Journal of Lie Theory
Volume **15** (2005) 379–391

Naturally graded p-filiform Lie algebras in arbitrary finite dimension

J. M. Cabezas and E. Pastor*

Communicated by K. H. Hofmann

Abstract. The present paper offers the classification of naturally graded p-filiform Lie algebras in arbitrary finite dimension n. For sufficiently high n, ($n \geq \max\{3p-1, p+8\}$), and for all admissible value of p the results are a generalization of Vergne's in case of filiform Lie algebras [11].
Mathematics subject classification 2000: 22E60, 17B30, 17B70
Keywords and Phrases: nilpotent Lie algebra, filiform, naturally graded

1. Introduction

A *Lie algebra* $(\mathfrak{g}, \mu)$ is a vector space $\mathfrak{g}$ over a field $\mathbf{K}$, with a bilinear mapping $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted $(X, Y) \to \mu(X, Y) = [X, Y]$ and called *bracket product*, verifying

$$[X, X] = 0,$$
$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all the elements X, Y, Z of $\mathfrak{g}$. The second condition is called *Jacobi Identity* and denoted by $\mathrm{Jac}(X, Y, Z)$.

The *dimension* of the Lie algebra is the dimension of the vector space $\mathfrak{g}$. In this paper, all Lie algebras $\mathfrak{g}$ will be complex and with finite dimension. By taking a basis $(X_0, X_1, \ldots, X_{n-1})$ in $\mathfrak{g}$, the algebra is completely determined by its *structure constants*, that is, for the set of complex constants $\{C_{ij}^k\}$, defined by $[X_i, X_j] = \sum_{k=0}^{n-1} C_{ij}^k X_k$. Then, we can identify the algebra $\mathfrak{g}$ and its law μ. Thus, the set $\mathcal{L}_n$ of laws of Lie algebras is an affine algebraic set defined by the polynomials expressions

$$C_{ij}^k = -C_{ji}^k \tag{1}$$

$$\sum_{l=0}^{n-1} (C_{ij}^l C_{kl}^s + C_{jk}^l C_{il}^s + C_{ki}^l C_{jl}^s) = 0 \tag{2}$$

* Partially supported by the DGICYT, BFM2000-1047 of the Ministerio de Ciencia y Tecnología of Spain and the PAICYT, FQM143, of Junta de Andalucía (Spain) and the research project 1/UPV 00066.163-E-15322/2003 of the Univ. País Vasco (Spain).

ISSN 0949–5932 / $2.50 © Heldermann Verlag

(In this paper, we indicate by $X_i \notin [X_j, X_k]$ that $C^i_{jk} = 0$)

One could think of a programme of classifying all Lie algebras by considering the above equations to be solved for unknown structure constants. This turns out to be a very complicated problem because of the non-linearity of (2). In fact, the general classification is an open problem.

The Levi's theorem decomposes the classification of Lie algebras into the classification of semisimple and solvable Lie algebras. The classification of the former is well-known from the works of Killing and Cartan in 1914 and the classification of solvable ones is actually reduced, module the study of derivations, to that of the nilpotent Lie algebras. We only know the classification up to dimension 7 [8].

The geometric approach to study nilpotent Lie algebras is one of the main methods which has been developed over the last few years. The set $\mathcal{N}_n$ of nilpotent algebra laws is an affine algebraic variety; two Lie algebras are isomorphic if and only if they belong to the same orbit of naturally acting general linear group. In this approach the notion of filiform Lie algebra appears in a natural way, the subset $\mathcal{F}_n$ of filiform laws is an open set in $\mathcal{N}_n$. We only know the classification of filiform Lie algebras up to dimension 11 [6].

The family of p-filiform Lie algebras is a large family of Lie algebras, comprising the filiform ones as a particular class. A nilpotent Lie algebra $\mathfrak{g}$, of dimension n, is called *p-filiform* if its characteristic sequence is $(n-p, 1, \ldots, 1)$. It follows that every p-filiform Lie algebra, of dimension n, has nilindex $n-p$ (the converse is not true), and that $1 \leq p \leq n-1$. Indeed, the filiform and quasifiliform Lie algebras are the p-filiform ones with $p = 1, 2$ respectively. If $p = n-1$ the situation is trivial because the family is just reduced to the abelian algebra in the appropriate dimension.

Filiform Lie algebras have maximal nilindex among the nilpotent Lie algebras having the same dimension. The p-filiform algebras play a similar role to that of the filiform Lie algebras in each dimension when the nilindex is smaller than the maximal one. We know the complete classification of p-filiform Lie algebras up to dimension 8 [5], and also know the classification for $p = n-2, n-3, n-4$ and (partially) $n-5$ for arbitrary finite dimension n [4], [1], [3]. However, the classification of p-filiform Lie algebras of dimension n, with $p = 1$ or near to 1, seems to be very difficult. As an application we have also studied some cohomological properties of certain p-filiform Lie algebras in which the determination of the algebra of derivations of each p-filiform Lie algebra is essential [2].

In the cohomological study of the variety of laws of nilpotent Lie algebras established by Vergne [11], the classification of a class of graded filiform Lie algebras plays a crucial role. The classification obtained by Vergne allows an easy description of filiform Lie algebras. The graduation considered by Vergne is provided in a natural way from the descending central sequence of any nilpotent Lie algebra. Accordingly, the algebras obtained in this way are called naturally graded Lie algebras.

If $\mathfrak{g}$ is a nilpotent Lie algebra of dimension n, it is naturally filtered by the descending central sequence $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^i(\mathfrak{g}) = [\mathfrak{g}, \mathcal{C}^{i-1}(\mathfrak{g})]$, $i \geq 1$. There exists a graded Lie algebra $\operatorname{gr}\mathfrak{g}$ associated with $\mathfrak{g}$ and with this filtration. In general $\mathfrak{g} \not\simeq \operatorname{gr}\mathfrak{g}$.

Definition 1.1. Let $\mathfrak{g}$ be a nilpotent Lie algebra. It is said that $\mathfrak{g}$ is *naturally graded* when $\mathfrak{g} \simeq \operatorname{gr} \mathfrak{g}$.

Vergne has obtained the classification in the case of 1-filiform Lie algebras in arbitrary dimension. She proves that, up to isomorphisms, there is only one naturally graded filiform Lie algebra for each odd dimension (denoted by $\mathcal{L}_n$) and two of them for each even dimension (denoted by $\mathcal{L}_n$ and $\mathcal{Q}_n$).
$\mathcal{L}_n$ is the Lie algebra defined in the basis $\{X_0, X_1, \ldots, X_{n-1}\}$ by

$$\left\{ \ [X_0, X_i] = X_{i+1} \qquad 1 \le i \le n-2. \right.$$

$\mathcal{Q}_n$ is the Lie algebra defined in the basis $\{X_0, X_1, \ldots, X_{n-1}\}$, with $n = 2q$, by

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le n-2, \\ [X_i, X_{n-1-i}] = (-1)^{i-1} X_{n-1} & 1 \le i \le q-1. \end{cases}$$

This fact also allows other authors to deal with different aspects of the theory. For example, using such graded filiform Lie algebras, in [9] Goze and Khakimdjanov give the geometric description of the characteristically nilpotent filiform Lie algebras.

Thus, it is clear that knowing the graded algebras of a certain class of nilpotent algebras provides valuable information towards our knowledge of the structure of such a class. They can later facilitate the study of several problems that can appear within the whole of the class. For example, in order to classify the families of nilpotent Lie algebras, the structure of the law of each algebra is partially determined by the law of the naturally graded Lie algebra associated with it.

The case $p = 2$ was clarified by Gómez and Jiménez-Merchán in [7]. For each dimension $n \ge 10$, we have one terminal algebra and one or two families of non-split algebras, depending on whether $n-2$ is even or odd; these families generalise the types $\mathcal{L}_n$ and $\mathcal{Q}_n$), and are denoted by
$\mathcal{L}(n, r)$ (r odd, $3 \le r \le 2\left\lfloor \frac{n-1}{2} \right\rfloor - 1$) and $\mathcal{Q}(n, r)$ (n odd; r odd, $3 \le r \le n-4$). However, for dimensions 7 and 9 there are one and two algebras, respectively.

The situation for 3-filiform Lie algebras [10] is a generalization of filiform and quasifiliform cases. For $n \ge 11$, there is one terminal family of algebras depending on one parameter. Besides the previous ones, there are one or two families that depend on two parameters when $n-3$ is even or odd, respectively, and denoted by $\mathcal{L}(n, r_1, r_2)$ (r_1, r_2 odd, $3 \le r_1 < r_2 \le n-3$) and $\mathcal{Q}(n, r_1, r_2)$ (n even, r_1, r_2 odd, $3 \le r_1 < r_2 \le n-5$). When $n = 5$, the Heisenberg algebra $\mathcal{H}_2$ appears. When $n = 8$ and $n = 9$ one more algebra emerges in each dimension. When $n = 10$, one has three algebras and one infinite family depending on one parameter.

In this paper, we obtain the complete classification of naturally graded p-filiform Lie algebras in arbitrary finite dimension for all admissible values of p. The authors of this paper would like to thank professor J.R. Gómez for his valuable help.

2. Naturally graded p-filiform Lie algebras structure

In this section we will obtain a first approximation to the structure of naturally graded p-filiform Lie algebras.

Generalities.

Let $\mathfrak{g}$ be a naturally graded p-filiform Lie algebra and let

$$\{X_0,\ X_1,\dots,X_{n-p},\ Y_1,\ Y_2,\dots,Y_{p-1}\}$$

be an adapted basis of $\mathfrak{g}$, (that is, $X_0 \in \mathfrak{g} - [\mathfrak{g},\mathfrak{g}]$;
$[X_0, X_i] = X_{i+1}$ for $1 \le i \le n-p-1$; $[X_0, X_{n-p}] = [X_0, Y_j] = 0,\ 1 \le j \le p-1$).
That implies

$$\mathcal{C}^i(\mathfrak{g}) \supset \langle X_{i+1}, X_{i+2}, \dots, X_{n-p}\rangle,\ 1 \le i \le n-p-1.$$

Lemma 2.1. *Let $\{X_0,\ X_1,\dots,X_{n-p-1},\ Y_1,\ Y_2,\dots,Y_{p-1}\}$ be an adapted basis of the p-filiform Lie algebra $\mathfrak{g}$ of dimension n. Then,*

$$X_1 \notin \mathcal{C}^1(\mathfrak{g}) \textit{ and } Y_j \notin \mathcal{C}^{n-p}(\mathfrak{g}),\ 1 \le j \le p-1$$

Proof. Obviously, $Y_j \notin \mathcal{C}^{n-p}(\mathfrak{g}),\ 1 \le j \le p-1$, otherwise $\mathfrak{g}$ would not be a p-filiform Lie algebra. $\mathrm{Jac}(X_0, Y_i, Y_j),\ 1 \le i < j \le p-1$, implies that $X_1 \notin [Y_i, Y_j]$.

$X_1 \notin [X_i, Y_j],\ 1 \le i \le n-p,\ 1 \le j \le p-1$.

Nilpotency implies straightforwardly that $X_1 \notin [X_1, Y_j]$. Let $i_j,\ 2 \le j \le n-p$, be the first value verifying $X_1 \in [X_{i_j}, Y_j]$, for each Y_j. Then, $\mathrm{Jac}(X_0, X_{i_j-1}, Y_j)$ implies a contradiction.

$X_1 \notin [X_i, X_j],\ 1 \le i < j \le n-p$.

From the nilpotency we deduce that $X_1 \notin [X_1, X_j],\ 2 \le j \le n-p$. If $X_1 \in [X_i, X_{n-p}],\ 2 \le i \le n-p$, then $\mathrm{Jac}(X_0, X_{i-1}, X_{n-p})$ implies that $X_1 \in Im(adX_0)$, and this is impossible. Let $[X_r, X_s]$ be the first bracket such that $X_1 \in [X_r, X_s],\ 2 \le r < s$. Then, $\mathrm{Jac}(X_0, X_{r-1}, X_s)$ implies a contradiction. ■

We immediately obtain the following corollary.

Corollary 2.2. *Let $\mathfrak{g}$ be a naturally graded p-filiform Lie algebra of dimension n, $\mathrm{gr}\,\mathfrak{g} = \bigoplus_{i\in\mathbf{Z}} \mathfrak{g}_i$ and let $\{X_0,\ X_1,\dots,X_{n-p},\ Y_1,\ Y_2,\dots,Y_{p-1}\}$ be an adapted basis of $\mathfrak{g}$. Then,*

$$\begin{array}{lcll} \mathfrak{g}_1 & \supset & \langle X_0, X_1\rangle, & \\ \mathfrak{g}_i & \supset & \langle X_i\rangle, & \textit{if}\quad 2 \le i \le n-p, \\ \mathfrak{g}_i & = & \{0\}, & \textit{if}\quad i \le 0 \quad \textit{or} \quad i \ge n-p+1. \end{array}$$

We have proved that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_{n-p}$ with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, for $i+j \le n-p$, satisfying $2 \le \dim(\mathfrak{g}_1) \le p+1$ and $1 \le \dim(\mathfrak{g}_i) \le p,\ 2 \le i \le n-p$. We will denote each case as follows

$$(n, r_1, r_2, \dots, r_{p-1}),\ 1 \le r_1 \le r_2 \le \dots \le r_{p-1} \le n-p,$$

to highlight the integers $r_j,\ 1 \le j \le p-1$, which indicate the positions of the coordinates of $(\dim(\mathfrak{g}_1), \dim(\mathfrak{g}_2), \dots, \dim(\mathfrak{g}_{n-p}))$ different from those of $(2, 1, \dots, 1)$; that is, $Y_j \in \mathfrak{g}_{r_j},\ 1 \le j \le p-1$. Analogously, $\mu(n, r_1, r_2, \dots, r_{p-1})$ represents the family of laws.

Conditions for r_j, $1 \le j \le p-1$.

We prove that $r_j,\ 1 \le j \le p-1$, are pairwise, not equal and all of them odd. We also study the non-admissible cases and the cases in which split algebras are obtained.

Proposition 2.3. *Let $\mathfrak{g}$ be a naturally graded p-filiform Lie algebra of law $\mu(n, r_1, r_2, \dots, r_{p-1})$. Then, $1 \le r_1 < r_2 < \dots < r_{p-1} \le n-p$.*

Proof. The proof involves many cases, namely, the following:
a) $(n,1,1,\dots,1)$,

b) $(n,r,r,\dots,r)$, $2\le r\le n-p$,

c) $(n,r_1,r_1,\dots,r_1,r_{s+1},r_{s+2},\dots,r_{p-1})$, $1\le r_1=r_2=\dots=r_s<r_{s+1}\le r_{s+2}\le \dots\le r_{p-1}\le n-p$,

c1) $(n,1,1,\dots,1,2,r_{s+2},\dots,r_{p-1})$, $2\le r_{s+2}\le r_{s+3}\le\dots\le r_{p-1}\le n-p$,

c2) $(n,1,1,\dots,1,r_{s+1},r_{s+2},\dots,r_{p-1})$, $3\le r_{s+1}\le r_{s+2}\le\dots\le r_{p-1}\le n-p$,

c3) $(n,r,r,\dots,r,r_{s+1},r_{s+2},\dots,r_{p-1})$, $2\le r<r_{s+1}\le r_{s+2}\le\dots\le r_{p-1}\le n-p$,

d) $(n,r_1,r_2,\dots,r_s,r_s,\dots,r_s=r_t,r_{t+1},r_{t+2},\dots,r_{p-1})$,
$1\le r_1<r_2<\dots<r_s=r_{s+1}=\dots=r_t<r_{t+1}\le r_{t+2}\le\dots\le r_{p-1}\le n-p$,

d1) $r_s=r_{s+1}=\dots=r_t=2$, d2) $r_s=r_{s+1}=\dots=r_t\ge 3$,

e) $(n,r_1,r_2,\dots,r_{s-1},r_s,r_s,\dots,r_s)$,
$1\le r_1<r_2<\dots<r_{s-1}<r_s=r_{s+1}=\dots=r_{p-1}\le n-p$,

e1) $(n,1,2,2,\dots,2)$, e2) $(n,r_1,r_2,\dots,r_{s-1},r_s,r_s,\dots,r_s)$, $r_s\ge 3$ In order to obtain the proof we use the Jacobi identities and appropriate arguments about p-filiformity along with a general change of basis.

Cases a) and b) show that all $r_j,\ 1\le j\le p-1$, are not equally pairwise.
Case c) shows that several r_j are not equal at the beginning of the sequence.
Case d) shows that several r_j are not equal in the middle.
Case e) shows that several r_j are not equal in the end of the sequence. ■

Proposition 2.4. *Let $\mathfrak{g}$ be a naturally graded p-filiform Lie algebra of law $\mu(n,r_1,r_2,\dots,r_{p-1})$. Then, $3\le r_1<r_2<\dots<r_{p-1}\le n-p$ such that all $r_j,\ 1\le j\le p-1$, are odd.*

Proof. From the Jacobi identities and several changes of basis it follows that if $r_1=1$ then $\mathfrak{g}$ is a split algebra. From the Jacobi identities it also follows that $r_1=2$ is not possible (that implies $Y_1\notin\mathcal{C}^1(\mathfrak{g})\Rightarrow r_1=1$).

By using finite induction for j, we obtain that all $r_j,\ 1\le j\le p-1$, are odd. ■

Structure theorem.

In this section we obtain the general structure of laws of naturally graded p-filiform Lie algebras in arbitrary finite dimension.

Theorem 2.5. (The structure of naturally graded p-filiform Lie algebras) *Any naturally graded non-split p-filiform Lie algebra of dimension n, $n\ge 3p-1$, is isomorphic to one which its law can be expressed, in an adapted basis*

$$\{X_0,\ X_1,\dots,X_{n-p},\ Y_1,\ Y_2,\dots,Y_{p-1}\}$$

by

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le n-p-1 \\ [X_i, X_j] = a_{ij} X_{i+j} & 1 \le i < j \le n-p-i, \ i+j \ne r_k, \\ & \ 1 \le k \le p-1 \\ [X_i, X_{r_j-i}] = a_{i,r_j-i} X_{r_j} + (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2}, \ 1 \le j \le p-1 \\ [X_i, Y_j] = b_j X_{i+r_j} & 1 \le i \le n-p-r_j, \ 1 \le j \le p-1 \\ [Y_i, Y_j] = c_{ij} X_{n-p} & \text{if } r_i + r_j = n-p, \\ & \ 1 \le i < j \le n-p-r_i \end{cases}$$

satisfying that $3 \le r_1 < r_2 < \ldots < r_{p-1} \le n-p, \ r_j$ *odd for all* $1 \le j \le p-1$.

Proof. In this case the general expression of the family of laws is

$$\begin{cases} [X_0, X_i] = X_{i+1} & 1 \le i \le n-p-1 \\ [X_i, X_j] = a_{ij} X_{i+j} & 1 \le i < j \le n-p-i, \ i+j \ne r_k, \\ & \ 1 \le k \le p-1 \\ [X_i, X_{r_j-i}] = a_{i,r_j-i} X_{r_j} + \alpha^j_{i,r_j-i} Y_j & 1 \le i \le \frac{r_j-1}{2}, \ 1 \le j \le p-1 \\ [X_i, Y_j] = b_{ij} X_{i+r_j} & 1 \le i \le n-p-r_j, \ i+r_j \ne r_k, \\ & \ 1 \le j,k \le p-1 \\ [X_{r_k-r_j}, Y_j] = b_{r_k-r_j,j} X_{r_k} + \beta^j_{r_k-r_j,j} Y_j & 1 \le j,k \le p-1, \ r_k > r_j \\ [Y_i, Y_j] = c_{ij} X_{r_i+r_j} & \text{if } r_i + r_j \le n-p, \\ & \ 1 \le i < j \le n-p-r_i \end{cases}$$

(The undefined brackets, except antisymmetry, are null).

From $\mathrm{Jac}(X_0, X_{r_k-r_j-1}, Y_j)$, $1 \le j,k \le p-1$, $r_k > r_j$, it follows that $\beta^j_{r_k-r_j,j} = 0$ and $b_{r_k-r_j,j} = b_{r_k-r_j-1,j}$.

From $\mathrm{Jac}(X_0, X_i, Y_j)$, $1 \le i \le n-p-r_j-1$, $i \ne r_k - r_j - 1$, $1 \le j \le k-1$, $2 \le k \le p-1$, we obtain that $b_{ij} = b_j$, $1 \le i \le n-p-r_j$ (actually, the case $i = r_k - r_j - 1$ has to be studied separately).

From $\mathrm{Jac}(X_0, X_i, X_{r_j-1-i})$, $1 \le i < \frac{r_j-3}{2}, 1 \le j \le p-1$, it follows that

$$\alpha^j_{i,r_j-i} = (-1)^{i-1} \alpha_j, \ 1 \le i \le \frac{r_j-1}{2}$$

Obviously, $\alpha_j \ne 0$, $1 \le j \le p-1$. If this is not so, that implies $Y_j \notin \mathcal{C}^1(\mathfrak{g}) \Rightarrow r_j = 1$, which is a contradiction.

From $\mathrm{Jac}(X_0, Y_i, Y_j)$, $1 \le i < j \le n-p-r_i$, it follows that $c_{ij} = 0$ if $r_i + r_j \ne n-p$.

By applying the change of basis defined by

$$\begin{cases} X'_i = X_i & 0 \le i \le n-p \\ Y'_j = \alpha_j Y_j & 1 \le j \le p-1 \end{cases}$$

we can suppose that $\alpha_j = 1$, $1 \le j \le p-1$. ∎

3. Classification (General case)

Firstly, we present some examples of naturally graded p-filiform Lie algebras. We finish this section with the maint classification theorem.

3.1. Examples. All the examples are expressed by their law which refer to an adapted basis $\{X_0, X_1, \ldots, X_{n-p}, Y_1, Y_2, \ldots, Y_{p-1}\}$.

Further on, we will prove that almost all non-split naturally graded p-filiform Lie algebras are the only ones listed below.

$\mathcal{L}(n, r_1, r_2, \ldots, r_{p-1})$

$(r_j \text{ odd}, 1 \leq j \leq p-1,\ 3 \leq r_1 < r_2 < \ldots < r_{p-1} \leq n-p)$

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \leq i \leq n-p-1 & \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2}, & 1 \leq j \leq p-1 \end{cases}$$

$\mathcal{Q}(n, r_1, r_2, \ldots, r_{p-1})$

$(r_j \text{ odd}, 1 \leq j \leq p-1,\ 3 \leq r_1 < r_2 < \ldots < r_{p-1} \leq n-p-2,\ n-p \text{ odd})$

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \leq i \leq n-p-1 & \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2}, & 1 \leq j \leq p-1 \\ [X_i, X_{n-p-i}] &= (-1)^{i-1} X_{n-p} & 1 \leq i \leq \frac{n-p-1}{2} & \end{cases}$$

$\tau(n, r_1, r_2, \ldots, r_{p-2}, n-p-1)$

$(r_j \text{ odd}, 1 \leq j \leq p-2,\ 3 \leq r_1 < r_2 < \ldots < r_{p-2} \leq n-p-3,\ n-p \text{ even})$

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \leq i \leq n-p-1 \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2},\ 1 \leq j \leq p-2 \\ [X_i, X_{n-p-1-i}] &= (-1)^{i-1}(X_{n-p-1} + Y_{p-1}) & 1 \leq i \leq \frac{n-p-2}{2} \\ [X_i, X_{n-p-i}] &= (-1)^{i-1} \frac{(n-p-2i)}{2} X_{n-p} & 1 \leq i \leq \frac{n-p-2}{2} \\ [X_1, Y_{p-1}] &= \frac{(p+2-n)}{2} X_{n-p} & \end{cases}$$

$\tau(n, r_1, r_2, \ldots, r_{p-2}, n-p-2)$

$(r_j \text{ odd}, 1 \leq j \leq p-2,\ 3 \leq r_1 < r_2 < \ldots < r_{p-2} \leq n-p-4,\ n-p \text{ odd})$

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \leq i \leq n-p-1 \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2},\ 1 \leq j \leq p-2 \\ [X_i, X_{n-p-2-i}] &= (-1)^{i-1}(X_{n-p-2} + Y_{p-1}) & 1 \leq i \leq \frac{n-p-3}{2} \\ [X_i, X_{n-p-1-i}] &= (-1)^{i-1} \frac{(n-p-1-2i)}{2} X_{n-p-1} & 1 \leq i \leq \frac{n-p-3}{2} \\ [X_i, X_{n-p-i}] &= (-1)^{i}(i-1) \frac{(n-p-1-i)}{2} X_{n-p} & 2 \leq i \leq \frac{n-p-1}{2} \\ [X_i, Y_{p-1}] &= \frac{(p+3-n)}{2} X_{n-p-2+i} & 1 \leq i \leq 2 \end{cases}$$

Lemma 3.1. *The algebras* $\mathcal{L}(n, r_1, \ldots, r_{p-1})$, $\mathcal{Q}(n, r_1, \ldots, r_{p-1})$, $\tau(n, r_1, \ldots, r_{p-2}, n-p-1)$, $\tau(n, r_1, \ldots, r_{p-2}, n-p-2)$ *are pairwise non-isomorphic.*

Proof. We can see how these algebras are pairwise non-isomorphic by considering the subalgebra $\mathcal{D}^2(\mathfrak{g}) = [\mathcal{D}^1(\mathfrak{g}), \mathcal{D}^1(\mathfrak{g})]$, $\mathcal{D}^1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$. ∎

3.2. Main Theorem.

In this section we obtain the complete classification of naturally graded non-split p-filiform Lie algebras for the general case (n must be high: $n \geq \max\{3p-1, p+8\}$).

Theorem 3.2. *Let $\mathfrak{g}$ be a non-split $n-$ dimensional naturally graded p-filiform Lie algebra, with $p > 1$ and $n \geq \max\{3p-1, p+8\}$. Let $\mu(n, r_1, r_2, \ldots, r_{p-1})$ be the law of $\mathfrak{g}$ ($3 \leq r_1 < r_2 < \ldots < r_{p-1} \leq n-p$, and all r_j, $1 \leq j \leq p-1$, are odd) . Then,*

a) If $r_{p-1} = n-p$, then $\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{p-2}, n-p)$

b) If $r_{p-1} = n-p-1$, then
$\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{p-2}, n-p-1)$ *or* $\mathfrak{g} \simeq \tau(n, r_1, r_2, \ldots, r_{p-2}, n-p-1)$

c) If $r_{p-1} = n-p-2$, then
$\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{p-2}, n-p-2)$ *or* $\mathfrak{g} \simeq \mathcal{Q}(n, r_1, r_2, \ldots, r_{p-2}, n-p-2)$
or $\mathfrak{g} \simeq \tau(n, r_1, r_2, \ldots, r_{p-2}, n-p-2)$

d) If $2p-1 \leq r_{p-1} \leq n-p-3$, then
$n-p$ *odd* $\Rightarrow$ $\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{p-1})$ *or* $\mathfrak{g} \simeq \mathcal{Q}(n, r_1, r_2, \ldots, r_{p-1})$
$n-p$ *even* $\Rightarrow$ $\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{p-1})$

Proof. We prove that by using finite induction for p.
When $p = 2$ and $p = 3$, the theorem is proved in [7], [10].
We suppose that the results are true for $p = k$ and we will prove them for $p = k+1$.

In order to prove that the results are true for $p = k+1$ it is necessary to distinguish the following cases: I) $r_{p-1} = n-p$, II) $r_{p-1} = n-p-1$, III) $r_{p-1} = n-p-2$, IV) $r_{p-1} = n-p-3$, V) $2p-1 \leq r_{p-1} \leq n-p-4$.

$\boxed{\textbf{Case I: } r_{p-1} = n-p}$

$p = k+1$ implies that $r_k = n-k-1$.

Let $\mathfrak{g}$ be a naturally graded $(k+1)$-filiform Lie algebra, $dim(\mathfrak{g}) = n$, $n \geq \max\{3k+2, k+9\}$, which law is $\mu(n, r_1, r_2, \ldots, r_{k-2}, r_{k-1}, n-k-1)$ satisfying $3 \leq r_1 < r_2 < \ldots < r_{k-1} < n-k-2 < r_k = n-k-1$ and all r_j, $1 \leq j \leq k$, are odd.

The natural graduation associated to $\mathfrak{g}$ is

$$
\begin{array}{lcll}
\mathfrak{g}_1 & = & \langle X_0, X_1 \rangle & \\
\mathfrak{g}_i & = & \langle X_i \rangle & 2 \leq i \leq n-k-2,\ i \neq r_j,\ 1 \leq j \leq k \\
\mathfrak{g}_{r_j} & = & \langle X_{r_j}, Y_j \rangle & 1 \leq j \leq k-1 \\
\mathfrak{g}_{r_k} & = & \langle X_{n-k-1}, Y_k \rangle = \mathcal{C}^{n-k-2}(\mathfrak{g}) &
\end{array}
$$

The quotient algebra $\mathfrak{g}' = \mathfrak{g}/\mathcal{C}^{n-k-2}(\mathfrak{g})$ is a naturally graded k-filiform Lie algebra of dimension $n-2$ and law $\mu(n-2, r_1, r_2, \ldots, r_{k-2}, r_{k-1})$ satisfying $3 \leq r_1 < r_2 < \ldots < r_{k-2} < r_{k-1} \leq n-k-3$.

The law of $\mathfrak{g}'$ is known by induction hypothesis. Later on, we consider the central extensions of $\mathfrak{g}'$ and we obtain the law of $\mathfrak{g}$.

As $r_k = n-k-1$ is odd we deduce that $n-k-3$ is also odd. Thus, only the following subcases must be considered:

i) $\mathfrak{g}' \simeq \mathcal{L}(n-2, r_1, r_2, \ldots, r_{k-2}, r_{k-1})$, $r_{k-1} \leq n-k-3$

ii) $\mathfrak{g}' \simeq \tau(n-2, r_1, r_2, \ldots, r_{k-2}, n-k-3)$

Subcase i) : $\mathfrak{g}' \simeq \mathcal{L}(n-2, r_1, r_2, \ldots, r_{k-2}, r_{k-1})$, $r_{k-1} \leq n-k-3$

In an adapted basis, $\{X_0, X_1, \ldots, X_{n-k-2},\ Y_1,\ Y_2, \ldots, Y_{k-1}\}$ the law of $\mathfrak{g}'$ can be given by $\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \le i \le n-k-3 \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2},\ 1 \le j \le k-1 \end{cases}$

We obtain the law of $\mathfrak{g}$ by extending $\mathfrak{g}'$ by $\langle X_{n-k-1}, Y_k \rangle$. In an adapted basis $\{X_0, X_1, \ldots, X_{n-k-2}, X_{n-k-1}, Y_1, Y_2, \ldots, Y_{k-1}, Y_k\}$, we can express the law of $\mathfrak{g}$ by

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \le i \le n-k-3 \\ [X_0, X_{n-k-2}] &= \alpha X_{n-k-1} + \beta Y_k & \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2},\ 1 \le j \le k-1 \\ [X_i, X_{n-k-1-i}] &= a_i X_{n-k-1} + d_i Y_k & 1 \le i \le \frac{n-k-2}{2} \\ [X_{n-k-1-r_j}, Y_j] &= b_j X_{n-k-1} + e_j Y_k & 1 \le j \le k-1 \end{cases}$$

From $\mathrm{Jac}(X_0, X_i, X_{n-k-2-i})$, $1 \le i \le \frac{n-k-4}{2}$, and $\mathrm{Jac}(X_0, X_{n-k-2-r_j}, Y_j)$, $1 \le j \le k-1$, it follows that $\begin{cases} a_i &= (-1)^{i-1} a & 1 \le i \le \frac{n-k-2}{2} \\ d_i &= (-1)^{i-1} d & 1 \le i \le \frac{n-k-2}{2} \\ b_j &= 0 & 1 \le j \le k-1 \\ e_j &= 0 & 1 \le j \le k-1 \end{cases}$

Thus, the law of $\mathfrak{g}$ is determined by

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \le i \le n-k-3 \\ [X_0, X_{n-k-2}] &= \alpha X_{n-k-1} + \beta Y_k & \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2},\ 1 \le j \le k-1 \\ [X_i, X_{n-k-1-i}] &= (-1)^{i-1}(a X_{n-k-1} + d Y_k) & 1 \le i \le \frac{n-k-2}{2} \end{cases}$$

Now let $s = rank \begin{pmatrix} \alpha & \beta \\ a & d \end{pmatrix}$.

$s = 0 \Rightarrow \alpha = \beta = a = d = 0$. Then, we conclude that $\mathfrak{g}$ is a split algebra.

$s = 1 \Rightarrow \exists t \in \mathbf{C}$ such that $a X_{n-k-1} + d Y_k = t(\alpha X_{n-k-1} + \beta Y_k)$

With the change of basis defined by

$$\begin{cases} X_i' = X_i,\ 0 \le i \le n-k-2, \quad X_{n-k-1}' = \alpha X_{n-k-1} + \beta Y_k \\ Y_1' = Y_k, \quad Y_k' = Y_1, \quad Y_j' = Y_j,\ \ 2 \le j \le k \end{cases}$$

it follows that $Y_1 \notin [\mathfrak{g}, \mathfrak{g}]$ and $Y_1 \in \mathcal{Z}(\mathfrak{g})$. Then $r_1 = 1$, which is a contradiction.

In case $s = 2$, a change of basis leads to

$$\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{k-1}, r_k = n-k-1)$$

Subcase ii) : $\mathfrak{g}' \simeq \tau(n-2, r_1, r_2, \ldots, r_{k-2}, n-k-3)$

The law of $\mathfrak{g}'$ is defined by

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \le i \le n-k-3 \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2},\ 1 \le j \le k-2 \\ [X_i, X_{n-k-3-i}] &= (-1)^{i-1}(X_{n-k-3} + Y_{k-1}) & 1 \le i \le \frac{n-k-4}{2} \\ [X_i, X_{n-k-2-i}] &= (-1)^{i-1} \frac{(n-k-2-2i)}{2} X_{n-k-2} & 1 \le i \le \frac{n-k-4}{2} \\ [X_1, Y_{k-1}] &= \frac{(k+4-n)}{2} X_{n-k-2} & \end{cases}$$

The law of $\mathfrak{g}$ can be expressed by

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \le i \le n-k-3 \\ [X_0, X_{n-k-2}] &= \alpha X_{n-k-1} + \beta Y_k & \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2},\ 1 \le j \le k-2 \\ [X_i, X_{n-k-3-i}] &= (-1)^{i-1}(X_{n-k-3} + Y_{k-1}) & 1 \le i \le \frac{n-k-4}{2} \\ [X_i, X_{n-k-2-i}] &= (-1)^{i-1} \frac{(n-k-2-2i)}{2} X_{n-k-2} & 1 \le i \le \frac{n-k-4}{2} \\ [X_i, X_{n-k-1-i}] &= a_i X_{n-k-1} + d_i Y_k & 1 \le i \le \frac{n-k-2}{2} \\ [X_{n-k-1-r_j}, Y_j] &= b_j X_{n-k-1} + e_j Y_k & 1 \le j \le k-1 \\ [X_1, Y_{k-1}] &= \frac{(k+4-n)}{2} X_{n-k-2} & \end{cases}$$

From $\mathrm{Jac}(X_0, X_i, X_{n-k-2-i})$, $1 \le i \le \frac{n-k-4}{2}$, a simple process of finite induction leads to $\begin{cases} a_i &= (-1)^{i-1}(a_1 - \frac{(n-k-2-i)}{2}(i-1)\alpha), & 1 \le i \le \frac{n-k-2}{2} \\ d_i &= (-1)^{i-1}(d_1 - \frac{(n-k-2-i)}{2}(i-1)\beta), & 1 \le i \le \frac{n-k-2}{2} \end{cases}$

From $\mathrm{Jac}(X_0, X_1, Y_{k-1})$ it follows that $b_{k-1} = \frac{(k+4-n)}{2}\alpha, \quad e_{k-1} = \frac{(k+4-n)}{2}\beta$.

From $\mathrm{Jac}(X_0, X_{n-k-2-r_j}, Y_j)$, $1 \le j \le k-2$, it follows that $b_j = e_j = 0$.
Finally, from $\mathrm{Jac}(X_1, X_2, X_{n-k-4})$ it follows that $\frac{(-n+k+8)}{2} a_1 = 0$, $\frac{(-n+k+8)}{2} d_1 = 0$.

As $n \ne k+8$ it is deduced that $a_1 = d_1 = 0$. Thus,
$a_i = (-1)^i \frac{(n-k-2-i)}{2}(i-1)\alpha, \qquad d_i = (-1)^i \frac{(n-k-2-i)}{2}(i-1)\beta, \qquad 1 \le i \le \frac{n-k-2}{2}$.

If $(\alpha, \beta) = (0,0)$ or $\alpha \ne 0$, by using changes of basis it follows that $r_1 = 1$, which is a contradiction. If $\alpha = 0$ and $\beta \ne 0$, we obtain that $r_k = 1$ which is also a contradiction. Thus, $\mathfrak{g} \not\simeq \tau(n, r_1, r_2, \ldots, r_k)$.

We conclude that in terminal case $r_{p-1} = n-p$ ($r_k = n-k-1$) the algebra $\mathfrak{g}$ is isomorphic to $\mathcal{L}(n, r_1, r_2, \ldots, r_{k-1}, r_k)$.

$\boxed{\textbf{Case II: } r_{p-1} = n-p-1}$

Let $\mathfrak{g}$ be a naturally graded $(k+1)$-filiform Lie algebra, $dim(\mathfrak{g}) = n$, $n \ge \max\{3k+2, k+9\}$, which law is $\mu(n, r_1, r_2, \ldots, r_{k-2}, r_{k-1}, n-k-2)$.

Then, the quotient algebra $\mathfrak{g}' = \mathfrak{g}/\mathcal{C}^{n-k-2}(\mathfrak{g}) = \mathfrak{g}/\langle X_{n-k-1} \rangle$ is a naturally graded $(k+1)$-filiform Lie algebra of dimension $n-1$ and which law is $\mu(n-1, r_1, r_2, \ldots, r_{k-1}, n-k-2 = (n-1)-(k+1))$.

Thus, the law of $\mathfrak{g}' \simeq \mathcal{L}(n-1, r_1, r_2, \ldots, r_{k-1}, r_k = n-k-2)$ is given by

$$\begin{cases} [X_0, X_i] &= X_{i+1} & 1 \le i \le n-k-3 & \\ [X_i, X_{r_j-i}] &= (-1)^{i-1} Y_j & 1 \le i \le \frac{r_j-1}{2}, & 1 \le j \le k-1 \\ [X_i, X_{n-k-2-i}] &= (-1)^{i-1} Y_k & 1 \le i \le \frac{n-k-3}{2} & \end{cases}$$

The law of $\mathfrak{g}$ is defined by

$$\begin{cases} [X_0, X_i] & = & X_{i+1} & 1 \leq i \leq n-k-3 \\ [X_0, X_{n-k-2}] & = & \alpha X_{n-k-1} & \\ [X_0, Y_k] & = & \beta X_{n-k-1} & \\ [X_i, X_{r_j-i}] & = & (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2}, \quad 1 \leq j \leq k-1 \\ [X_i, X_{n-k-2-i}] & = & (-1)^{i-1} Y_k & 1 \leq i \leq \frac{n-k-3}{2} \\ [X_i, X_{n-k-1-i}] & = & a_i X_{n-k-1} & 1 \leq i \leq \frac{n-k-3}{2} \\ [X_{n-k-1-r_j}, Y_j] & = & b_j X_{n-k-1} & 1 \leq j \leq k-1 \\ [X_1, Y_k] & = & b_k X_{n-k-1} & \\ [Y_i, Y_j] & = & c_{ij} X_{n-k-1} & \text{if } r_i + r_j = n-k-1, \quad 1 \leq i < j \leq k-1 \end{cases}$$

$\mathrm{Jac}(X_0, X_i, X_{n-k-2-i})$, $1 \leq i \leq \frac{n-k-3}{2} \Rightarrow a_i = (-1)^{i-1}(\frac{n-2i-k-1}{2})\beta$.
$\mathrm{Jac}(X_0, X_{n-k-2-r_j}, Y_j)$, $1 \leq j \leq k-1 \Rightarrow b_j = 0$.
$\mathrm{Jac}(X_1, X_2, X_{n-k-4})$, $\mathrm{Jac}(X_1, X_{r_i-1}, Y_j)$, $1 \leq i < j \leq k-1 \Rightarrow b_k = 0$, $c_{ij} = 0$.

The law of $\mathfrak{g}$ can be expressed as

$$\begin{cases} [X_0, X_i] & = & X_{i+1} & 1 \leq i \leq n-k-3 \\ [X_0, X_{n-k-2}] & = & \alpha X_{n-k-1} & \\ [X_0, Y_k] & = & \beta X_{n-k-1} & \\ [X_i, X_{r_j-i}] & = & (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2}, \quad 1 \leq j \leq k-1 \\ [X_i, X_{n-k-2-i}] & = & (-1)^{i-1} Y_k & 1 \leq i \leq \frac{n-k-3}{2} \\ [X_i, X_{n-k-1-i}] & = & (-1)^{i-1}(\frac{n-2i-k-1}{2})\beta X_{n-k-1} & 1 \leq i \leq \frac{n-k-3}{2} \end{cases}$$

If $(\alpha, \beta) = (0, 0)$ then $dim(\mathcal{Z}(\mathfrak{g})) = p+2$, which is a contradiction. In order to prove the remaining ones we only need to use simple changes of basis. Thus, $\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{k-1}, r_k = n-k-2)$ or $\mathfrak{g} \simeq \tau(n, r_1, r_2, \ldots, r_{k-1}, r_k = n-k-2)$.

Case III: $r_{p-1} = n-p-2$

Now, it can be assumed that
$\mathfrak{g}' \simeq \mathcal{L}(n-1, r_1, r_2, \ldots, r_{k-1}, n-k-3)$ or $\mathfrak{g}' \simeq \tau(n-1, r_1, r_2, \ldots, r_{k-1}, n-k-3)$.

If $\mathfrak{g}' \simeq \mathcal{L}(n-1, r_1, r_2, \ldots, r_{k-1}, n-k-3)$ it is proved that $\mathfrak{g} \simeq \mathcal{L}(n, r_1, r_2, \ldots, r_{k-1}, n-k-3)$, $\mathfrak{g} \simeq \mathcal{Q}(n, r_1, r_2, \ldots, r_{k-1}, n-k-3)$ or there is a contradiction.

If $\mathfrak{g}' \simeq \tau(n-1, r_1, r_2, \ldots, r_{k-1}, n-k-3)$ it is proved that $\mathfrak{g} \simeq \tau(n, r_1, r_2, \ldots, r_{k-1}, n-k-3)$ or there is a contradiction.

Case IV: $r_{p-1} = n-p-3$

This is a particular case of the general case. It is separately studied for technical reasons.

Now, it can be assumed that $\mathfrak{g}' \simeq \mathcal{L}(n-1, r_1, \ldots, r_{k-1}, n-k-4)$, $\mathfrak{g}' \simeq \mathcal{Q}(n-1, r_1, \ldots, r_{k-1}, n-k-4)$ or $\mathfrak{g}' \simeq \tau(n-1, r_1, \ldots, r_{k-1}, n-k-4)$.

$\mathfrak{g}' \simeq \mathcal{L}(n-1, r_1, \ldots, r_{k-1}, n-k-4) \Rightarrow \mathfrak{g} \simeq \mathcal{L}(n, r_1, \ldots, r_{k-1}, n-k-4)$.

$\mathfrak{g}' \simeq \mathcal{Q}(n-1, r_1, \ldots, r_{k-1}, n-k-4) \Rightarrow$ contradiction.

$\mathfrak{g}' \simeq \tau(n-1, r_1, \ldots, r_{k-1}, n-k-4) \Rightarrow$ contradiction.

Remark 3.3. The case $\mathfrak{g}' \simeq \tau(n-1, r_1, r_2, \ldots, r_{k-1}, n-k-4)$ is specially complicated from a technical viewpoint but, in essence, it is analogous to the above cases.

$$\boxed{\textbf{Case V: } 2p-1 \leq r_{p-1} \leq n-p-4}$$

Now, the algebra $\mathfrak{g}$ is a naturally graded $(k+1)$-filiform Lie algebra, $dim(\mathfrak{g}) = n$, which law is $\mu(n, r_1, r_2, \ldots, r_k)$. The algebra $\mathfrak{g}' = \mathfrak{g}/\mathcal{C}^{n-k-2}(\mathfrak{g}) = \mathfrak{g}/\langle X_{n-k-1}\rangle$ is a naturally graded $(k+1)$-filiform Lie algebra of dimension $n-1$ and law $\mu(n-1, r_1, r_2, \ldots, r_k)$.

Two subcases may be distinguished: $n-k$ is either even or odd. The proof is analogous to the above cases. ∎

4. Exceptional Algebras

In order to classify naturally graded p-filiform Lie algebras, the dimension n must be $\geq 3p-1$. For all the expressions to be true it is also necessary that $n \geq p+8$. In this section we study the cases where $n < \max\{3p-1, p+8\}$. There are some difficulties and other algebras appear: the exceptional algebras.

As $3p-1 > p+8$ for $p \geq 5$, it follows that $\max\{3p-1, p+8\} = 3p-1$ for $p \geq 5$. Consequently, we only have to study the case $p \leq 4$.

The cases $p = 2$ and $p = 3$ have already been studied in [7], [10]. Thus, we only consider $p = 4$ when $n < \max\{3.4-1, 4+8\} = 12$.

As $3 \leq r_1 < r_2 < r_3 \leq n-p = 11-4 = 7$, it is deduced that only case $\mu(11, 3, 5, 7)$ may be studied.

Theorem 4.1. *The naturally graded non-split 4-filiform Lie algebras of dimension 11, are $\mathcal{L}(11,3,5,7)$, $\epsilon^1(11,3,5,7)$ and $\epsilon^2(11,3,5,7)$, being*

$$\epsilon^1(11,3,5,7): \left\{ \begin{array}{lcll} [X_0, X_i] & = & X_{i+1} & 1 \leq i \leq 6 \\ [X_i, X_{r_j-i}] & = & (-1)^{i-1} Y_j & 1 \leq i \leq \frac{r_j-1}{2}, \qquad 1 \leq j \leq 3, \\ & & & (r_1, r_2, r_3) = (3,5,7) \\ [X_i, Y_1] & = & X_{i+3} & 1 \leq i \leq 4 \\ [X_i, Y_2] & = & X_{i+5} & 1 \leq i \leq 2 \end{array} \right.$$

$$\epsilon^2(11,3,5,7): \left\{ \begin{array}{lcll} [X_0, X_i] & = & X_{i+1} & 1 \leq i \leq 6 \\ [X_1, X_2] & = & Y_1 & \\ [X_i, X_{5-i}] & = & (-1)^{i-1}(X_5 + Y_2) & 1 \leq i \leq 2 \\ [X_i, X_{7-i}] & = & (-1)^i(\frac{(i-3)(i-4)}{2} X_7 - Y_3) & 1 \leq i \leq 3 \\ [X_i, X_{6-i}] & = & (-1)^i (3-i) X_6 & 1 \leq i \leq 2 \\ [X_i, Y_2] & = & -2X_{i+5} & 1 \leq i \leq 2 \end{array} \right.$$

Remark 4.2. The exceptional naturally graded non-split p-filiform Lie algebras are

i) $p = 2 \Rightarrow \epsilon(7,3)$, $\epsilon^1(9,5)$ and $\epsilon^2(9,5)$ [7].

ii) $p = 3 \Rightarrow \mathcal{H}_2$, $\epsilon(8,3,5)$, $\epsilon(9,3,5)$, $\epsilon^{1,\gamma}(10,3,5)$, $\gamma = re^{i\theta}$, $r \geq 0$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, $\epsilon^{2,0}(10,3,5)$, $\epsilon^{2,1}(10,3,5)$ and $\epsilon^{2,3}(10,3,5)$ [10].

iii) $p = 4 \Rightarrow \epsilon^1(11,3,5,7)$ and $\epsilon^2(11,3,5,7)$.

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J. M. Cabezas
Dpto. Matemática Aplicada
E. U. de Ingeniería
Universidad del País Vasco
Nieves Cano, 12
01006 Vitoria (Spain)
mapcamaj@vc.ehu.es

E. Pastor
Dpto. Matemática Aplicada
E. U. de Ingeniería
Universidad del País Vasco
Nieves Cano, 12
01006 Vitoria (Spain)
mappasae@vc.ehu.es

Received February 12, 2004
and in final form November 17, 2004