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Ideals of finite codimension in contact Lie algebra

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Abstract. Ideals of germs of vector fields leaving 0 fixed in R^{2n+1}, of finite codimension in contact Lie algebra contain the ideal of germs infinitely flat at 0. We give an application.

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1. Introduction

In this paper, we characterize ideals of germs of vector fields X with $X(0) = 0$ in R^{2n+1}, of finite codimension in a contact Lie algebra. Let $\alpha = dx_{2n+1} + \frac{1}{2}\sum_{i=1}^{n}(x_i dx_{n+i} - x_{n+i}dx_i)$ denote the canonical contact form on R^{2n+1}, and let χ_α denote the contact Lie algebra of germs of vector fields leaving 0 fixed. Also, denote by χ_α^∞ the ideal of χ_α, of infinitely flat germs at 0. Finally, let F denote the space of germs of smooth functions. The main theorem is:

Theorem 1.1. *Let I be a finite codimension ideal of χ_α. Then I contains χ_α^∞ .*

As a consequence of this result we obtain a reduction of the action of the group of germs of origin preserving contact diffeomorphisms to the action of the group of infinite jets (at the origin) on a Lie group; we pass to the linear version. We encounter such an action when we deal with the theory of natural fiber bundles (see [1] , [2] [3]). In [2] and [3], the authors used Borel lemma and Whitney extension theorem to prove the reduction of the above action. In the category of manifolds endowed with geometric structures, Borel lemma and Whitney extension theorem fail. The interest of finding appropriate methods for this cases was raised in [2] . Our result applies in the category of contact manifolds.

2. Fundamental lemma

The proof of Theorem1 is based on the following lemma.

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Lemma 2.1. *Let V be a space of finite codimension in a real linear space E and ψ an endomorphism of E such that:*

1) $\psi(V) \subset V$

2) *for any $b \in R$, $\psi + bI$ is onto in E.*

3) *for any $b, c \in R$ with $b^2 - 4c < 0$, $\psi^2 + b\psi + cI$ is onto in E. Then $V = E$.*

Proof. Since the subspace V of E is invariant by the endomorphism $\psi: E \to E$, there exists a unique endomorphism $\bar{\psi}$: $E/V \to E/V$ such that $\pi o \psi = \bar{\psi}\ o \pi$, where $\pi: E \to E/V$ denotes the canonical projection. If V is of non zero finite codimension in E, the quotient space E/V is of non zero finite dimension and $\bar{\psi}$ admits an eigenvalue. If this eigenvalue is real, there exists $b \in R$ such that $\bar{\psi} + bI_{E/V}$ is not onto in E/V. So $\psi + bI_E$ cannot be onto in E, since if it is this contradicts the property $\pi o \psi = \bar{\psi}\ o\pi$. And if this eigenvalue is complex, there are b and $c \in R$ such that $b^2 - 4c < 0$ and $\bar{\psi}^2 + b\ \bar{\psi} + cI_{E/V}$ is not onto in E/V. The same argument as above shows that $\psi^2 + b\psi + cI_E$ cannot be onto in E. This proves Lemma 2. ∎

3. Preparatory lemmas

We know that $\varphi: \chi_\alpha \to F$ defined by $\varphi(X) = i_X\alpha$ (where $i_X\alpha$ denotes the interior product by X) is an isomorphism. φ induces a Lie algebra structure on F, of which the bracket operation is called a Jacobi bracket and will be denoted by $\{\,\}$, it has the following representation

$$\{f, g\} = \sum_{i=1}^{n} (\delta f/\delta x_i . \delta g/\delta x_{n+i} - \delta f/\delta x_{n+i} . \delta g/\delta x_i) - g\partial f/\delta x_{2n+1} + f\partial g/\partial x_{2n+1}$$

where $\delta/\delta x_i = \partial/\partial x_i + \frac{1}{2}x_{n+i}\partial/\partial x_{2n+1}$, $\delta/\delta x_{n+i} = \partial/\partial x_{n+i} - \frac{1}{2}x_i\partial/\partial x_{2n+1}$.

Let $k(t)$ be a bounded continuous function for all $0 \leq t \leq 1$ and $x = (x_1, \ldots, x_{2n+1})$.

Lemma 3.1. *For any $b \in R$ and $h \in F$ with $J_o^\infty h = 0$, the germ of the function g defined by the integral*

$$g(x) = \int_0^1 t^b k(t) h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})dt \tag{1}$$

is smooth and infinitely flat at 0.

Proof. Since h is infinitely flat at 0, for any positive integer m, there exist constants $\delta > 0$ and $M > 0$ such that for all x with $|x| \leq \delta$, we have

$$\left| h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})) \right| \leq M\delta^m t^{\frac{1}{2}m}.$$

By hypothesis there exists a constant c such $|k(t)| \leq c$ for any $0 \leq t \leq 1$; then

$$\left|k(t)t^b h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})\right| \leq cMt^{b+\frac{m}{2}} = f_m(t).$$

Since m is arbitrary, we choose m_o such that $b + \frac{m_o}{2} - 1 > 0$; consequently the integral defining the function g converges uniformly in a neighborhood of 0. Now, we shall show that the function g is C^∞ and infinitely flat at 0. For any multi-indice $\alpha \in N^{2n+1}$ we have

$$D_x^\alpha(h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})) = t^{|\alpha|/2}(D_x^\alpha h)(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})$$

where $|\alpha|$ means the length of α. Since $D_x^\alpha h$ is infinitely flat at 0, there exist constants $\delta > 0$ and $M > 0$ such that:

$$\left|(D_x^\alpha h)(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})\right| \leq Mt^{1/2(m+|\alpha|)}\delta^m$$

provided that $|x| \leq \delta$. Since m is arbitrary, we choose m_o such that $|\alpha| + m_o + b - 1 > 0$. Consequently, for any multi-indice $\alpha \in N^{2n+1}$,

$$\int_0^1 t^b k(t) D_x^\alpha h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})dt$$

converges uniformly in a neighborhood of 0, so the function g defined by (1) is smooth. Now since $D_x^\alpha h(0) = 0$ and $D_x^\alpha h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})$ converges uniformly to $D_x^\alpha h(0)$ with respect to x, we pass to the limit and obtain $D_x^\alpha g(0) = 0$. This proves Lemma 3. ■

Let z be the $(2n+1)^{th}$ coordinate x_{2n+1}.

Lemma 3.2. *For any $b \in R$ and $h \in F$ with $J_0^\infty h = 0$, there exists $g \in F$ with $J_0^\infty g = 0$ such that*

$$\{z, g\} + bg = h \qquad (2)$$

Proof. Equation (2) is equivalent to

$$\frac{1}{2}\sum_{i=1}^{n}(x_i\partial g/\partial x_i + x_{n+i}\partial g/\partial x_{n+i}) + x_{2n+1}\partial g/\partial x_{2n+1} + (b-1)g = h$$

Consider the function g defined by

$$g(x) = \int_0^1 t^{b-2}h(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1})dt.$$

By Lemma 3, $g(x)$ is smooth and infinitely flat at 0. Since

$$t\frac{dh}{dt}(t^{1/2}x_1, ..., t^{1/2}x_{2n}, tx_{2n+1}) = \frac{1}{2}\sum_{i=1}^{n} t^{1/2}(x_i\partial h/\partial x_i + x_{n+i}\partial h/\partial x_{n+i}) + tx_{2n+1}\partial h/\partial x_{2n+1}$$

it follows that

$$\begin{array}{c}\frac{1}{2}\ \sum_{i=1}^{n}(x_i\partial g/\partial x_i+x_{n+i}\partial g/\partial x_{n+i})+x_{2n+1}\partial g/\partial x_{2n+1}= \\ \int_0^1 t^{b-1}\frac{dh}{dt}(t^{1/2}x_1,...,t^{1/2}x_{2n},tx_{2n+1})dt\end{array} \tag{3}$$

$$=\Big[t^{b-1}h(t^{1/2}x_1,...,t^{1/2}x_{2n},tx_{2n+1})\Big]_0^1-(b-1)\int_0^1 t^{b-2}h(t^{1/2}x_1,...,t^{1/2}x_{2n},tx_{2n+1})dt.$$

The infinitely flatness of h at 0 leads us to write for any $b\in R$

$$\lim_{t\to 0} t^{b-1}h(t^{1/2}x_1,...,t^{1/2}x_{2n},tx_{2n+1})=0.$$

Hence

$$\frac{1}{2}\sum_{i=1}^{n}(x_i\partial g/\partial x_i+x_{n+i}\partial g/\partial x_{n+i})+x_{2n+1}\partial g/\partial x_{2n+1}=h-(b-1)g(x).$$

This checks Lemma 4. ■

Lemma 3.3. *For any $b,\ c\in R$ with $b^2-4c<0$ and any $h\in F$ with $J_0^\infty h=0$, there exists $g\in F$ with $J_0^\infty g=0$ such that*

$$\{z,\{z,g\}\}+b\{z,g\}+cg=h. \tag{4}$$

Proof. Similarly as in Lemma 4, we must find $g\in F$ with $J_0^\infty g=0$ satisfying equation(4). Let

$$k(t)=\frac{-2t^{-b/2}}{\sqrt{4c-b^2}}\sin(\sqrt{4c-b^2}/2)\log|t|\,.$$

Note that $k(t)$ is the solution of the Cauchy problem

$$\left\{\begin{array}{c} t^2k''(t)+(1+b)tk'(t)+ck(t)=0 \\ k'(1)=-1 \quad k(1)=0.\end{array}\right. \tag{5}$$

Consider the function g defined by the integral

$$g(x)=\int_o^1 t^{b-2}k(t)h(t^{1/2}x_1,...,t^{1/2}x_{2n},tx_{2n+1})dt. \tag{6}$$

For simplicity we write (.) for $(t^{1/2}x_1,...,t^{1/2}x_{2n},tx_{2n+1})$. By Lemma 3 $g(x)$ given by (6) is smooth and infinitely flat at 0. On the other hand from formula (3) we obtain

$$\begin{array}{c}\{z,g\}+g=\frac{1}{2}\ \sum_{i=1}^{n}(x_i\partial g/\partial x_i+x_{n+i}\partial g/\partial x_{n+i})+x_{2n+1}\partial g/\partial x_{2n+1} \\ =\int_o^1 t^{b-1}k(t)\frac{dh}{dt}(.)dt,\end{array}$$

and

$$\{z,\{z,g\}\} = g + \int_o^1 t^{b-1}k(t)\frac{d}{dt}(t\frac{dh}{dt} - 2h)\,(.)\,dt.$$

An integration by parts gives us

$$\begin{gathered}\{z,\{z,g\}\} = \left[t^{b-1}k(t)(t\tfrac{dh}{dt} - 2h)\,(.)\right]_0^1 + \\ \int_0^1 t^{b-2}\left((2b-1)k(t) + 2tk'(t)\right)h(.)dt - \\ \int_0^1 t^{b-2}\left((b-1)k(t) + tk'(t)\right)t\tfrac{dh}{dt}(.)dt,\end{gathered}$$

and

$$\begin{gathered}\int_0^1 t^{b-1}((b-1)k(t) + tk'(t))\tfrac{dh}{dt}(.)dt = \\ \left[t^{b-1}((b-1)k(t) + tk'(t))h(.)\right]_0^1 - \int_0^1 t^{b-2}\left(t^2k''(t) + (2b-1)tk'(t) + (b-1)^2k(t)\right)h(.)dt.\end{gathered}$$

Then

$$\begin{aligned}\{z,\{z,g\}\} &= h(x) + \int_0^1 t^{b-2}\left(t^2k''(t) + (2b+1)tk'(t) + b^2k(t)\right)h(.)dt \\ &= h(x) - b\,\{z,g\} - cg + \int_0^1 t^{b-2}\left(t^2k''(t) + (b+1)tk'(t) + ck(t)\right)h(.)dt.\end{aligned}$$

Since $k(t)$ is solution of (5), Lemma 5 is proved. ■

4. Proof of the main result

We apply Lemma 2 to $E = \chi_\alpha^\infty$ and $I^\infty = \chi_\alpha^\infty \cap I$. From well known facts in linear algebra, we obtain

$$\dim(\chi_\alpha^\infty/\chi_\alpha^\infty \cap I) = \dim((\chi_\alpha^\infty + I)/I) \leq \dim(\chi_\alpha/I) < +\infty.$$

So I^∞ is of finite codimension in χ_α^∞ . The endomorphism $\psi = [X_o, .]$, where $X_o(x_1, ..., x_{2n+1}) = \sum_{i=1}^{2n+1} x_i \frac{\partial}{\partial x_i}$ $(X_o = \varphi^{-1}(z))$, satisfies obviously the first assumption of Lemma 2, and by Lemma 4 and 5, the conditions of Lemma 2 are fulfilled. Consequently $\chi_\alpha^\infty = I^\infty$ i.e. $\chi_\alpha^\infty \subset I$. ■

5. Application

Let Γ be a pseudogroup of local diffeomorphisms of R^n.

Definition 5.1. A left action of a pseudogroup Γ on a Lie group G, is a functorial assignment to each $f \in \Gamma$ with domain U a smooth map $\bar{f}: U \times G \to G$ such that the following axioms are satisfied

1) For any $\xi \in U$ and any $a, b \in G$,

$$\bar{f}\ (\xi, ab) = \bar{f}\ (\xi, a)b.$$

2) For any open set V of U

$$\overline{f\ |_V} = \bar{f}\ |_{V\times G}\ .$$

3) For any $f, g \in \Gamma$,

$$\overline{g \circ f}(\xi, y) = \bar{g}\ (f(\xi), \bar{f}\ (\xi, y)).$$

4) Let I be an open interval of the real line R. If $f: I \times U \to R^n$ is a smooth map such that $\forall t \in I,\ f_t \in \Gamma$, where $f_t(x) = f(t, x)$, the map

$$I \times U \times G \to G$$
$$(t, (\xi, y)) \to \bar{f}_t\ (\xi, y)$$

is smooth.

Remark. It is obvious from axiom 2) that the action is local, i.e. depends only on germs.

Let P be the pseudogroup of contact diffeomorphisms and P_o the group of those fixing 0 and $L(G)$ be the Lie algebra of a Lie group G. Suppose that P acts on G; let $X \in \chi_\alpha$ and $\phi_t = exp(tX)$ be the flow generated by X. So $\phi_t \in P_o$ for any $t \in I$ (since X leaves 0 invariant). Let $(\bar{\phi}_t)_t$ be the flow of diffeomorphisms (translations) induced by the action of the pseudogroup P on G. Consider the vector field defined by:

$$\bar{X} = \frac{d}{dt}\,|_{t=0}\ (\bar{\phi}_t)_o$$

where $(\bar{\phi}_t)_o(x) = \bar{\phi}_t\ (0, x)$, then we obtain a Lie algebra homomorphism:

$$H : \chi_\alpha \to L(G),\ H(X) = \bar{X}$$

Proposition 5.2. *The homomorphism H depends only on the infinite jet $J_o^\infty X$ of X at 0.*

Proof. The kernel I of H is a finite codimension ideal of χ_α, so by Theorem1 its contains χ_α^∞ and then H depends only on $J_o^\infty X$. ■

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