U_2 \in \mathbb{C}^{n-p}$ and X_2 is an arbitrary $p \times (n-p)$-matrix. With the help of this isomorphism we obtain from [14, page 62ff] a basis V_{ijl}, $1 \le i < j < l \le n$, of $\mathcal{K}^1(so(p, n-p) \times_s \mathbb{R}^n)$:

$$V_{ijl} = \begin{cases} X_{ij}Z_l + X_{jl}Z_i + X_{li}Z_j & \text{for} \quad l \le p \ \text{ or } \ i \ge p+1 \\ X_{ij}Z_l - Y_{jl}Z_i + Y_{li}Z_j & \text{for} \quad j \le p < l \\ Y_{ij}Z_l + X_{jl}Z_i - Y_{li}Z_j & \text{for} \quad i \le p < j \end{cases}.$$

The dual space of $V = \mathbb{R}^n$ will be identified with $\mathbb{R}^n$ via

$$B(x, y) := \sum_{i=1}^{p} x_i y_i - \sum_{i=p+1}^{n} x_i y_i\,.$$

Since H leaves B invariant, $\rho = \rho^*$.

Let $E := \langle Z_k, Z_{k+1}, \ldots, Z_{k+d-1}\rangle$, $1 < k \leq k+d-1 < n$, that is, $1 < k < n+1-d$ (in particular, $d < n-1$), $U := E^{\perp}$. Conditions (i) and (ii) in the beginning of section 5 are satisfied.

We follow [13] and introduce a system of coordinates on a neighborhood of E in $\mathcal{E}$. Let $G(d,n)$ denote the set of d-planes in $\mathbb{R}^n$, $G_{d,n} \subseteq G(d,n)$ the subset of d-dimensional subspaces. For every multiindex $j = (j_1, \ldots, j_d)$, $1 \leq j_1 < \ldots < j_d \leq n$, let

$$U_j := \left\{ \sigma \in G_{d,n} \;\middle|\; \begin{array}{c} \text{the orthogonal projection of } \sigma \text{ on the subspace} \\ \text{spanned by } Z_{j_1}, \ldots, Z_{j_d} \text{ is bijective} \end{array} \right\}.$$

Let $1 \leq j_{d+1} < \ldots < j_n \leq n$ be such that $\{j_{d+1}, \ldots, j_n\} \cup \{j_1, \ldots, j_d\} = \{1, \ldots, n\}$. Let $\sigma \in U_j$, $v_{j_\alpha} \in \sigma$ be the inverse image of Z_{j_α} under the orthogonal projection onto $\operatorname{span}\{Z_{j_1}, \ldots, Z_{j_d}\}$ $(1 \leq \alpha \leq d)$. It holds:

$$V_{j_\alpha} = Z_{j_\alpha} + \sum_{\kappa=d+1}^{n} \sigma_{j_\kappa j_\alpha} Z_{j_\kappa}$$

with real coefficients $\sigma_{j_\kappa j_\alpha}$. In what follows, the range of α, κ will always be $[1,d], [d+1,n]$ respectively. We define the $n \times d$-matrix Σ by taking $v_{j_1}, \ldots, v_{j_d}$ for its columns (expressed in the basis $Z_1, \ldots, Z_n$). The open set U_j can now be parametrized by Σ, that is, by $(\sigma_{j_\kappa j_\alpha})_{\kappa,\alpha}$. Since $\cup_j U_j = G_{d,n}$, we obtain in this way an atlas for $G_{d,n}$.

The space $G(d,n)$ is a vector bundle over $G_{d,n}$. Let $\pi : G(d,n) \to G_{d,n}$ denote the projection. We take $\xi \in G(d,n)$ and put $\sigma := \pi(\xi) \in G_{d,n}$. If $\sigma \in U_j$, $\{y\} := \xi \cap \langle Z_{j_{d+1}}, \ldots, Z_{j_n}\rangle$, it holds: $\xi = y + \sigma$. We agree to write $y = \sum_{\kappa=d+1}^{n} y_{j_\kappa} Z_{j_\kappa}$ and parametrize $\pi^{-1}(U_j)$ by the vector

$$\big((\sigma_{j_\kappa j_\alpha})_{\kappa,\alpha}, y_{j_{d+1}}, \ldots, y_{j_n}\big) \in \mathbb{R}^{d(n-d)} \times \mathbb{R}^{n-d}. \tag{16}$$

However, not every d-plane in $\pi^{-1}(U_j)$ can be found in $\mathcal{E}$. In fact, $\pi(\mathcal{E})$ consists precisely of the subspaces σ, where the signature of B equals its signature on E. We conclude that $\mathcal{E} \subseteq G(d,n)$ is open, so we can restrict the above parametrization to $\pi^{-1}(U_j) \cap \mathcal{E}$ (and obtain in this way an atlas of $\mathcal{E}$).

The action of $SO(p, n-p)$ on $\mathbb{R}^n$ possesses property (G) of section 6, and consequently the differential operators $d\nu(V_{ijl})$ characterize the range of $\mathcal{S}(\mathbb{R}^n)$. However, the computation of these operators in the coordinates (16) is very lengthy and tedious (see [15, page 36-38]). Their second order terms are linear combinations of

$$\mathcal{T}_{j_\kappa j_\alpha j_\lambda} := \frac{\partial^2}{\partial \sigma_{j_\kappa j_\alpha} \partial y_{j_\lambda}} - \frac{\partial^2}{\partial \sigma_{j_\lambda j_\alpha} \partial y_{j_\kappa}},$$

their first order ones consist of derivatives in the y-coordinates, and the coefficients of all terms only depend on $\sigma_{j_\kappa j_\alpha}$. Nevertheless, we can prove the following result, which is analogous to that of Richter for the classical case [14, page 73ff]:

Proposition 9.1. *A function $\varphi \in \mathcal{S}(\mathcal{E})$ satisfies $d\nu(V_{ijl})\varphi = 0$ for $1 \leq i < j < l \leq n$ iff for every multiindex $j = (j_1, \ldots, j_d)$ as above, φ_0 satisfies $\mathcal{T}_{j_\kappa j_\alpha j_\lambda}\varphi_0 = 0$ for $1 \leq \alpha \leq d$, $d+1 \leq \kappa, \lambda \leq n$ on $\pi^{-1}(U_j) \cap \mathcal{E}$, where φ_0 is a modified function, defined with the help of the $n \times d$-matrix Σ:*

$$\varphi_0(\Sigma, y) := |\det \Sigma^T I_{p,n-p}\Sigma|^{-\frac{1}{2}}\varphi(\Sigma, y)$$

($I_{p,n-p}$ denoting the matrix of B in the basis $Z_1, \ldots, Z_n$).

Proof. — The condition is necessary: If $\varphi \in \mathcal{S}_D(\mathcal{E})$, there exists $f \in \mathcal{S}(\mathbb{R}^n)$ such that $\hat{f} = \varphi$. By looking at the integration measure on the various d-planes f is integrated on, we conclude that

$$\varphi_0(\Sigma, y) = \int_{\mathbb{R}^d} f(\Sigma x + y)dx_1 \ldots dx_d\,, \quad x = (x_1, \ldots, x_d)^T\,.$$

The relations $\mathcal{T}_{j_\kappa j_\alpha j_\lambda}\varphi_0 = 0$ now follow immediately.

— The condition is sufficient: We compute the Fourier transform of φ ($A \in SO(p, n-p)$, $u \in U$):

$$\tilde{\varphi}(A \cdot E, Au) = \int_U \varphi(A \cdot E, Au')e^{-i\langle Au, Au'\rangle}du' = \int_U \varphi(\Sigma, y)e^{-i\langle Au, y\rangle}du'$$

in the notation of (16),

$$= |\det \Sigma^T I_{p,n-p}\Sigma|^{-\frac{1}{2}} \int_{\mathbb{R}^{n-d}} \varphi(\Sigma, y)e^{-i\langle Au, y\rangle}dy_{j_{d+1}} \ldots dy_{j_n}$$

$$= \int_{\mathbb{R}^{n-d}} \varphi_0(\Sigma, y)e^{-i\langle Au, y\rangle}dy_{j_{d+1}} \ldots dy_{j_n}\,.$$

In a completely analogous way to that of Richter [13, page 252-254] it can now be shown that $\tilde{\varphi}(A \cdot E, Au)$ only depends on Au, so there exists a function $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\tilde{\varphi}(A \cdot E, Au) = F(Au)$. The proof now follows by remark 2 to lemma 5.2 and the subsequent computation of the Fourier transforms. ■

4. The group of orthogonal matrices

$$H = \left\{ \left(\begin{array}{cccc} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & x_4 & -x_3 \\ x_3 & -x_4 & x_1 & x_2 \\ x_4 & x_3 & -x_2 & x_1 \end{array} \right) \middle|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\} < SO(4)$$

operates on $V = \mathbb{R}^4$, which is identified with $(\mathbb{R}^4)^*$ via the canonical scalar product. We have $\rho = \rho^*$.

The choice $E = \langle Z_1 \rangle$ (notation as above) shows how indispensable assumption (ii), section 5, is for our considerations. The other assumptions are all satisfied. For $\xi \neq 0$ we have $H_\xi = \{e\}$. Thus, there should be no need of differential operators. But since every point ξ lies on more than one hyperplanes of the form $h \cdot E^\perp$, the moment conditions fail to characterize the range of $\mathcal{S}(\mathbb{R}^4)$. A counterexample can be constructed as in [6, page 605].

Now let $E = \langle Z_1, Z_2 \rangle$. All assumptions are satisfied. The action does not have property (G) of section 6. We have instead the extreme situation where two planes of the form $h \cdot E^\perp$ either coincide or are complementary. Here we cannot dispense with the moment conditions. Otherwise, $\mathcal{S}(\mathbb{R}^4)^\wedge = \mathcal{S}(\mathcal{E})$ would hold, but this is contradicted by the following counterexample:

Let $g \in C_c^\infty(\mathbb{R}^4)$, $g(0) \neq 0$. Let (x_1, x_2, x_3, x_4) denote the matrix in H, which has this vector in its first column. $x_3^2 + x_4^2$ is constant on the coset $(x_1, x_2, x_3, x_4) H_E$. For $\xi \in (x_1, x_2, x_3, x_4) \cdot E^\perp$, $\xi \neq 0$, it holds:

$$x_3^2 + x_4^2 = \frac{\xi_1^2 + \xi_2^2}{\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2}\,.$$

Now take $\varphi \in \mathcal{S}(\mathcal{E})$ such that

$$\tilde{\varphi}(\,(x_1, x_2, x_3, x_4)H_E, \xi) = (x_3^2 + x_4^2) g(\xi) = \frac{(\xi_1^2 + \xi_2^2) g(\xi)}{\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2} =: F(\xi)$$

for $\xi \in (x_1, x_2, x_3, x_4) \cdot E^\perp$, $\xi \neq 0$. Obviously, φ is not an element of $\mathcal{S}(V)^\wedge$, because F cannot be continuously extended to 0.

Next, H is being enriched by the central matrices of the form

$$\begin{pmatrix} \cos t & -\sin t & & \\ \sin t & \cos t & \multicolumn{2}{c}{0} \\ \multicolumn{2}{c}{0} & \cos t & -\sin t \\ & & \sin t & \cos t \end{pmatrix}. \tag{17}$$

We study the interesting case $E = \langle Z_2, Z_3 \rangle$. The matrices

$$X_1 = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} \multicolumn{2}{c}{0} & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & \multicolumn{2}{c}{0} \\ 0 & -1 & & \end{pmatrix}, \quad Z = \begin{pmatrix} \multicolumn{2}{c}{0} & 0 & 1 \\ & & -1 & 0 \\ 0 & 1 & \multicolumn{2}{c}{0} \\ -1 & 0 & & \end{pmatrix}$$

form a basis of the Lie algebra of H. Now $\mathcal{K}^1(\mathfrak{h} \times_s \mathbb{R}^4)$ is spanned by

$$P = X_1(Z_3^2 + Z_4^2) + X_2(Z_1^2 + Z_2^2) + Y(Z_1Z_4 - Z_2Z_3) - Z(Z_1Z_3 + Z_2Z_4)\,.$$

For $w = w_1Z_1 + w_2Z_4 \in E^\perp \setminus \{0\}$ we have: $\mathfrak{h}_w = \langle w_2^2 X_1 + w_1^2 X_2 + w_1w_2 Y \rangle$. Therefore, every 2-plane $h \cdot E^\perp$ through w has the form $\exp[t(w_2^2X_1 + w_1^2X_2 + w_1w_2Y)] \cdot E^\perp$, $t \in \mathbb{R}$. Since $\bar{w} := w_1Z_2 - w_2Z_3$ is orthogonal to all these planes ($\mathfrak{h}_{\bar{w}} = \mathfrak{h}_w$ and $\bar{w} \perp E^\perp$), the action does not have property (G). Nevertheless, we can still dispense with the moment conditions. The latter are satisfied if the functions F_m in (15) are polynomials. So we proceed as follows.

Let $F \in C^\infty(\mathbb{R}^4 \setminus \{0\})$ have the property that its restriction to every plane $h \cdot E^\perp$ is a homogeneous polynomial of degree m. We shall show that F is a homogeneous polynomial of degree m.

At first, we establish this for the hyperplane $\mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^4$.

Let (x_1, x_2, x_3, x_4, t) denote the matrix (x_1, x_2, x_3, x_4) multiplied by the matrix (17). Then $(x_1, x_2, x_3, x_4, t) \cdot E^\perp$ lies in $\mathbb{R}^3 \times \{0\}$ iff

$$x_3 \sin t + x_4 \cos t = x_1 \cos t + x_2 \sin t = 0\,.$$

In that case,

$$h \cdot w = \frac{1}{\cos t} \begin{pmatrix} -w_1 x_2 \sin 2t + w_2 x_3 \sin 2t \\ w_1 x_2 \cos 2t - w_2 x_3 \cos 2t \\ w_1 x_3 + w_2 x_2 \\ 0 \end{pmatrix}$$

for $\cos t \neq 0$ and $h \cdot E^\perp = \langle Z_2, Z_3 \rangle$ for $\cos t = 0$. We obtain all planes through $(0,0,1)$ in $\mathbb{R}^3$. Since F, restricted to each of them, is a homogeneous polynomial, we can write

$$F(\xi_1, \xi_2, \xi_3, 0) = A_0(\xi_1, \xi_2) + A_1(\xi_1, \xi_2)\xi_3 + \ldots + A_m(\xi_1, \xi_2)\xi_3^m\,.$$

If we write down this equation for $m+1$ different values of $\xi_3 \neq 0$, we obtain a linear system in $A_0, \ldots, A_m$. By Cramer's rule we deduce that the A_i's are C^∞-functions and, therefore, so is $F|_{\mathbb{R}^3 \times \{0\}}$. Now since $F|_{\mathbb{R}^3 \times \{0\}}$ is homogeneous, it is a polynomial. We denote it by P_0.

If $F(\xi_1, \xi_2, \xi_3, \xi_4) \not\equiv P_0(\xi_1, \xi_2, \xi_3)$, let $0 < k_0 \leq m$ be the highest possible exponent, such that

$$F_1(\xi_1, \xi_2, \xi_3, \xi_4) := \frac{F(\xi_1, \xi_2, \xi_3, \xi_4) - P_0(\xi_1, \xi_2, \xi_3)}{\xi_4^{k_0}}$$

is a C^∞-function on $\mathbb{R}^4 \setminus \{0\}$. This function is a homogeneous polynomial of the smaller degree $m - k_0$ on every plane $h \cdot E^\perp$ that is not contained in $\mathbb{R}^3 \times \{0\}$. This means that every expression of the form

$$(\frac{\partial}{\partial w_1})^{m-k_0+1-j} (\frac{\partial}{\partial w_2})^j F_1(h \cdot w)$$

vanishes for $h = (x_1, x_2, x_3, x_4, t)$, $(x_3 \sin t + x_4 \cos t, x_1 \cos t + x_2 \sin t) \neq (0,0)$. By continuity, it vanishes for every h. Consequently, F_1 is a homogeneous polynomial on every plane $h \cdot E^\perp \subseteq \mathbb{R}^3 \times \{0\}$ too.

We repeat the argument with F_1 instead of F, define a polynomial P_1, a number k_1 and a function F_2 with which we continue in the same manner. After finitely many steps

$$(P_0, k_0) \to (P_1, k_1) \to \ldots \to (P_n, k_n)$$

we obtain a function $F_{n+1} \in C^\infty(\mathbb{R}^4 \setminus \{0\})$, constant on every plane $h \cdot E^\perp$. Since all planes $h \cdot E^\perp \in \mathbb{R}^3 \times \{0\}$ contain $(0,0,1)$, F_{n+1} is constant on $\mathbb{R}^3 \times \{0\}$. Since every other plane $h \cdot E^\perp$ meets $\mathbb{R}^3 \times \{0\}$, it is globally constant. We obtain:

$$F(\xi_1, \xi_2, \xi_3, \xi_4) = F_{n+1} \xi_4^{\sum_{i=0}^n k_i} + \sum_{j=0}^n P_{n-j}(\xi_1, \xi_2, \xi_3) \xi_4^{\sum_{i=0}^{n-j-1} k_i}$$

$(\sum_{i=0}^n k_i = m)$.

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Received July 9, 1997
and in final form June 5, 1998