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Moduli for Spherical Maps and Minimal Immersions of Homogeneous Spaces

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Abstract. The DoCarmo-Wallach theory studies isometric minimal immersions $f\colon G/K \to S^n$ of a compact Riemannian homogeneous space G/K into Euclidean n-spheres for various n. For a given domain G/K, the moduli space of such immersions is a compact convex body in a representation space for the Lie group G. In 1971 DoCarmo and Wallach gave a lower bound for the (dimension of the) moduli for $G/K = S^m$, and conjectured that the lower bound was achieved. In 1997 the author proved that this was true. The DoCarmo-Wallach conjecture has a natural generalization to all compact Riemannian homogeneous domains G/K. The purpose of the present paper is to show that for G/K a nonspherical compact rank 1 symmetric space this generalized conjecture is false. The main technical tool is to consider spherical functions of subrepresentations of $C^\infty(G/K)$, express them in terms of Jacobi polynomials, and use a recent linearization formula for products of Jacobi polynomials.

1. Introduction and Statement of Results

Let $M = G/K$ be a Riemannian homogeneous space, where G is a compact Lie group and K a closed subgroup. Then G acts on the space $C^\infty(M)$ of (real valued) functions on M in a natural way: $g \cdot \xi = \xi \circ g^{-1}$, $g \in G$, $\xi \in C^\infty(M)$. This action preserves the L^2-scalar product on $C^\infty(M)$ defined by the volume element v_M. Let $\mathcal{H} \subset C^\infty(M)$ be a G-submodule. We call a map $f\colon M \to S_V$ into the unit sphere S_V of a Euclidean vector space V a spherical $\mathcal{H}$-map if its components $\alpha \circ f$, $\alpha \in V^*$, belong to $\mathcal{H}$. The Dirac delta $\delta\colon M \to S_{\mathcal{H}^*}$ [5] defined by evaluating the elements of $\mathcal{H}$ on points of M is the universal example of a spherical $\mathcal{H}^*$-map. (The scalar product on $\mathcal{H}^*$ is induced by the L^2-scalar product on $\mathcal{H}$ suitably scaled.)

Remark. If $M = G/K$ is naturally reductive, and $\mathcal{H} \subset C^\infty(M)$ is irreducible then $\mathcal{H}$ is contained in an eigenspace V_λ of the Laplacian $\triangle^M$ for some eigenvalue λ [25]. In particular, the components of an $\mathcal{H}$-map $f\colon M \to S_V$ are eigenfunctions of the Laplacian with a common eigenvalue. Thus an $\mathcal{H}$-map is a λ-eigenmap in the sense of Eells-Sampson [8], a harmonic map with constant energy density $\lambda/2$.

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In general, a DoCarmo-Wallach type argument [6] shows that the set of (congruence classes of) full spherical $\mathcal{H}$-maps $f\colon M \to S_V$, for various V, can be parametrized by a moduli space $\mathcal{L}(\mathcal{H})$, a compact convex body in a G-submodule $\mathcal{E}(\mathcal{H})$ of the symmetric square $S^2(\mathcal{H})$ (Propositions 3.1-3.2 in Section 3 below). (The map f is full if its image is not contained in a proper great sphere of the range [3], and congruent maps differ by an isometry between the ranges.) In what follows, we identify $S^2(\mathcal{H})$ with the space of linear endomorphisms of $\mathcal{H}$. Then the moduli space is given by

$$\mathcal{L}(\mathcal{H}) = \{C \in \mathcal{E}(\mathcal{H}) \,|\, C + I \geq 0\},$$

where $\geq$ means positive semidefinite, and I is the identity. The origin of $\mathcal{E}(\mathcal{H})$ is in the interior of $\mathcal{L}(\mathcal{H})$, and it corresponds to δ.
The G-module homomorphism

$$\Psi^0\colon S^2(\mathcal{H}) \to C^\infty(M)$$

given by multiplication has image $\mathcal{H}\cdot\mathcal{H} \subset C^\infty(M)$ consisting of (finite) sums of products of functions in $\mathcal{H}$. The DoCarmo-Wallach parametrization of $\mathcal{L}(\mathcal{H})$ implies that the kernel of Ψ^0 is $\mathcal{E}(\mathcal{H})$ (Proposition 3.3). We thus have

$$\mathcal{E}(\mathcal{H}) = S^2(\mathcal{H})/(\mathcal{H}\cdot\mathcal{H}),$$

as G-modules. To determine $\mathcal{E}(\mathcal{H})$ (and thereby to compute $\dim\mathcal{L}(\mathcal{H}) = \dim\mathcal{E}(\mathcal{H})$) amounts to decomposing $\mathcal{H}\cdot\mathcal{H}$ into irreducible components.
Let $M = G/K$ be a compact rank 1 symmetric space. Then M is the Euclidean m-sphere S^m, one of the projective spaces $\mathbf{R}P^m$, $\mathbf{C}P^m$, $\mathbf{H}P^m$, or the Cayley projective plane $\mathbf{Ca}P^2$ [2]. It is well-known that $C^\infty(M)$ has a multiplicity one decomposition into irreducible components, and each component $\mathcal{H} \subset C^\infty(M)$ is the full eigenspace V_λ of the Laplacian corresponding to an eigenvalue λ [15-17]. Our first result is the following:

Theorem. *A Let $M = G/K$ be a compact rank 1 symmetric space, $\mathcal{H} \subset C^\infty(M)$ an irreducible G-submodule. We write $\mathcal{H} = V_{\lambda_p}$, where λ_p is the p-th eigenvalue of the Laplacian on $\mathcal{H}$. Then we have*

$$V_{\lambda_p}\cdot V_{\lambda_p} = \begin{cases} \sum_{j=0}^{p} V_{\lambda_{2j}} & \text{if } M = S^m \\ \sum_{j=0}^{2p} V_{\lambda_j} & \text{otherwise.} \end{cases}$$

In particular, the dimension of the moduli space is given by

$$\dim\mathcal{L}(V_{\lambda_p}) = \begin{cases} n(\lambda_p)(n(\lambda_p)+1)/2 - \sum_{j=0}^{p} n(\lambda_{2j}) & \text{if } M = S^m, \\ n(\lambda_p)(n(\lambda_p)+1)/2 - \sum_{j=0}^{2p} n(\lambda_j) & \text{otherwise,} \end{cases}$$

where $n(\lambda_p) = \dim V_{\lambda_p}$.

Since $n(\lambda_p)$ is known for each case of M (second table in Section 2), an explicit formula can be derived for the dimension of $\mathcal{L}(V_{\lambda_p})$. If the dimension is zero then the moduli space reduces to a point, and we have rigidity. This means that the corresponding spherical V_{λ_p}-maps are rigid in the sense that any full f is congruent to the Dirac delta δ.

Corollary. *Let M and $\mathcal{H} = V_{\lambda_p}$ be as in Theorem A. Then the cases when $\dim \mathcal{L}(V_{\lambda_p})$ is trivial are summarized in the following table:*

M	m	p
S^m	$m \geq 2$	$p = 1$
S^m, $\mathbf{R}P^m$	$m = 2$	$p \geq 1$
$\mathbf{C}P^m$, $\mathbf{H}P^m$, $\mathbf{Ca}P^2$	$m = 2$	$p = 1$

Remark. Rigidity of a spherical V_{λ_1}-map $f\colon S^m \to S_V$ is obvious since f is the restriction of a linear map, and thereby it is an isometry. Rigidity of spherical V_{λ_p}-maps $f\colon M \to S_V$ for $M = S^2$, $\mathbf{R}P^2$ is due to Calabi [3] (stated only for minimal immersions). (In general, a spherical V_{λ_p}-map $f\colon \mathbf{R}P^m \to S_V$ is a spherical $V_{\lambda_{2p}}$-map $\tilde{f}\colon S^m \to S_V$ factored through the twofold projection $S^m \to \mathbf{R}P^m$.)
A rigidity result of DoCarmo-Wallach [6,25] asserts that a minimal immersion $f\colon M \to S_V$ of a compact analytic manifold M is rigid among minimal immersions, if the (geometric) degree of f is < 4. For $M = \mathbf{C}P^m, \mathbf{H}P^m, \mathbf{Ca}P^2$ as in the corollary, the degree of $\delta\colon M \to S_{V_{\lambda_p}}$ is $2p$ [18]. Notice however that the corollary gives rigidity among all spherical V_{λ_1}-maps not just minimal immersions.

We now return to the general setting. Let G be a compact Lie group. An orthogonal G-module $\mathcal{H}$ is a Euclidean vector space on which G acts linearly via orthogonal transformations. In other words, $\mathcal{H}$ is a representation space for G, and it is endowed with a G-invariant scalar product.
Let K be a closed subgroup. A class 1 representation of (G, K) is an irreducible orthogonal G-module $\mathcal{H}$ so that there is a nonzero vector $\chi_0 \in \mathcal{H}$ fixed by K. It is well known that, for $M = G/K$ Riemannian homogeneous, the irreducible components of $C^\infty(M)$ are class 1 representations of (G, K).
Since all components of $C^\infty(M)$ are class 1 with respect to (G, K) it is natural to ask whether $\mathcal{H} \cdot \mathcal{H} \subset C^\infty(M)$ contains all class 1 components of $S^2(\mathcal{H})$. We reformulate this by introducing $\bar{\mathcal{E}}(\mathcal{H})$ as the sum of those irreducible G-submodules of $S^2(\mathcal{H})$ that are not class 1 with respect to (G, K). The very existence of the homomorphism Ψ^0 above implies that

$$\bar{\mathcal{E}}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}),$$

and the question is whether equality holds. For $M = S^m$, the answer is yes, and it follows from the (multiplicity one) decomposition for $S^2(V_{\lambda_p})$ derived by DoCarmo and Wallach [6]. (For a simple proof, see also [14].)
Our next result shows that the answer is negative for $M = \mathbf{C}P^m$.

Theorem. *B Let $(G, K) = (U(m+1), U(m) \times U(1))$, and $M = \mathbf{C}P^m$, $m \geq 2$, the complex projective m-space. Let $\mathcal{H} = V_{\lambda_p}$, $p \geq 2$. Then $\mathcal{E}(V_{\lambda_p})$ contains class 1 submodules with respect to $(U(m+1), U(m) \times U(1))$. Equivalently*

$$\bar{\mathcal{E}}(V_{\lambda_p}) \neq \mathcal{E}(V_{\lambda_p}).$$

More precisely, we have

$$\sum_{q=2}^{2p-2} \frac{1}{2}\left(\min(q, 2p-q) + \frac{(-1)^q - 1}{2}\right) V_{\lambda_q} \subset \mathcal{E}(V_{\lambda_p}).$$

Remark. For $M = \mathbf{C}P^m$, Theorems A-B correct Theorem 4.2 of Chapter III in [22]. The formula there should give only a lower bound for $\mathcal{E}(V_{\lambda_p})$ and for the dimension of the moduli space $\mathcal{L}(V_{\lambda_p})$. This is due to the fact that the osculating spaces of $\delta\colon \mathbf{C}P^m \to S_{V^*_{\lambda_p}}$ are reducible as $U(m)$-modules (Theorem 3.3 of Chapter II).

Assume now that $M = G/K$ is isotropy irreducible. This means that K acts on the tangent space $T_o(M)$, $o = \{K\}$, irreducibly by the isotropy representation. Then, for an irreducible G-submodule $\mathcal{H} \subset C^\infty(M)$, the Dirac delta $\delta\colon M \to S_{\mathcal{H}^*}$ is a minimal immersion inducing the $\lambda/\dim M$-multiple of the original Riemannian metric on M [25].

DoCarmo and Wallach proved that the set of (congruence classes of) full minimal immersions $f\colon M \to S_V$, for various V, and with induced Riemannian metric the $\lambda/\dim M$-multiple of the original, can be parametrized by a moduli space $\mathcal{M}(\mathcal{H})$, a compact convex body in a G-submodule $\mathcal{F}(\mathcal{H})$ of $S^2(\mathcal{H})$ (Proposition 3.4). The moduli space is given by

$$\mathcal{M}(\mathcal{H}) = \{C \in \mathcal{F}(\mathcal{H}) \,|\, C + I \geq 0\},$$

where $\geq$ means positive semidefinite.

We now recall the definition of induced representations [25]. If $\mathcal{W}$ is a K-module then $\mathrm{Ind}_K^G(\mathcal{W})$ denotes the linear space of continuous maps $\phi\colon G \to \mathcal{W}$ which satisfy $\phi(kg) = k \cdot \phi(g)$, $k \in K$, $g \in G$. The action of G on $\mathrm{Ind}_K^G(\mathcal{W})$ given by $g \cdot \phi(g') = \phi(gg')$, $g, g' \in G$, defines a G-module structure on $\mathrm{Ind}_K^G(\mathcal{W})$. We call $\mathrm{Ind}_K^G(\mathcal{W})$ the G-module induced from the K-module $\mathcal{W}$.

DoCarmo and Wallach constructed a homomorphism

$$\Psi\colon S^2(\mathcal{H}) \to \mathrm{Ind}_K^G(S^2(\mathfrak{p})),$$

where K-module $S^2(\mathfrak{p})$ is the symmetric square of the isotropy representation of $M = G/K$. The kernel of Ψ is $\mathcal{F}(\mathcal{H})$.

Let $\bar{\mathcal{F}}(\mathcal{H})$ denote the sum of those components of $S^2(\mathcal{H})$ that, when restricted to K, do not contain any irreducible K-submodules of $S^2(\mathfrak{p})$. Frobenius reciprocity [25] says that

$$\bar{\mathcal{F}}(\mathcal{H}) \subset \mathcal{F}(\mathcal{H}). \tag{1}$$

Thus, once the irreducible decomposition of $S^2(\mathcal{H})$ is known, this gives a lower bound on the dimension of the moduli $\mathcal{M}(\mathcal{H})$.

DoCarmo and Wallach carried this out for $M = S^m$, and $\mathcal{H} = V_{\lambda_p}$. Identifying the irreducible components of $\bar{\mathcal{F}}(V_{\lambda_p})$, for $m \geq 3$ and $p \geq 4$, they obtained the lower estimate

$$\dim \mathcal{M}(V_{\lambda_p}) = \dim \mathcal{F}(V_{\lambda_p}) \geq \dim \bar{\mathcal{F}}(V_{\lambda_p}) \geq \dim \bar{\mathcal{F}}(V_{\lambda_4}) \geq 18.$$

They conjectured that equality holds in (1). This has been resolved by the author in [21] using different methods. (For a recent algebraic proof, see [26].) For the lowest dimensional moduli space, $\mathcal{M}(V_{\lambda_4})$ with $m = 3$, see [23].

Once again, it is natural to ask whether equality holds in (1) in general, or at least for compact rank 1 symmetric spaces $M = G/K$. Our last result is to show that the answer is negative for $M = \mathbf{C}P^m$, and $\mathcal{H} = V_{\lambda_p}$.

Theorem. *C Let* $m \geq 3$, $(G,K) = (U(m+1), U(m) \times U(1))$, $M = U(m+1)/(U(m) \times U(1)) = \mathbf{C}P^m$, *and* $\mathcal{H} = V_{\lambda_p}$. *Then, for* $p = 3$ *and* $m \not\equiv 1 \pmod 4$, *or for* $p \geq 4$, *we have*

$$\bar{\mathcal{F}}(V_{\lambda_p}) \neq \mathcal{F}(V_{\lambda_p}).$$

The striking difference between the spherical and complex projective cases is that $S^2(V_{\lambda_p})$, $V_{\lambda_p} \subset C^\infty(S^m)$, has a multiplicity one decomposition into irreducible components, but according to the multiplicity formulas developed by Barbasch [22], this fails for $S^2(V_{\lambda_p})$, $V_{\lambda_p} \subset C^\infty(\mathbf{C}P^m)$.

2. Zonal Spherical Functions and Jacobi Polynomials

In this section we describe the main idea of the proof of Theorem A as well as assemble some preliminary facts.

Let $M = G/K$ be a compact rank 1 symmetric space. As noted above, an irreducible G-submodule $\mathcal{H} \subset C^\infty(M)$ is class 1 with respect to the pair (G,K). We call a K-fixed vector $\chi_0 \in \mathcal{H}$ a zonal spherical function [15,25]. It is well-known that a zonal spherical function is unique up to a constant multiple [2].

Let χ_0 be a zonal spherical function of $\mathcal{H}$. Its square $\chi_0^2 \in \mathcal{H}\cdot\mathcal{H}$ is also fixed by K. Since $C^\infty(M)$ has a multiplicity one decomposition into irreducible components, as an element of $C^\infty(M)$, χ_0^2 decomposes into a sum

$$\chi_0^2 = \sum_{j=1}^{n} \chi_j,$$

where each χ_j belongs to a unique irreducible component $\mathcal{H}_j \subset C^\infty(M)$. Clearly, χ_j is a zonal spherical function of $\mathcal{H}_j$. Since $\chi_j \in \mathcal{H}_j$ is a component of $\chi_0^2 \in \mathcal{H}\cdot\mathcal{H}$, by Schur's lemma, $\mathcal{H}_j$ projects nontrivially to $\mathcal{H}\cdot\mathcal{H}$, and we obtain

$$\sum_{j=1}^{n} \mathcal{H}_j \subset \mathcal{H}\cdot\mathcal{H}.$$

In Section 4 we will prove Theorem A by showing that equality holds here.

We now illustrate this in a different setting by a simple example.

Example. Let G be a compact Lie group viewed as a symmetric space $G \times G/G^*$ of compact type, where $G^* \subset G \times G$ is the diagonal [15,16,24]. (The map $(g_1, g_2)G^* \mapsto g_1 g_2^{-1}$, $g_1, g_2 \in G$, identifies $G \times G/G^*$ with G.) The space $C^\infty(G \times G/G^*, \mathbf{C})$ of complex valued smooth functions on $G \times G/G^*$ has a multiplicity one decomposition into irreducible components. A component, a complex irreducible $G \times G$-submodule of $C^\infty(G \times G/G^*, \mathbf{C})$, has the form $\mathcal{H}^* \otimes \mathcal{H}$, where $\mathcal{H}$ is a complex irreducible G-module. The G^*-fixed vectors in $\mathcal{H}^* \otimes \mathcal{H}$ can be identified with the (multiples of a normalized) character χ_0 of $\mathcal{H}$ [24]. Given χ_0, according to our procedure, we need to decompose the square χ_0^2 into a sum of (nonzero) characters

$$\chi_0^2 = \sum_{j=1}^{n} c_j \chi_j.$$

By elementary character theory, this decomposition corresponds to the decomposition of the tensor product

$$\mathcal{H} \otimes \mathcal{H} = \sum_{j=1}^{n} c_j \mathcal{H}_j$$

as a G-module, where χ_j is the character of $\mathcal{H}_j$ and $c_j \in \mathbf{N}$ is the multiplicity of $\mathcal{H}_j$ in $\mathcal{H} \otimes \mathcal{H}$. Since $\chi_0^2 \in (\mathcal{H}^* \otimes \mathcal{H}) \cdot (\mathcal{H}^* \otimes \mathcal{H})$, Schur's lemma tells us that

$$\sum_{j=1}^{n} \mathcal{H}_j^* \otimes \mathcal{H}_j \subset (\mathcal{H}^* \otimes \mathcal{H}) \cdot (\mathcal{H}^* \otimes \mathcal{H})$$

as $G \times G$-modules. We claim that equality holds here. Indeed, consider the natural extension of Ψ^0 above

$$\Psi^0 \colon (\mathcal{H}^* \otimes \mathcal{H}) \otimes (\mathcal{H}^* \otimes \mathcal{H}) \to (\mathcal{H}^* \otimes \mathcal{H}) \cdot (\mathcal{H}^* \otimes \mathcal{H})$$

given by multiplication. The domain of Ψ^0, as a $G \times G$-module, can be decomposed as

$$\begin{aligned}(\mathcal{H}^* \otimes \mathcal{H}) \otimes (\mathcal{H}^* \otimes \mathcal{H}) &= (\mathcal{H}^* \otimes \mathcal{H}^*) \otimes (\mathcal{H} \otimes \mathcal{H}) \\ &= \left(\sum_{j=1}^{n} c_j \mathcal{H}_j^*\right) \otimes \left(\sum_{l=1}^{n} c_l \mathcal{H}_l\right) = \sum_{j,l=1}^{n} c_j c_l \left(\mathcal{H}_j^* \otimes \mathcal{H}_l\right).\end{aligned}$$

Finally, by Schur's lemma again $\mathcal{H}_j^* \otimes \mathcal{H}_l$ contains a G^*-fixed vector if and only if $j = l$ [24]. The claim follows.

We now return to our compact rank 1 symmetric space $M = G/K$. As noted in Section 1, the full eigenspace V_λ of the Laplacian $\triangle^M$ corresponding to an eigenvalue λ is an irreducible G-module. Moreover, if $\{\lambda_p\}_{p=0}^{\infty}$ denotes the sequence of eigenvalues of $\triangle^M$ in increasing order, then we have

$$C^\infty(M) = \sum_{p=0}^{\infty} V_{\lambda_p}.$$

By the above, $V_{\lambda_p} \subset C^\infty(M)$ contains a zonal spherical function χ_0, unique up to a constant multiple.

We now recall that, for fixed $\alpha, \beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}$, $n \geq 0$, form an orthogonal series on $[-1, 1]$ with respect to the weight function $(1 - x)^\alpha (1 - x)^\beta$ [1]. The polynomial $P_n^{(\alpha,\beta)}$ can be defined by

$$(1 - x)^\alpha (1 - x)^\beta P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} [(1 - x)^{n+\alpha} (1 - x)^{n+\beta}].$$

With a suitable choice of parameters on M, the zonal function χ_0 of a component V_{λ_p} of $C^\infty(M)$ is a constant multiple of $P_n^{(\alpha,\beta)}$ with α, β, n depending on V_{λ_p} [4,10,24]. (For example, $n = p$ in all cases but $M = \mathbf{R}P^m$ for which $n = 2p$.) The classification of compact rank 1 symmetric spaces M, the eigenvalues of $\triangle^M$ [2], and the Jacobi polynomials corresponding to the zonal spherical harmonics χ_0 are summarized in the following table:

(G,K)	$M=G/K$	λ_p	χ_0
$(SO(m+1),SO(m))$	S^m	$p(p+m-1)$	$P_p^{(m/2-1,m/2-1)}$
$(SO(m+1),O(m))$	$\mathbf{R}P^m$	$2p(2p+m-1)$	$P_{2p}^{(m/2-1,m/2-1)}$
$(U(m+1),U(m)\times U(1))$	$\mathbf{C}P^m$	$4p(p+m)$	$P_p^{(m-1,0)}$
$(Sp(n+1),(Sp(n)\times Sp(1))$	$\mathbf{H}P^m$	$4p(p+2m+1)$	$P_p^{(2m-1,1)}$
$(F_4,Spin(9))$	$\mathbf{Ca}P^2$	$4p(p+11)$	$P_p^{(7,3)}$

The multiplicities $n(\lambda_p)=\dim V_{\lambda_p}$ are given as follows:

M	$n(\lambda_p)$
S^m	$\binom{p+m}{m}-\binom{p+m-2}{m}$
$\mathbf{R}P^m$	$\binom{2p+m}{m}-\binom{2p+m-2}{m}$
$\mathbf{C}P^m$	$\binom{p+m}{m}^2-\binom{p+m-1}{m}^2$
$\mathbf{H}P^m$	$\frac{2p+2m+1}{2m(2m+1)}\binom{p+2m}{2m-1}\binom{p+2m-1}{2m-1}$
$\mathbf{Ca}P^2$	$\frac{2p+11}{1320}\binom{p+10}{7}\binom{p+7}{7}$

To simplify the treatment and to avoid some overlapping cases, we will assume that $m\geq 2$.

Since, up to parametrization, the zonals are Jacobi polynomials, we need to obtain a decomposition of the square $(P_p^{(\alpha,\beta)})^2$ into a sum of Jacobi polynomials:

$$(P_p^{(\alpha,\beta)})^2=\sum_{j=0}^{2p}c(j,p;\alpha,\beta)P_j^{(\alpha,\beta)}. \tag{2}$$

More generally, a formula of the type

$$P_p^{(\alpha,\beta)}P_q^{(\alpha,\beta)}=\sum_{j=|p-q|}^{p+q}c(j,p,q;\alpha,\beta)P_j^{(\alpha,\beta)}. \tag{3}$$

is usually called "linearization of the product."

For $\alpha=\beta$, the Jacobi polynomial $P_p^{(\alpha,\beta)}$ is, up to normalization, the ultraspherical (or Gegenbauer) polynomial C_p^ν, where $\nu-1/2=\alpha=\beta$. (The precise formula is given in (20) below.) Linearization of the product of ultraspherical polynomials dates back to the early twentieth century, and the coefficients $c(j,p,q;\lambda-1/2,\lambda-1/2)$ have been calculated explicitly [1,7,24]. For our purposes, we need only that $c(j,p,q;\lambda-1/2,\lambda-1/2)$ is positive if and only if $|p-q|\leq j\leq p+q$ and $j\equiv p+q \pmod 2$.

For Jacobi polynomials in general linearization proved to be much more difficult and the exact decomposition formula is fairly recent [19]. A general and sharp positivity result for the coefficients $c(j,p,q;\alpha,\beta)$ (covering the remaining cases in the table above for $m\geq 2$) is due to Gasper [11,12]. It states that if $\alpha,\beta>-1$, $a=\alpha+\beta+1$, $b=\alpha-\beta$, then $c(j,p,q;\alpha,\beta)>0$ provided that (α,β) is in the interior of the set

$$V=\{(\alpha,\beta)\,|\,\alpha\geq\beta,\ a(a+5)(a+3)^2\geq(a^2-7a-24)b^2\}.$$

Note that Theorem 1 in [12] states nonnegativity of the coefficients for $(\alpha,\beta)\in V$. As Professor Gasper communicated to the author [13], a closer inspection of his proof of Theorem 1 in [12], pp. 585-591, shows strict positivity of the coefficients if (α,β) is in the interior of V. Another proof of the positivity follows by using the $\{\,\}_9F_8$ series representations for the linearization coefficients in [19] (formula (3.9)).

3. Generalities on the Moduli

Let G be a compact Lie group and $\mathcal{H}$ an orthogonal G-module. We define

$$\mathcal{K}(\mathcal{H}) = \{C \in S^2(\mathcal{H}) \,|\, C + I \geq 0\}.$$

We write $\mathcal{K} = \mathcal{K}(\mathcal{H})$ if there is no danger of confusion. $\mathcal{K}$ is a G-invariant set in $S^2(\mathcal{H})$, where the G-module structure on $S^2(\mathcal{H})$ is extended from that of $\mathcal{H}$. Since $C + I \geq 0$ is a convex condition, $\mathcal{K}$ is a convex set. The interior of $\mathcal{K}$ consists of those endomorphisms C that satisfy $C + I > 0$. It follows that $\mathcal{K}$ has a nonempty interior, and hence it is a convex body in $S^2(\mathcal{H})$. Notice that $\mathcal{K}$ is noncompact since the multiples λI, $\lambda \geq -1$, are contained in $\mathcal{K}$. We call $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the general moduli space for $\mathcal{H}$.

We let $S^2_0(\mathcal{H})$ denote the G-submodule of $S^2(\mathcal{H})$ comprised of the traceless symmetric endomorphisms of V. We define

$$\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap S^2_0(\mathcal{H}) = \{C \in S^2_0(\mathcal{H}) \,|\, C + I \geq 0\}.$$

The eigenvalues of the symmetric endomorphisms in $\mathcal{K}$ are greater or equal to -1. Hence the eigenvalues of the endomorphisms in $\mathcal{K}_0$ are contained in $[-1, \dim \mathcal{H} - 1]$. It follows that $\mathcal{K}_0$ is compact, and a convex body in $S^2_0(\mathcal{H})$. We call $\mathcal{K}_0 = \mathcal{K}_0(\mathcal{H})$ the reduced moduli space for $\mathcal{H}$.

We now give an interpretation of the moduli as parameter spaces for certain maps. We let M be a compact Riemannian manifold, and G a compact Lie group of isometries of M. (G is a closed subgroup of the full isometry group of M.) The space $C^\infty(M)$ of smooth functions on M is a representation space for G, where $g \in G$ acts on $\xi \in C^\infty(M)$ by $g \cdot \xi = \xi \circ g^{-1}$. We fix a finite dimensional G-submodule $\mathcal{H} \subset C^\infty(M)$. We endow $\mathcal{H}$ with the scaled L^2-scalar product

$$\langle \chi_1, \chi_2 \rangle = \frac{\dim \mathcal{H}}{\mathrm{vol}\,(M)} \int_M \chi_1 \chi_2 v_M, \quad \chi_1, \chi_2 \in \mathcal{H}, \tag{4}$$

where v_M is the Riemannian volume form on M, and $\mathrm{vol}\,(M) = \int_M v_M$ is the volume of M. With this scalar product $\mathcal{H}$ becomes an orthogonal G-module.

A smooth map $f\colon M \to V$ into a Euclidean vector space V is said to be full if the image of f spans V. A component of f is $\alpha \circ f \in C^\infty(M)$, where $\alpha \in V^*$. The space of components of f is defined as

$$V_f = \{\alpha \circ f \,|\, \alpha \in V^*\} \subset C^\infty(M).$$

The map f is full if and only if the linear map $f^*\colon V^* \to V_f$, given by precomposition with f, is an isomorphism. Since V is Euclidean we also have $V \cong V^* \cong V_f$. Note that any map can be made full by restricting its range to the linear span of the image.

Two maps $f_1\colon M \to V_1$ and $f_2\colon M \to V_2$ are said to be congruent if there is a linear isometry $U\colon V_1 \to V_2$ such that $f_2 = U \circ f_1$.

With $\mathcal{H}$ as above, $f\colon M \to V$ is said to be an $\mathcal{H}$-map if $V_f \subset \mathcal{H}$. Note that any smooth map $f\colon M \to V$ is an $\mathcal{H}$-map for $\mathcal{H}$ the smallest G-invariant linear subspace in $C^\infty(M)$ that contains V_f.

The Dirac delta as a map $\delta_{\mathcal{H}}\colon M \to \mathcal{H}^*$ is defined in the usual way

$$\delta_{\mathcal{H}}(x)(\chi) = \chi(x), \quad x \in M,\ \chi \in \mathcal{H}.$$

The component of $\delta_{\mathcal{H}}$ corresponding to $\chi \in \mathcal{H} = \mathcal{H}^{**}$ is $\langle \delta_{\mathcal{H}}, \chi \rangle = \chi$. Hence, $V_{\delta_{\mathcal{H}}} = \mathcal{H}$ and $\delta_{\mathcal{H}}$ is a full $\mathcal{H}$-map.

In what follows we will identify $\mathcal{H}$ with its dual $\mathcal{H}^*$ via the scalar product on $\mathcal{H}$. With respect to an orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$, $\dim \mathcal{H} = N+1$, the Dirac delta as a map $\delta_{\mathcal{H}} \colon M \to \mathcal{H}$ can be written as

$$\delta_{\mathcal{H}}(x) = \sum_{j=0}^{N} \chi^j(x)\chi^j, \quad x \in M. \tag{5}$$

Indeed, for $\chi \in \mathcal{H}$, we have

$$\langle \delta_{\mathcal{H}}(x), \chi \rangle = \chi(x) = \sum_{j=0}^{N} \langle \chi, \chi^j \rangle \chi^j(x) = \left\langle \sum_{j=0}^{N} \chi^j(x)\chi^j, \chi \right\rangle.$$

The Dirac delta $\delta_{\mathcal{H}}$ is equivariant with respect to the homomorphism $\rho_{\mathcal{H}} \colon G \to O(\mathcal{H})$ that defines the orthogonal G-module structure on $\mathcal{H} \cong \mathcal{H}^*$.

For a full $\mathcal{H}$-map $f \colon M \to V$, we have $f = A \circ \delta_{\mathcal{H}}$, where $A \colon \mathcal{H} \to V$ is a surjective linear map. We associate to f the symmetric linear endomorphism

$$\langle f \rangle = A^*A - I \in S^2(\mathcal{H}).$$

It is clear that $\langle f \rangle$ depends only on the congruence class of f. Since A^*A is always positive semidefinite, we also have $\langle f \rangle \in \mathcal{K}(\mathcal{H})$. A DoCarmo-Wallach type argument shows that $f \mapsto \langle f \rangle$ gives rise to a one-to-one correspondence between the set of congruence classes of full $\mathcal{H}$-maps and the general moduli space $\mathcal{K}(\mathcal{H})$ [6,25].

Let $f \colon M \to V$ be a full $\mathcal{H}$-map. With respect to an orthonormal basis in V, f can be written in components as $f = (f^0, \dots, f^n)$, $\dim V = n+1$. With the orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ as above, $A \colon \mathcal{H} \to V$ becomes an $(n+1) \times (N+1)$-matrix with entries a_{kj}, $k = 0, \dots, n$, $j = 0, \dots, N$. In components, $f = A \circ \delta_{\mathcal{H}}$ can be written as

$$f^k = \sum_{j=0}^{N} a_{kj}\chi^j, \quad k = 0, \dots, n.$$

We now calculate

$$\operatorname{trace}(\langle f \rangle + I) = \operatorname{trace} A^*A = \sum_{k=0}^{n} \sum_{j=0}^{N} a_{kj}^2 = \sum_{k=0}^{n} |f^k|^2.$$

We conclude that, in terms of the scaled L^2-scalar product (4) on $\mathcal{H}$, the parameter point $\langle f \rangle \in \mathcal{K}(\mathcal{H})$ is traceless if and only if

$$\int_M \sum_{k=0}^{n} (f^k)^2 v_M = \operatorname{vol}(M). \tag{6}$$

We call f normalized if (6) is satisfied. Clearly, $\delta_{\mathcal{H}}$ is normalized.

It is also clear that, by suitable scaling, any nontrivial map can be normalized.

Summarizing, we obtain the following:

Proposition 3.1. *Let M be a compact Riemannian manifold with compact group G of isometries. Given a finite dimensional G-submodule $\mathcal{H}$ of $C^\infty(M)$, the set of congruence classes of full $\mathcal{H}$-maps $f\colon M \to V$ can be parametrized by the general moduli space $\mathcal{K}(\mathcal{H})$. The reduced moduli $\mathcal{K}_0(\mathcal{H})$ parametrizes the normalized $\mathcal{H}$-maps.*

An $\mathcal{H}$-map $f\colon M \to V$ is called spherical if the image of f is contained in the unit sphere S_V of V. A finite dimensional G-module $\mathcal{H} \subset C^\infty(M)$ is called δ-spherical if $\delta_{\mathcal{H}}$ is spherical. Due to the scaling of the L^2-scalar product in (4), $\mathcal{H}$ is δ-spherical if and only if

$$\sum_{j=0}^{N} (\chi^j)^2 = 1$$

on M, where $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ is an orthonormal basis.
If $M = G/K$ is homogeneous then any $\mathcal{H} \subset C^\infty(M)$ is δ-spherical. This is because $\delta_{\mathcal{H}}$ is equivariant, and thereby its image is a G-orbit in $\mathcal{H}$ necessarily contained in $S_{\mathcal{H}}$.
Let $\mathcal{H}$ be a δ-spherical G-module. A full $\mathcal{H}$-map $f\colon M \to V$ is spherical if and only if

$$|f(x)|^2 - |\delta_{\mathcal{H}}(x)|^2 = \langle (A^*A - I)\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle = \langle \langle f\rangle, \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x)\rangle = 0,$$

for all $x \in M$. Here $\odot$ denotes the symmetric tensor product. We define

$$\mathcal{E}(\mathcal{H}) = \{\delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x) \mid x \in M\}^\perp \subset S^2(\mathcal{H}). \tag{7}$$

The previous computation shows that an $\mathcal{H}$-map $f\colon M \to V$ is spherical if and only if $\langle f\rangle \in \mathcal{E}(\mathcal{H})$.
Once again, since $\delta_{\mathcal{H}}$ is equivariant, $\mathcal{E}(\mathcal{H}) \subset S^2(\mathcal{H})$ is a G-submodule.
We obtain the following:

Proposition 3.2. *Let M be a compact Riemannian manifold with compact group G of isometries, and $\mathcal{H} \subset C^\infty(M)$ a δ-spherical G-submodule. Then the set of congruence classes of full spherical $\mathcal{H}$-maps $f\colon M \to S_V$ can be parametrized by the moduli space*

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap \mathcal{E}(\mathcal{H}).$$

Moreover $\mathcal{L}(\mathcal{H})$ is a compact convex body in $\mathcal{E}(\mathcal{H})$.

Compactness follows since spherical maps are automatically normalized:

$$\mathcal{L}(\mathcal{H}) \subset \mathcal{K}_0(\mathcal{H}) \Rightarrow \mathcal{E}(\mathcal{H}) \subset S_0^2(\mathcal{H}),$$

so that

$$\mathcal{L}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{E}(\mathcal{H}).$$

Remark. Let $M = G/K$ be a compact naturally reductive Riemannian homogeneous space, and $V_\lambda \subset C^\infty(M)$ the eigenspace of $\triangle^M$ corresponding to an eigenvalue λ. Recall from Section 1 that a λ-eigenmap $f\colon M \to S_V$ is a spherical V_λ-map.

Let $\mathcal{H} \subset C^\infty(M)$ be a finite dimensional G-submodule. Then $\mathcal{H} \subset V_\lambda$ for some λ. Proposition 3.2 asserts that $\mathcal{L}(\mathcal{H})$ parametrizes the congruence classes of full λ-eigenmaps $f\colon M \to S_V$ with components in $\mathcal{H} \subset V_\lambda$. In particular, $\mathcal{L}(V_\lambda)$ parametrizes the congruence classes of all full λ-eigenmaps $f\colon M \to S_V$.

Returning to the general situation, let $\mathcal{H} \subset C^\infty(M)$ be a δ-spherical G-module. We define

$$\Psi^0 = \Psi^0_{\mathcal{H}}\colon S^2(\mathcal{H}) \to C^\infty(M) \tag{8}$$

by

$$\Psi^0(C)(x) = \langle C\delta_{\mathcal{H}}(x), \delta_{\mathcal{H}}(x)\rangle = \langle C, \delta_{\mathcal{H}}(x) \odot \delta_{\mathcal{H}}(x)\rangle, \quad x \in M.$$

Since $\delta_{\mathcal{H}}$ is equivariant, Ψ^0 is a homomorphism of G-modules. By (7), we have

$$\ker \Psi^0 = \mathcal{E}(\mathcal{H}). \tag{9}$$

We claim that the image of Ψ^0 is the G-submodule

$$\mathcal{H} \cdot \mathcal{H} = \text{ span}\,\{\chi_1\chi_2 \,|\, \chi_1, \chi_2 \in \mathcal{H}\} \subset C^\infty(M).$$

Indeed, using (5) in the definition of Ψ^0, we obtain

$$\Psi^0(C) = \sum_{j,l=0}^{N} c_{jl}\chi^j\chi^l,$$

where $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$ is an orthonormal basis, and the c_{jl}'s are the matrix entries of $C \in S^2(\mathcal{H})$. The claim follows.

Note that $\mathcal{H} \cdot \mathcal{H}$ always contains the trivial G-module, a consequence of δ-sphericality.

We obtain the following:

Proposition 3.3. *Let $\mathcal{H} \subset C^\infty(M)$ be a δ-spherical G-module. Then the G-module homomorphism*

$$\Psi^0\colon S^2(\mathcal{H}) \to \mathcal{H} \cdot \mathcal{H}$$

is onto, and has kernel $\mathcal{E}(\mathcal{H})$. In particular, $\mathcal{H} \cdot \mathcal{H}$ is (isomorphic to) a G-submodule of $S^2(\mathcal{H})$ and we have

$$\mathcal{E}(\mathcal{H}) \cong S^2(\mathcal{H})/(\mathcal{H} \cdot \mathcal{H})$$

as G-modules.

Let $K \subset G$ be a closed subgroup. Recall that an irreducible orthogonal G-module $\mathcal{V}$ is called class 1 with respect to the pair (G, K) if $\mathcal{V}$ contains a nonzero K-fixed vector, or equivalently, if $\mathcal{V}|_K$ contains the trivial representation.

We now assume that $M = G/K$ is Riemannian homogeneous. As noted in Section 1, any irreducible G-submodule of $C^\infty(M)$ is class 1 with respect to (G, K). Conversely, any class 1 G-module $\mathcal{V}$ with respect to (G, K) is isomorphic to an irreducible G-submodule of $C^\infty(M)$ [25].

Let $\mathcal{H} \subset C^\infty(M)$ be a δ-spherical G-submodule. We define $\bar{\mathcal{E}}(\mathcal{H}) \subset S^2(\mathcal{H})$ as the sum of those irreducible G-submodules in $S^2(\mathcal{H})$ that are not class 1 with respect to (G,K). By the description of class 1 modules above and (8)-(9), we see that

$$\bar{\mathcal{E}}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}). \tag{10}$$

Equality holds if and only if the sum of all irreducible G-submodules in $S^2(\mathcal{H})$ that are class 1 with respect to (G,K), is isomorphic to $\mathcal{H}\cdot\mathcal{H}$.

A map $f\colon M \to V$ is said to be conformal if

$$\langle f_*(X), f_*(Y)\rangle = c\langle X,Y\rangle, \quad X,Y \in T(M),$$

where $c > 0$ is a constant. Then c is called the conformality constant of f. We say that a finite dimensional G-module $\mathcal{H} \subset C^\infty(M)$ is δ-conformal if $\delta_{\mathcal{H}}$ is conformal.
Using (5), we have

$$(\delta_{\mathcal{H}})_*(X) = X\delta_{\mathcal{H}} = \sum_{j=1}^{N} X(\chi^j)\chi^j, \quad X \in T(M). \tag{11}$$

Thus, $\mathcal{H}$ is δ-conformal if and only if

$$\sum_{j=0}^{N} X(\chi^j)Y(\chi^j) = c\langle X,Y\rangle, \quad X,Y \in T(M), \tag{12}$$

holds for any orthonormal basis $\{\chi^j\}_{j=0}^N \subset \mathcal{H}$.
Let $f\colon M \to V$ be a conformal map as above, and assume that $V_f \subset V_\lambda$ for some eigenvalue λ of $\triangle^M$. Then $f\colon M \to V$ is an isometric immersion with respect to c times the original metric on M. By Takahashi's theorem [20] f maps into a sphere rS_V for some r. Calculating $\triangle^M(|f|^2)$, we obtain $c = r^2\lambda/\dim M$. If f is normalized then $r=1$ and we get $c = \lambda/\dim M$. Again by Takahashi, we obtain that $f\colon M \to S_V$ is an isometric minimal immersion of the $\lambda/\dim M$-multiple of the metric on M.

Let $\mathcal{H} \subset V_\lambda$ be a δ-conformal G-submodule. By definition, $\delta_{\mathcal{H}}\colon M \to \mathcal{H}$ is conformal with $V_{\delta_{\mathcal{H}}} = \mathcal{H} \subset V_\lambda$ so that the argument above applies. Since $\delta_{\mathcal{H}}$ is automatically normalized, we obtain that $\delta_{\mathcal{H}}\colon M \to S_{\mathcal{H}}$ is an isometric minimal immersion of the $\lambda/\dim M$-multiple of the metric on M. In particular, $\mathcal{H}$ is δ-spherical.

Remark. Let $M = G/K$ be isotropy irreducible. Then any irreducible G-submodule $\mathcal{H} \subset C^\infty(M)$ is δ-conformal. Indeed, (12) holds because $\sum_{j=0}^N d\chi^j \odot d\chi^j$ is a G-invariant bilinear form on $\mathcal{H}$. Its coordinate representation in (5) shows that $\delta_{\mathcal{H}}\colon M \to S_{\mathcal{H}^*}$ is the standard minimal immersion [6,25].

A DoCarmo-Wallach type argument gives the following:

Proposition 3.4. *Let $\mathcal{H} \subset V_\lambda \subset C^\infty(M)$ be a δ-conformal G-submodule. Then the congruence classes of isometric minimal $\mathcal{H}$-immersions $f\colon M \to S_V$ (with respect to the $\lambda/\dim M$-multiple of the metric on M) are parametrized by the compact convex body*

$$\mathcal{M}(\mathcal{H}) = \mathcal{K}_0(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}), \tag{13}$$

in the G-module

$$\mathcal{F}(\mathcal{H}) = \{X\delta_{\mathcal{H}} \odot Y\delta_{\mathcal{H}} \,|\, X, Y \in T(M)\}^\perp \subset S^2(\mathcal{H}). \tag{14}$$

We also have

$$\mathcal{F}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H}), \tag{15}$$

so that

$$\mathcal{M}(\mathcal{H}) = \mathcal{L}(\mathcal{H}) \cap \mathcal{F}(\mathcal{H}).$$

Let $M = G/K$ be a naturally reductive homogeneous space with orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{g}$ and $\mathfrak{k}$ are the Lie algebras of G and K, and $\mathfrak{p}$ is identified with the tangent space $T_o(M)$, $o = \{K\}$. The subgroup K acts on its Lie algebra $\mathfrak{k}$ by the adjoint representation, and, under the identification $\mathfrak{p} \cong T_o(M)$, this action corresponds to the action of K on $T_o(M)$ via the isotropy representation.

As noted in Section 1, if $\mathcal{W}$ is any (finite dimensional) orthogonal K-module then the induced G-module $\mathrm{Ind}_K^G(\mathcal{W})$ is comprised of continuous maps $f\colon G \to \mathcal{W}$ that satisfy $f(kg) = k \cdot f(g)$, $g \in G$, $k \in K$. Precomposition of these maps by right multiplication on G defines the G-module structure on $\mathrm{Ind}_K^G(\mathcal{W})$. In addition, integration with respect to the Haar measure on G and the scalar product on $\mathcal{W}$ define a G-invariant scalar product on $\mathrm{Ind}_K^G(\mathcal{W})$.

By Frobenius reciprocity, we have

$$\mathrm{Hom}_G(\mathcal{V}, \mathrm{Ind}_K^G(\mathcal{W})) = \mathrm{Hom}_K(\mathcal{V}|_K, \mathcal{W}),$$

where $\mathcal{V}$ is an orthogonal G-module and $\mathcal{W}$ is an orthogonal K-module.

Let $\mathcal{H} \subset C^\infty(M)$ be a δ-conformal G-module. Restricting the differential of $\delta_{\mathcal{H}}$ to $\mathfrak{p} = T_o(M)$ gives a K-equivariant linear imbedding $(\delta_{\mathcal{H}})_*\colon \mathfrak{p} \to \mathcal{H}$. We identify the K-module $\mathfrak{p}$ with the image, and think of $\mathfrak{p}$ as a K-submodule of $\mathcal{H}|_K$. Notice that this can also be thought of as the inclusion $\mathfrak{p} \subset \mathcal{H}^* \cong \mathcal{H}$ given by the action of the tangent vectors at o to M on the elements of $\mathcal{H}$ by directional differentiation.

We define

$$\Psi\colon S^2(\mathcal{H}) \to \mathrm{Ind}_K^G(S^2(\mathfrak{p})) \tag{16}$$

as follows. For $C \in S^2(\mathcal{H})$, we let $\Psi(C)\colon G \to S^2(\mathfrak{p})$ be the map defined by $\Psi(C)(g) = \pi(g \cdot C)$, where $\pi\colon S^2(\mathcal{H}) \to S^2(\mathfrak{p})$ is the orthogonal projection, a homomorphism of K-modules.

We have

$$\ker \Psi = \mathcal{F}(\mathcal{H}).$$

Indeed, for $C \in S^2(\mathcal{H})$, $\Psi(C) = 0$ if and only if $\langle g \cdot C, S^2(\mathfrak{p})\rangle = 0$ for all $g \in G$, if and only if $\langle C, g \cdot S^2(\mathfrak{p})\rangle = 0$ for all $g \in G$. In view of the identification $\mathfrak{p} \subset \mathcal{H}|_K$, this holds if and only if

$$\langle C, S^2((\delta_{\mathcal{H}})_*(T_x(M)))\rangle = 0, \quad x \in M.$$

This is equivalent to $C \in \mathcal{F}(\mathcal{H})$.

4. Proofs of Theorems A-C.

Proof of Theorem A. We first let $(G, K) = (SO(m+1), S(m))$, $SO(m) = SO(m) \oplus [1] \subset SO(m+1)$, with $M = G/K = S^m$ the Euclidean m-sphere, and $\mathcal{H} = V_{\lambda_p}$. The eigenspace V_{λ_p} corresponding to $\lambda_p = p(p+m-1)$ is $\mathcal{H}^p$, the irreducible $SO(m+1)$-module of spherical harmonics of order p on S^m.

We let $\mathcal{P}^p$ denote the $SO(m+1)$-module of homogeneous polynomials on $\mathbf{R}^{m+1}$ of degree p (with the usual action $g \cdot \xi = \xi \circ g^{-1}$, $g \in SO(m+1)$, $\xi \in \mathcal{P}^p$). By homogeneity, a polynomial in $\mathcal{P}^p$ is uniquely determined by its restriction to $S^m \subset \mathbf{R}^{m+1}$.

We also think of a spherical harmonic χ of order p on S^m as a harmonic homogeneous polynomial on $\mathbf{R}^{m+1}$ of degree p. (The equivalence of these two representations is given by restriction from $\mathbf{R}^{m+1}$ to S^m, and comparison of the Euclidean and spherical Laplacians. We suppress the restriction if there is no danger of confusion.) This way $\mathcal{H}^p$ becomes an $SO(m+1)$-submodule of $\mathcal{P}^p$. We have the orthogonal decomposition

$$\mathcal{P}^p = \mathcal{H}^p \oplus \mathcal{P}^{p-2} \cdot \rho^2 = \sum_{k=0}^{[p/2]} \mathcal{H}^{p-2k} \cdot \rho^{2k}, \tag{17}$$

where $\rho(x) = |x|$, $x = (x_0, \dots, x_m) \in \mathbf{R}^{m+1}$ [15,24].

Since $\mathcal{H}^p \subset \mathcal{P}^p$, we have

$$\mathcal{H}^p \cdot \mathcal{H}^p \subset \mathcal{P}^{2p} = \sum_{j=0}^{p} \mathcal{H}^{2j}$$

as $SO(m+1)$-modules. Theorem A for $M = S^m$ states that equality holds, and this is what we need to show.

We define the harmonic projection operator as the orthogonal projection $H\colon \mathcal{P}^p \to \mathcal{H}^p$ with kernel $\ker H = \mathcal{P}^{p-2} \cdot \rho^2$ [24]. It is given explicitly by

$$H(\xi) = \xi + \sum_{j=1}^{[p/2]} \frac{(-1)^j (p-1)\dots(p-j)}{j! \lambda_{2(p-1)} \dots \lambda_{2(p-j)}} \triangle^j \xi \cdot \rho^{2j}, \quad \xi \in \mathcal{P}^p. \tag{18}$$

Since $SO(m)$ fixes x_m, a zonal spherical harmonic in $\mathcal{H}^p$ is $H(x_m^p)$. By (18), it is given by

$$\begin{aligned} H(x_m^p) &= x_m^p \\ &+ \sum_{j=1}^{[p/2]} \frac{(-1)^j (p-1)\dots(p-j) p(p-1)\dots(p-2j+1)}{j! \lambda_{2(p-1)} \dots \lambda_{2(p-j)}} x_m^{p-2j} \rho^{2j}. \end{aligned}$$

Rewriting the coefficients in terms of the Gamma function, we obtain

$$H(x_m^p) = \frac{p!}{2^p\Gamma\left(p+\frac{m-1}{2}\right)} \sum_{j=0}^{[p/2]} \frac{(-1)^j \Gamma\left(p+\frac{m-1}{2}-j\right)}{j!(p-2j)!} (2x_m)^{p-2j}\rho^{2j}.$$

Up to a normalizing factor, this is the ultraspherical polynomial C_p^ν with $\nu = (m-1)/2$ [1,24]:

$$H(x_m^p) = \frac{p!\Gamma\left(\frac{m-1}{2}\right)}{2^p\Gamma\left(p+\frac{m-1}{2}\right)} \rho^p C_p^{(m-1)/2}(\cos\theta), \tag{19}$$

where $x_m/\rho = \cos\theta$. In terms of the Jacobi polynomials, we have

$$C_p^\nu = \frac{(2\nu)_p}{(\nu+1/2)_p} P_p^{(\nu-1/2,\nu-1/2)}, \tag{20}$$

where $(a)_p = \Gamma(a+p)/\Gamma(a)$. The choice of the zonal spherical harmonic χ_0 for $M = S^m$ specified in the first table of Section 2 follows. The linearization of the product formula for ultraspherical polynomials [7] reads as

$$\begin{aligned} C_p^\nu C_q^\nu \quad = \quad & \sum_{k=0}^{\min(p,q)} \frac{(p+q+\nu-2k)}{(p+q+\nu-k)} \\ & \times \frac{(\nu)_k(\nu)_{p-k}(\nu)_{q-k}(2\nu)_{p+q-k}}{k!(p-k)!(q-k)!(\nu)_{p+q-k}} \frac{(p+q-2k)!}{(2\nu)_{p+q-2k}} C_{p+q-2k}^\nu. \end{aligned}$$

We now let $p = q$ and $\nu = (m-1)/2$. In view of (20), the linearization formula above reduces to (2) with $\alpha = \beta = m/2-1$, and we also obtain an explicit formula for the linearization coefficients. This immediately shows that $c(j,p;m/2-1,m/2-1)$ is nonzero if and only if $0 \le j \le 2p$ is even. By (19), evaluating (2) on $\cos\theta$, the Jacobi polynomials become zonal spherical harmonics. Suppressing the argument $\cos\theta$, by definition, $(P_p^{(m/2-1,m/2-1)})^2 \in \mathcal{H}^p \cdot \mathcal{H}^p$. The restriction of the orthogonal projection $\mathcal{P}^{2p} \to \mathcal{H}^{2j}$, $j = 0,\ldots,p$, to $\mathcal{H}^p \cdot \mathcal{H}^p \subset \mathcal{P}^{2p}$ maps $(P_p^{(m/2-1,m/2-1)})^2$ to a nonzero constant multiple of $P_{2j}^{(m/2-1,m/2-1)}$ since $c(2j,p;m/2-1,m/2-1)$ is nonzero. Schur's lemma implies that $\mathcal{H}^{2j}$ must be a component of $\mathcal{H}^p \cdot \mathcal{H}^p$ for $j = 0,\ldots,p$. Theorem A follows for $M = S^m$.

For $M = \mathbf{R}P^m$, the real projective m-space, the eigenspace V_{λ_p} corresponding to the p-th eigenvalue $\lambda_p = 2p(2p+m-1)$ of the Laplacian $\triangle^{\mathbf{R}P^m}$ can be identified with $\mathcal{H}^{2p}$. Theorem A follows from the spherical case above.

Next we let $(G,K) = (U(m+1), U(m)\times U(1))$ with $\mathbf{C}P^m = U(m+1)/(U(m)\times U(1))$, the complex projective m-space. Let $\mathcal{P}^{p,q}$ denote the space of complex homogeneous polynomials of bidegree (p,q) on $\mathbf{C}^{m+1}$. An element $\xi \in \mathcal{P}^{p,q}$ is a complex valued homogeneous polynomial that has degree p in the variables $z_0,\ldots,z_m \in \mathbf{C}$ and degree q in the variables $\bar{z}_0,\ldots,\bar{z}_m \in \mathbf{C}$. By homogeneity, ξ can be thought of as a function on the unit sphere $S^{2m+1} \subset \mathbf{C}^{m+1}$.

The space $\mathcal{P}^{p,p}$ is the complexification of a real $U(m+1)$-submodule, and this real submodule is also denoted by the same symbol. An element in $\mathcal{P}^{p,p}$ can be thought of as a function on $\mathbf{C}P^m$.

The decomposition in (17) gives

$$\mathcal{P}^{p,q} = \mathcal{H}^{p,q} \oplus \mathcal{P}^{p-1,q-1}\cdot\rho^2 = \sum_{k=0}^{\min(p,q)} \mathcal{H}^{p-k,q-k}\cdot\rho^{2k},$$

where $\rho = |z|$, $z = (z_0, \ldots, z_m) \in \mathbf{C}^{m+1}$, and $\mathcal{H}^{p,q}$ is the space of complex harmonic homogeneous polynomials of bidegree (p,q) on $\mathbf{C}^{m+1}$. Then $\mathcal{H}^{p,q}$ is a complex irreducible $U(m+1)$-module. For real valued polynomials we also have

$$\mathcal{P}^{p,p} = \sum_{k=0}^{p} \mathcal{H}^{p-k,p-k} \cdot \rho^{2k},$$

as real $U(m+1)$-modules. Here $\mathcal{P}^{p,p}$ is the space of real valued homogeneous polynomials of bidegree (p,q) on $\mathbf{C}^{m+1}$, and $\mathcal{H}^{j,j}$ is the eigenspace V_{λ_j} corresponding to the j-th eigenvalue $\lambda_j = 4j(j+m)$.
Since $\mathcal{H}^{p,p} \subset \mathcal{P}^{p,p}$, we have

$$\mathcal{H}^{p,p} \cdot \mathcal{H}^{p,p} \subset \mathcal{P}^{2p,2p} = \sum_{j=0}^{2p} \mathcal{H}^{j,j} \tag{21}$$

as real $U(m+1)$-modules. To prove Theorem A for $M = \mathbf{C}P^m$, it remains to show that equality holds.
Since $U(m)$ fixes z_m and the center $U(1)$ acts on $\mathcal{H}^{p,p}$ trivially, a zonal spherical harmonic in $\mathcal{H}^{p,p}$ is $H(|z_m|^{2p})$. Here the harmonic projection operator H is the restriction of the harmonic projection for the spherical case above. We have

$$H(|z_m|^{2p}) = \frac{p!(p+m-1)!}{(2p+m-1)!} \rho^{2p} P_p^{(m-1,0)}(\cos(2\theta)),$$

where $|z_m|/\rho = \cos\theta$. (See also [24], formula (5') in Chapter 11.3.2, Vol.2.) In the linearization of the square $(P_p^{(m-1,0)})^2$ all coefficients $c(j,p;m-1,0)$, $j = 0, \ldots, 2p$, are positive for $m \geq 2$. As in the spherical case it follows that $\mathcal{H}^{j,j}$, $j = 0, \ldots, 2p$, are $U(m+1)$-submodules of $\mathcal{H}^{p,p} \cdot \mathcal{H}^{p,p}$. The equality in (21) follows.
The cases of the quaternionic projective space $\mathbf{H}P^m$ and the Cayley projective plane $\mathbf{Ca}P^2$ are entirely analogous [10]. The zonal spherical functions for $\mathbf{H}P^m$ are explicitly derived in [24] (cf. formula (14) in Chapter 11.7.4, Vol. 2). Another approach for the Cayley projective plane is to determine the highest weights of the class 1 modules with respect to the pair $(F_4, Spin(9))$ and to use the Weyl dimension formula for the multiplicities.
PROOF OF THEOREM B. One of the principal results of [22] (Theorem 4.1, p. 136) gives the multiplicity m of $\mathcal{H}^{q,q}$ in $S^2(\mathcal{H}^{p,p})$ as follows:

$$m\left[\mathcal{H}^{q,q} \colon S^2(\mathcal{H}^{p,p})\right] = \frac{1}{2}\left[\min(q, 2p-q) + 1 + \frac{1+(-1)^q}{2}\right].$$

By the definition of $\bar{\mathcal{E}}(\mathcal{H}^{p,p})$, we thus have

$$S^2(\mathcal{H}^{p,p}) = \sum_{q=0}^{2p} \frac{1}{2}\left(\min(q, 2p-q) + 1 + \frac{1+(-1)^q}{2}\right) \mathcal{H}^{q,q} \oplus \bar{\mathcal{E}}(\mathcal{H}^{p,p}).$$

On the other hand, by Proposition 3.3 and Theorem A, we have

$$\mathcal{E}(V_{\lambda_p}) = S^2(V_{\lambda_p})/(V_{\lambda_p} \cdot V_{\lambda_p}) = S^2(V_{\lambda_p})/\left(\sum_{q=0}^{2p} V_{\lambda_q}\right).$$

Since $V_{\lambda_q} = \mathcal{H}^{q,q}$, combining these two formulas, Theorem B follows.
PROOF OF THEOREM C. The proof is based on comparing the multiplicities of some irreducible components of the domain and the image of Ψ in (16) . To do this, we first complexify, and consider

$$\Psi\colon S^2(\mathcal{H}^{p,p}) \otimes_{\mathbf{R}} \mathbf{C} \to \operatorname{Ind}_{U(m)\times U(1)}^{U(m+1)}(S^2(\mathbf{C}^m) \otimes_{\mathbf{R}} \mathbf{C}), \tag{22}$$

where $\mathbf{C}^m = T_o(\mathbf{C}P^m)$ with $U(m)\times U(1)$-module structure given by $U(m)$ acting on $\mathbf{C}^m$ by matrix multiplication, and the center $U(1) \subset U(m+1)$ acting trivially. For the irreducible decompositions, recall that a complex irreducible $U(m+1)$-module $\mathcal{V}$ is given by its highest weight which, with respect to the standard maximal torus in $U(m+1)$, is an $(m+1)$-tuple with integral coefficients. We write $V^\rho = V^\rho_{U(m+1)}$, where $\rho = (\rho_1, \ldots, \rho_{m+1}) \in \mathbf{Z}^{m+1}$ with

$$\rho_1 \geq \rho_2 \geq \ldots \geq \rho_{m+1}.$$

The center $U(1) \subset U(m+1)$ acts by the weight $\sum_{j=1}^{m+1} \rho_j$.
For example, we have

$$\mathcal{H}^{p,q} = V^{(p,0,\ldots,0,-q)}.$$

The branching rule for restrictions from $U(m+1)$ to $U(m)$ takes the form

$$V^\rho_{U(m+1)}|_{U(m)} = \sum_\sigma V^\sigma_{U(m)},$$

where the summation runs over all $\sigma \in \mathbf{Z}^m$ for which

$$\rho_1 \geq \sigma_1 \geq \rho_2 \geq \ldots \geq \rho_m \geq \sigma_m \geq \rho_{m+1}.$$

The decomposition of the domain in (22) into irreducible components is one of the technical results in [22] (Theorem 4.1 on p. 136). For $m \geq 3$, we have

$$\begin{aligned} S^2(\mathcal{H}^{p,p}) \otimes_{\mathbf{R}} \mathbf{C} \;\cong\; & \sum_{b=0}^{2p} \sum_{c=0}^{\min(b,2p-b)} \sum_{d=0}^{\min(b,p,e)} \frac{n_0(b,c,d) + m_0(b,c,d)}{2} \\ \times\; & V^{(b,c,0,\ldots,0,-d,d-b-c)}. \end{aligned} \tag{23}$$

Here $e = \left[\frac{b+c}{2}\right]$, and

$$n_0(b,c,d) = \min(b-c, b-d, p-c, p-d, b+c-2d, 2p-b-c) + 1.$$

$m_0(b,c,d) = 0$ for $b \not\equiv c (\bmod 2)$, and for $b \equiv c (\bmod 2)$, we have

$$m_0(b,c,d) = \begin{cases} -1 & \text{if } b, d \text{ are odd and } m \equiv 1 (\bmod 4) \\ 1 & \text{otherwise.} \end{cases}$$

We now fix a component $V^{(b,c,0,\ldots,0,-d,d-b-c)}$ in $S^2(\mathcal{H}^{p,p}) \otimes_{\mathbf{R}} \mathbf{C}$. We need to determine the multiplicity

$$m\left[V^{(b,c,0,\ldots,0,-d,d-b-c)} : \operatorname{Ind}_{U(m)\times U(1)}^{U(m+1)}(S^2(\mathbf{C}^m) \otimes_{\mathbf{R}} \mathbf{C})\right]. \tag{24}$$

First note that the multiplicity in (24) is the dimension of the module

$$\mathrm{Hom}_{U(m)}\left(V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}, S^2(\mathbf{C}^m)\otimes_{\mathbf{R}}\mathbf{C}\right). \tag{25}$$

This follows by Frobenius reciprocity along with the fact that $U(1)$ acts trivially. In particular, the multiplicity in (24) is nonzero if and only if $V^{(b,c,0,\dots,0,-d,d-b-c)}$ is disjoint from $\bar{\mathcal{F}}(\mathcal{H}^{p,p})$.
As a real $SO(2m)$-module

$$S^2(\mathbf{C}^m) = S^2(\mathbf{R}^{2m}) = \mathcal{H}^0 \oplus \mathcal{H}^2.$$

Complexifying, and restricting to $U(m) \subset SO(2m)$, we obtain

$$S^2(\mathbf{C}^m)\otimes_{\mathbf{R}}\mathbf{C} = \mathcal{H}^0|_{U(m)} \oplus \mathcal{H}^2|_{U(m)} = \mathcal{H}^{0,0} \oplus \sum_{j=0}^{2}\mathcal{H}^{2-j,j}.$$

Thus (25) can be written as

$$\mathrm{Hom}_{U(m)}\left(V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}, \mathcal{H}^{0,0} \oplus \sum_{j=0}^{2}\mathcal{H}^{2-j,j}\right).$$

The dimension of this module is equal to

$$m\left[\mathcal{H}^{0,0} \colon V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}\right] + \sum_{j=0}^{2} m\left[\mathcal{H}^{2-j,j} \colon V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}\right].$$

By the branching rule, the first multiplicity is 1 if and only if $c = d = 0$ and zero otherwise. The remaining multiplicities can be obtained similarly using the branching rule. For $0 \le j \le 2$, we obtain

$$m\left[\mathcal{H}^{2-j,j} \colon V^{(b,c,0,\dots,0,-d,d-b-c)}|_{U(m)}\right] = \begin{cases} 1 & \text{if } b \ge 2-j \ge c \text{ and } -d \ge -j \ge d-b-c \\ 0 & \text{otherwise.} \end{cases}$$

Comparing this with (23), we see that, for $m \not\equiv 1 \pmod 4$, $p = 3$, $b = 3$, $c = d = 1$, the multiplicity of the component $V^{(3,1,0,\dots,0,-1,-3)}$ is 2 in the domain of Ψ and 1 in the image of Ψ. The same holds for $p = 4$, $b = 4$, $c = d = 1$. Theorem C follows.

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References

[1] Askey, R., "Orthogonal Polynomials and Special Functions," CBMS-NSF Reg. Conf. Ser. in Applied Math. **21** SIAM, 1975.

[2] Besse, A., "Manifolds all of whose Geodesics are Closed," Springer-Verlag, Berlin etc., 1978.

[3] Calabi, E., *Minimal immersions of surfaces in euclidean spheres*, J. Diff. Geom. **1** (1967), 111–125.

[4] Cartan, É., *Sur la détermination d'une système orthogonal complet dans un espace de Riemann symétrique clos*, Rend. Circ. Mat. Palermo **53** (1929), 217–252.

[5] Chavel, I., "Eigenvalues in Riemannian Geometry," Academic Press, New York etc. 1984.

[6] DoCarmo, M., and N. Wallach, *Minimal immersions of spheres into spheres*, Ann. of Math. **93** (1971), 43–62.

[7] Dougall, J., *A theorem of Sonine in Bessel functions, with two extensions to spherical harmonics*, Proc. Edinburgh Math. Soc. **37** (1919), 33–47.

[8] Eells, J., and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.

[9] Escher, Ch., and G. Weingart, *Orbits of $SU(2)$-representations and minimal isometric immersions*, Math. Ann. **316** (2000), 743–769.

[10] Gangolli, R., *Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévi's Brownian motion on several parameters*, Ann. Inst. H. Poincaré, Sect. B **3** (1967), 121–226.

[11] Gasper, G., *Linearization of the product of Jacobi polynomials I*, Can. J. Math. **22** (1970), 171–175.

[12] Gasper, G., *Linearization of the product of Jacobi polynomials II*, Can. J. Math. **22** (1970), 582–593.

[13] Gasper, G., private communication.

[14] Gauchman, H., and G. Toth, *Fine structure of the space of spherical minimal immersions*, Trans. Amer. Math. Soc. **348** (1996), 2441–2463.

[15] Helgason, S., "Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators, and Spherical Functions," Mathematical Surveys and Monographs **83** American Mathematical Society, Providence, 2000.

[16] Helgason, S., "Geometric Analysis on Symmetric Spaces," Mathematical Surveys and Monographs **39** American Mathematical Society, Providence, 1994.

[17] Helgason, S., *The Radon transform on Euclidean spaces, compact two-point homogeneous spaces and Grassmann manifolds*, Acta Math. **113** (1965), 153–180.

[18] Mashimo, K., *Degree of a standard isometric minimal immersions of the symmetric spaces of rank one into spheres*, Tsukuba J. Math. Vol. **5** (1981), 291–297.

[19] Rahman, M., *A non-negative representation of the linearization coefficients of the product of Jacobi polynomials*, Can. J. Math. **33** (1981), 915–928.

[20] Takahashi, T., *Minimal immersions of Riemannian manifolds*, J. Math. Soc. Japan **18** (1966), 380–385.

[21] Toth, G., *Eigenmaps and the space of minimal immersions between spheres*, Indiana Univ. Math. J. **46** (1997), 637–658.

[22] Toth, G., "Harmonic Maps and Minimal Immersions through Representation Theory," Academic Press, New York etc., 1990.

[23] Toth, G., and W. Ziller, *Spherical minimal immersions of the 3-sphere*, Comment. Math. Helv. **74** (1999), 1–34.

[24] Vilenkin, N.I., and A. U. Klimyk, "Representation of Lie Groups and Special Functions," Vols. 1-3, Kluwer, Dordrecht, 1991.

[25] Wallach, N., *Minimal immersions of symmetric spaces into spheres*, in "Symmetric Spaces", Marcel Dekker, New York (1972), 1–40.

[26] Weingart, G., "Geometrie der Modulräume minimaler isometrischer Immersionen der Sphären in Sphären," Bonner Mathematische Schriften **314**, Bonn, 1999.

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