Journal of Lie Theory
Volume **15** (2005) 227–234

Lifting smooth curves over invariants for representations of compact Lie groups, II

Andreas Kriegl, Mark Losik, Peter W. Michor, and Armin Rainer*

Communicated by E. B. Vinberg

Abstract. Any sufficiently often differentiable curve in the orbit space of a compact Lie group representation can be lifted to a once differentiable curve into the representation space.

1. Introduction

In [2] the following problem was investigated. Consider an orthogonal representation of a compact Lie group G on a real finite dimensional Euclidean vector space V. Let $\sigma_1, \ldots, \sigma_n$ be a system of homogeneous generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. Then the mapping $\sigma = (\sigma_1, \ldots, \sigma_n) : V \to \mathbb{R}^n$ induces a homeomorphism between the orbit space V/G and the semialgebraic set $\sigma(V)$. Suppose a smooth curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ in the orbit space is given (smooth as curve in $\mathbb{R}^n$), does there exist a smooth lift to V, i.e., a smooth curve $\bar{c} : \mathbb{R} \to V$ with $c = \sigma \circ \bar{c}$?

It was shown in [2] that a real analytic curve in V/G admits a local real analytic lift to V, and that a smooth curve in V/G admits a global smooth lift, if certain genericity conditions are satisfied. In both cases the lifts may be chosen orthogonal to each orbit they meet and then they are unique up to a transformation in G, whenever the representation of G on V is polar, i.e., admits sections.

In this paper we treat the same problem under weaker differentiability conditions for $c : \mathbb{R} \to V/G$ and without the mentioned genericity conditions. In section 3 we show that a continuous curve in the orbit space V/G allows a global continuous lift to V. As a consequence we can prove in section 4 that a sufficiently often differentiable curve in V/G can be lifted to a once differentiable curve in V. What we mean by sufficiently often differentiable will be specified there.

In the special case that the symmetric group S_n is acting on $\mathbb{R}^n$, in other words (see [2]), if smooth parameterizations of the roots of smooth curves of polynomials with all roots real are looked for, the following results were proved

* M.L., P.W.M. and A.R. were supported by 'Fonds zur Förderung der wissenschaftlichen Forschung, Projekt P 14195 MAT'

ISSN 0949–5932 / \$2.50

in [5]: Any differentiable lift of a C^{2n}-curve (of polynomials) $c : \mathbb{R} \to \mathbb{R}^n/S_n$ is actually C^1, and there always exists a twice differentiable but in general not better lift of c, if it is of class C^{3n}. Note that here the differentiability assumptions on c are not the weakest possible which is shown by the case $n = 2$, elaborated in [1] 2.1. The proof there is based on the fact that the roots of a C^n-curve of polynomials $c : \mathbb{R} \to \mathbb{R}^n/S_n$ may be chosen differentiable with locally bounded derivative; this is due to Bronshtein [4] and Wakabayashi [12]. Therefore, our long-term objective is to prove the existence of a twice differentiable lift also in the general setting. The key is the generalization of Bronshtein's and Wakabayashi's result which seems to be difficult.

The polynomial results have applications in the theory of partial differential equations and perturbation theory, see [6].

2. Preliminaries

2.1. The setting. Let G be a compact Lie group and let $\rho : G \to O(V)$ be an orthogonal representation in a real finite dimensional Euclidean vector space V with inner product $\langle \ | \ \rangle$. By a classical theorem of Hilbert and Nagata, the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V is finitely generated. So let $\sigma_1, \dots, \sigma_n$ be a system of homogeneous generators of $\mathbb{R}[V]^G$ of positive degrees $d_1, \dots, d_n$. We may assume that $\sigma_1 : v \mapsto \langle v|v\rangle$ is the inner product. Consider the *orbit map* $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$. Note that, if $(y_1, \dots, y_n) = \sigma(v)$ for $v \in V$, then $(t^{d_1}y_1, \dots, t^{d_n}y_n) = \sigma(tv)$ for $t \in \mathbb{R}$, and that $\sigma^{-1}(0) = \{0\}$. The image $\sigma(V)$ is a semialgebraic set in the categorical quotient $V/\!\!/G := \{y \in \mathbb{R}^n : P(y) = 0 \text{ for all } P \in I\}$ where I is the ideal of relations between $\sigma_1, \dots, \sigma_n$. Since G is compact, σ is proper and separates orbits of G, it thus induces a homeomorphism between V/G and $\sigma(V)$.

2.2. The slice theorem. For a point $v \in V$ we denote by G_v its isotropy group and by $N_v = T_v(G.v)^\perp$ the normal subspace of the orbit $G.v$ at v. It is well known that there exists a G-invariant open neighborhood U of v which is real analytically G-isomorphic to the crossed product (or associated bundle) $G \times_{G_v} S_v = (G \times S_v)/G_v$, where S_v is a ball in N_v with center at the origin. The quotient U/G is homeomorphic to S_v/G_v. It follows that the problem of local lifting curves in V/G passing through $\sigma(v)$ reduces to the same problem for curves in N_v/G_v passing through 0. For more details see [2], [8] and [10], theorem 1.1.

A point $v \in V$ (and its orbit $G.v$ in V/G) is called regular if the isotropy representation $G_v \to O(N_v)$ is trivial. Hence a neighborhood of this point is analytically G-isomorphic to $G/G_v \times S_v \cong G.v \times S_v$. The set V_{reg} of regular points is open and dense in V, and the projection $V_{\text{reg}} \to V_{\text{reg}}/G$ is a locally trivial fiber bundle. A non regular orbit or point is called singular.

2.3. Removing fixed points.

Let V^G be the space of G-invariant vectors in V, and let V' be its orthogonal complement in V. Then we have $V = V^G \oplus V'$, $\mathbb{R}[V]^G = \mathbb{R}[V^G] \otimes \mathbb{R}[V']^G$ and $V/G = V^G \times V'/G$.

2.4. Lemma. *Any lift $\bar{c}$ of a curve $c = (c_0, c_1)$ of class C^k $(k = 0, 1, \dots, \infty, \omega)$*

in $V^G \times V'/G$ has the form $\bar c = (c_0, \bar c_1)$, where $\bar c_1$ is a lift of c_1 to V' of class C^k ($k = 0, 1, \dots, \infty, \omega$). The lift $\bar c$ is orthogonal if and only if $\bar c_1$ is orthogonal. ■

2.5. Multiplicity. For a continuous function f defined near 0 in $\mathbb{R}$, let the *multiplicity* or *order of flatness* $m(f)$ at 0 be the supremum of all integers p such that $f(t) = t^p g(t)$ near 0 for a continuous function g. If f is C^n and $m(f) < n$, then $f(t) = t^{m(f)} g(t)$, where now g is $C^{n-m(f)}$ and $g(0) \neq 0$. Similarly, one can define multiplicity of a function at any $t \in \mathbb{R}$.

2.6. Lemma. *Let $c = (c_1, \dots, c_n)$ be a curve in $\sigma(V) \subseteq \mathbb{R}^n$, where c_i is C^{d_i}, for $1 \le i \le n$, and $c(0) = 0$. Then the following two conditions are equivalent:*

1. *$c_1(t) = t^2 c_{1,1}(t)$ near 0 for a continuous function $c_{1,1}$;*
2. *$c_i(t) = t^{d_i} c_{i,i}(t)$ near 0 for a continuous function $c_{i,i}$, for all $1 \le i \le n$.*

Proof. The proof of the nontrivial implication (1) ⇒ (2) is the same as in the smooth case with $r = 1$, see [2] 3.3. for details. ■

3. Lifting continuous curves over invariants

3.1. Proposition. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be continuous. Then there exists a global continuous lift $\bar c : \mathbb{R} \to V$ of c.*

This result is due to Montgomery and Yang [7] see also [3]. We present a short proof adapted to our setting:

Proof. We will make induction on the size of G. More precisely, for two compact Lie groups G' and G we denote $G' < G$, if

- $\dim G' < \dim G$ or
- if $\dim G' = \dim G$, then G' has less connected components than G has.

In the simplest case, when $G = \{e\}$ is trivial, we find $\sigma(V) = V/G = V$, whence we can put $\bar c := c$.

Let us assume that for any $G' < G$ and any continuous $c : \mathbb{R} \to V/G'$ there exists a global continuous lift $\bar c : \mathbb{R} \to V$ of c, where $G' \to O(V)$ is an orthogonal representation on an arbitrary real finite dimensional Euclidean vector space V.

We shall prove that then the same is true for G. Let $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be continuous. By lemma 2.3, we may remove the nontrivial fixed points of the G-action on V and suppose that $V^G = \{0\}$. The set $c^{-1}(0)$ is closed in $\mathbb{R}$ and, consequently, $c^{-1}(\sigma(V)\backslash\{0\}) = \mathbb{R}\backslash c^{-1}(0)$ is open in $\mathbb{R}$. Thus, we can write $c^{-1}(\sigma(V)\backslash\{0\}) = \bigcup_{i\in I}(a_i, b_i)$, a disjoint union, where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ with $a_i < b_i$ such that each (a_i, b_i) is maximal with respect to not containing zeros of c, and I is an at most countable set of indices. In particular, we have $c(a_i) = c(b_i) = 0$ for all $a_i, b_i \in \mathbb{R}$ appearing in the above presentation.

We assert that on each (a_i, b_i) there exists a continuous lift $\bar c : (a_i, b_i) \to V\backslash\{0\}$ of the restriction $c|_{(a_i,b_i)} : (a_i, b_i) \to \sigma(V)\backslash\{0\}$. In fact, since $V^G = \{0\}$, for all $v \in V\backslash\{0\}$ the isotropy groups G_v, acting orthogonally on N_v, satisfy $G_v < G$. Therefore, by induction hypothesis and by 2.2, we find local continuous lifts of $c|_{(a_i,b_i)}$ near any $t \in (a_i, b_i)$ and through all $v \in \sigma^{-1}(c(t))$. Suppose $\bar{c}_1 : (a_i, b_i) \supseteq (a, b) \to V\backslash\{0\}$ is a local continuous lift of $c|_{(a_i,b_i)}$ with maximal domain (a, b), where, say, $b < b_i$. Then there exists a local continuous lift $\bar{c}_2$ of $c|_{(a_i,b_i)}$ near b, and there is a $t_0 < b$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_0. Since $\bar{c}_1(t_0)$ and $\bar{c}_2(t_0)$ lie in the same orbit, there must exist a $g \in G$ such that $\bar{c}_1(t_0) = g.\bar{c}_2(t_0)$. But then,

$$\bar{c}_{12}(t) := \begin{cases} \bar{c}_1(t) & \text{for} \quad t \le t_0 \\ g.\bar{c}_2(t) & \text{for} \quad t \ge t_0 \end{cases}$$

is a local continuous lift of $c|_{(a_i,b_i)}$ defined on a larger interval than $\bar{c}_1$. Thus we have shown that each local continuous lift of $c|_{(a_i,b_i)}$ defined on an open interval $(a, b) \subseteq (a_i, b_i)$ can be extended to a larger interval whenever $(a, b) \subsetneq (a_i, b_i)$. This proves the assertion.

We put $\bar{c}|_{c^{-1}(0)} := 0$, since, by $\sigma^{-1}(0) = \{0\}$, this is the only choice. Then $\bar{c}$ is also continuous at points $t_0 \in c^{-1}(0)$ since $\langle \bar{c}(t)|\bar{c}(t)\rangle = \sigma_1(\bar{c}(t)) = c_1(t)$ converges to 0 as $t \to t_0$. ∎

4. Lifting differentiably

Throughout the whole section we let $d \geq 2$ be the maximum of all degrees of systems of minimal generators of invariant polynomials of all slice representations of ρ. Of these there are only finitely many isomorphism types.

4.1. Lemma. *A curve $c : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ of class C^d admits an orthogonal C^d-lift $\bar{c}$ in a neighborhood of a regular point $c(t_0) \in V_{\mathrm{reg}}/G$. It is unique up to a transformation from G.*

Proof. The proof works analogously as in the smooth case, see [2] 3.1. ∎

4.2. Theorem. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^d. Then for any $t_0 \in \mathbb{R}$ there exists a local lift $\bar{c}$ of c near t_0 which is differentiable at t_0.*

Proof. We follow partially the algorithm given in [2] 3.4. Without loss of generality we may assume that $t_0 = 0$. We show the existence of local lifts of c which are differentiable at 0 through any $v \in \sigma^{-1}(c(0))$. By lemma 2.3 we can assume $V^G = \{0\}$.

If $c(0) \neq 0$ corresponds to a regular orbit, then unique orthogonal C^d-lifts defined near 0 exist through all $v \in \sigma^{-1}(c(0))$, by lemma 4.1.

If $c(0) = 0$, then c_1 must vanish of at least second order at 0, since $c_1(t) \geq 0$ for all $t \in \mathbb{R}$. That means $c_1(t) = t^2 c_{1,1}(t)$ near 0 for a continuous function $c_{1,1}$ since c_1 is C^2. By the multiplicity lemma 2.5 we find that $c_i(t) = t^{d_i} c_{i,i}(t)$ near 0 for $1 \leq i \leq n$, where $c_{1,1}, c_{2,2}, \dots, c_{n,n}$ are continuous functions. We consider the following curve in $\sigma(V)$ which is continuous since $\sigma(V)$ is closed in $\mathbb{R}^n$, see [9]:

$$c_{(1)}(t) : \;=\; (c_{1,1}(t), c_{2,2}(t), \dots, c_{n,n}(t)) = (t^{-2}c_1(t), t^{-d_2}c_2(t), \dots, t^{-d_n}c_n(t)).$$

By proposition 3.1, there exists a continuous lift $\bar{c}_{(1)}$ of $c_{(1)}$. Thus, $\bar{c}(t) := t \cdot \bar{c}_{(1)}(t)$ is a local lift of c near 0 which is differentiable at 0:

$$\sigma(\bar{c}(t)) = \sigma(t \cdot \bar{c}_{(1)}(t)) = (t^2 c_{1,1}(t), \dots, t^{d_n} c_{n,n}(t)) = c(t),$$

and

$$\lim_{t\to 0} \frac{t \cdot \bar{c}_{(1)}(t)}{t} = \lim_{t\to 0} \bar{c}_{(1)}(t) = \bar{c}_{(1)}(0).$$

Note that $\sigma^{-1}(0) = \{0\}$, therefore we are done in this case.

If $c(0) \neq 0$ corresponds to a singular orbit, let v be in $\sigma^{-1}(c(0))$ and consider the isotropy representation $G_v \to O(N_v)$. By 2.2, the lifting problem reduces to the same problem for curves in N_v/G_v now passing through 0. ■

4.3. Lemma. *Consider a continuous curve $c : (a,b) \to X$ in a compact metric space X. Then the set A of all accumulation points of $c(t)$ as $t \searrow a$ is connected.*

Proof. On the contrary suppose that $A = A_1 \cup A_2$, where A_1 and A_2 are disjoint open and closed subsets of A. Since A is closed in X, also A_1 and A_2 are closed in X. There exist disjoint open subsets $A_1', A_2' \subseteq X$ with $A_1 \subseteq A_1'$ and $A_2 \subseteq A_2'$. Consider $F := X \backslash (A_1' \cup A_2')$ which is closed in X and hence compact. Since c visits A_1' and A_2' infinitely often and $c^{-1}(A_1')$ and $c^{-1}(A_2')$ are disjoint and open in $\mathbb{R}$, there exists a sequence $t_m \to a$ and $c(t_m) \in F$ for all m. By compactness of F, this sequence has a cluster point y in F. Hence y is in A by definition, which contradicts $F \cap A = \emptyset$. ■

4.4. Theorem. *Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be a curve of class C^d. Then there exists a global differentiable lift $\bar{c} : \mathbb{R} \to V$ of c.*

Proof. The proof, as the one of proposition 3.1, will be carried out by induction on the size of G.

If $G = \{e\}$ is trivial, then $\bar{c} := c$ is a global differentiable lift.

So let us assume that for any $G' < G$ and any $c : \mathbb{R} \to V/G'$ satisfying the differentiability conditions of the theorem there exists a global differentiable lift $\bar{c} : \mathbb{R} \to V$ of c, where $G' \to O(V)$ is an orthogonal representation on an arbitrary real finite dimensional Euclidean vector space V.

We shall prove that the same is true for G. Let $c = (c_1, \dots, c_n) : \mathbb{R} \to V/G = \sigma(V) \subseteq \mathbb{R}^n$ be of class C^d. We may assume that $V^G = \{0\}$, by lemma 2.3. As in the proof of proposition 3.1 we can write $c^{-1}(\sigma(V) \backslash \{0\}) = \bigcup_i (a_i, b_i)$, a disjoint union, where $a_i, b_i \in \mathbb{R} \cup \{\pm\infty\}$ with $a_i < b_i$. In particular, we have $c(a_i) = c(b_i) = 0$ for all $a_i, b_i \in \mathbb{R}$ appearing in the above presentation.

Claim: On each (a_i, b_i) there exists a differentiable lift $\bar{c} : (a_i, b_i) \to V \backslash \{0\}$ of the restriction $c|_{(a_i,b_i)} : (a_i, b_i) \to \sigma(V) \backslash \{0\}$. The lack of nontrivial fixed points guarantees that for all $v \in V \backslash \{0\}$ the isotropy groups G_v acting on N_v satisfy $G_v < G$. Therefore, by induction hypothesis and by 2.2, we find local differentiable lifts of $c|_{(a_i,b_i)}$ near any $t \in (a_i, b_i)$ and through all $v \in \sigma^{-1}(c(t))$. Suppose that $\bar{c}_1 : (a_i, b_i) \supseteq (a, b) \to V \backslash \{0\}$ is a local differentiable lift of $c|_{(a_i,b_i)}$ with maximal domain (a, b), where, say, $b < b_i$. Then there exists a local differentiable lift $\bar{c}_2$ of $c|_{(a_i,b_i)}$ near b, and there exists a $t_0 < b$ such that both $\bar{c}_1$ and $\bar{c}_2$ are defined near t_0. We may assume without loss that $\bar c_1(t_0) = \bar c_2(t_0) =: v_0$, by applying a transformation $g \in G$ to $\bar c_2$, say. We want to show that we can arrange the lift $\bar c_2$ in such a way that its derivative at t_0 matches with the derivative of $\bar c_1$ at t_0. We decompose $\bar c_i'(t_0) = \bar c_i'(t_0)^\top + \bar c_i'(t_0)^\perp$ into the parts tangent to the orbit $G.v_0$ and normal to it.

First we deal with the normal parts $\bar c_i'(t_0)^\perp \in V$. We consider the projection $p : G.S_{v_0} \cong G \times_{G_{v_0}} S_{v_0} \to G/G_{v_0} \cong G.v_0$ of the fiber bundle associated to the principal bundle $\pi : G \to G/G_{v_0}$. Then, for t close to t_0, $\bar c_1$ and $\bar c_2$ are differentiable curves in $G.S_{v_0}$, whence $p \circ \bar c_i$ $(i = 1, 2)$ are differentiable curves in G/G_{v_0} which admit differentiable lifts g_i into G with $g_i(t_0) = e$ (via the horizontal lift of a principal connection, say). Consequently, $t \mapsto g_i(t)^{-1}.\bar c_i(t) =: \tilde c_i(t)$ are differentiable lifts of $c|_{(a_i,b_i)}$ near t_0 which lie in S_{v_0}, whence $\tilde c_i'(t_0) = \frac{d}{dt}\big|_{t=t_0} (g_i(t)^{-1}.\bar c_i(t)) = -g_i'(t_0).v_0 + \bar c_i'(t_0) \in N_{v_0}$. So, $\bar c_i'(t_0)^\top = (g_i'(t_0).v_0)^\top = g_i'(t_0).v_0$, and so for the normal part we get $\bar c_i'(t_0)^\perp = \tilde c_i'(t_0)$.

Since $\tilde c_i$ lie in S_{v_0} we can change to the isotropy representation $G_{v_0} \to O(N_{v_0})$ (using the same letters σ_i for the generators of $\mathbb{R}[N_{v_0}]^{G_{v_0}}$). We can suppose that $v_0 = 0$, i.e., $c(t_0) = 0$.

Recall the continuous curve in $\sigma(V)$ defined in the proof of theorem 4.2 which depends on the point t_0:

$$c_{(1,t_0)}(t) := ((t - t_0)^{-2} c_1(t), (t - t_0)^{-d_2} c_2(t), \ldots, (t - t_0)^{-d_n} c_n(t)).$$

We find that for $i = 1, 2$:

$$\sigma(\tilde c_i'(t_0)) = \sigma\left(\lim_{t\to t_0} \frac{\tilde c_i(t) - \tilde c_i(t_0)}{t - t_0}\right) = \lim_{t\to t_0} \sigma\left(\frac{\tilde c_i(t)}{t - t_0}\right) = c_{(1,t_0)}(t_0).$$

So $\tilde c_1'(t_0)$ and $\tilde c_2'(t_0)$ are lying in the same orbit. This shows also that

- ♦ for any two lifts of c near $t_0 \in c^{-1}(0)$ which are one-sided differentiable at t_0 the derivatives at t_0 lie in the same G-orbit.

Thus, there must exist a $g_0 \in G_{v_0}$ such that $\bar c_1'(t_0)^\perp = \tilde c_1'(t_0) = g_0.\tilde c_2'(t_0) = g_0.\bar c_2'(t_0)^\perp = (g_0.\bar c_2)'(t_0)^\perp$.

Now we deal with the tangential parts. We search for a differentiable curve $t \mapsto g(t)$ in G with $g(t_0) = g_0$ and

$$\bar c_1'(t_0)^\top = \left(\tfrac{d}{dt}|_{t=t_0}(g(t).\bar c_2(t))\right)^\top = g'(t_0).v_0 + g_0.\bar c_2'(t_0)^\top.$$

But this linear equation can be solved for $g'(t_0)$, and, hence, the required curve $t \mapsto g(t)$ exists. Note that the normal parts still fit since

$$\left(\tfrac{d}{dt}|_{t=t_0}(g(t).\bar c_2(t))\right)^\perp = \left(g'(t_0).v_0 + g_0.\bar c_2'(t_0)\right)^\perp = 0 + g_0.\bar c_2'(t_0)^\perp = \bar c_1'(t_0)^\perp.$$

The two lifts $\bar c_1$ for $t \le t_0$ and $g.\bar c_2$ for $t \ge t_0$ fit together differentiably at t_0. This proves the claim.

Now let $\bar c : (a_i, b_i) \to V\backslash\{0\}$ be the differentiable lift of $c|_{(a_i,b_i)}$ constructed above. For $a_i \ne -\infty$, we put $\bar c(a_i) := 0$, the only choice. Consider the expression $\gamma(t) := \frac{\bar c(t)}{t-a_i}$ which is a differentiable curve in $V\backslash\{0\}$ for $t \in (a_i, b_i)$. We want to show that $\lim_{t\searrow a_i} \gamma(t)$ exists. For t sufficiently close to a_i we have

$$\sigma(\gamma(t)) = \sigma\left(\frac{\bar c(t)}{t - a_i}\right) = c_{(1,a_i)}(t) \to c_{(1,a_i)}(a_i) \qquad \text{as } t \searrow a_i,$$

where now $c_{(1,a_i)}(t) := ((t-a_i)^{-2}c_1(t), (t-a_i)^{-d_2}c_2(t), \dots, (t-a_i)^{-d_n}c_n(t))$. Let $\bar{c}_{(1,a_i)}$ be a corresponding continuous lift of $c_{(1,a_i)}$ which exists by proposition 3.1. This shows that the set A of all accumulation points of $(\gamma(t))_{t\searrow a_i}$ lies in the orbit $G.\bar{c}_{(1,a_i)}(a_i)$ through $\bar{c}_{(1,a_i)}(a_i)$. By lemma 4.3, A is connected. In particular, the limit $\lim_{t\searrow a_i}\gamma(t)$ must exist, if G is a finite group. In general let us consider the projection $p : G.S_{v_1} \cong G\times_{G_{v_1}} S_{v_1} \to G/G_{v_1} \cong G.v_1$ of a fiber bundle associated to the principal bundle $\pi : G \to G/G_{v_1}$, where we choose some $v_1 \in A$. For t close to a_i the curve $t \mapsto \gamma(t)$ is differentiable in $G.S_{v_1}$, whence $t\mapsto p(\gamma(t))$ defines a differentiable curve in G/G_{v_1} which admits a differentiable lift $t \mapsto g(t)$ into G. Now, $t \mapsto g(t)^{-1}.\gamma(t)$ is a differentiable curve in S_{v_1} whose accumulation points for $t \searrow a_i$ have to lie in $G.v_1 \cap S_{v_1} = \{v_1\}$, since $\sigma(g(t)^{-1}.\gamma(t)) = \sigma(\gamma(t))$. That means that $t \mapsto g(t)^{-1}.\bar{c}(t)$ defines a differentiable lift of $c|_{(a_i,b_i)}$, for $t > a_i$ close to a_i, whose one-sided derivative at a_i exists:

$$\lim_{t\searrow a_i}\frac{g(t)^{-1}.\bar{c}(t)}{t-a_i} = \lim_{t\searrow a_i} g(t)^{-1}.\gamma(t) = v_1.$$

Let $t \mapsto g(t)$ be extended smoothly to (a_i, b_i) so that near b_i it is constant and replace $t \mapsto \bar{c}(t)$ by $t \mapsto g(t)^{-1}\bar{c}(t)$. Thus

$$\bar{c}'(a_i) := \lim_{t\searrow a_i}\frac{\bar{c}(t)}{t-a_i} = v_1.$$

The same reasoning is true for $b_i \neq +\infty$. Thus we have extended $\bar{c}$ differentiably to the closure of (a_i, b_i).

Let us now construct a global differentiable lift of c defined on the whole of $\mathbb{R}$. For isolated points $t_0 \in c^{-1}(0)$ the two differentiable lifts on the neighboring intervals can be made to match differentiably, by applying a fixed $g \in G$ to one of them by ♦. Let E be the set of accumulation points of $c^{-1}(0)$. For connected components of $\mathbb{R}\setminus E$ we can proceed inductively to obtain differentiable lifts on them.

We extend the lift by 0 on the set E of accumulation points of $c^{-1}(0)$. Note that every lift $\tilde{c}$ of c has to vanish on E and is continuous there since $\langle\tilde{c}(t)|\tilde{c}(t)\rangle = \sigma_1(\tilde{c}(t)) = c_1(t)$. We also claim that any lift $\tilde{c}$ of c is differentiable at any point $t' \in E$ with derivative 0. Namely, the difference quotient $t \mapsto \frac{\tilde{c}(t)}{t-t'}$ is a lift of the curve $c_{(1,t')}$ which vanishes at t' by the following argument: Consider the local lift $\bar{c}$ of c near t' which is differentiable at t', provided by theorem 4.2. Let $(t_m)_{m\in\mathbb{N}} \subseteq c^{-1}(0)$ be a sequence with $t' \neq t_m \to t'$, consisting exclusively of zeros of c. Such a sequence always exists since $t' \in E$. Then we have

$$\bar{c}'(t') = \lim_{t\to t'}\frac{\bar{c}(t)-\bar{c}(t')}{t-t'} = \lim_{m\to\infty}\frac{\bar{c}(t_m)}{t_m-t'} = 0.$$

Thus $c_{(1,t')}(t') = \lim_{t\to t'}\sigma(\frac{\tilde{c}(t)}{t-t'}) = \sigma(\bar{c}'(t')) = 0$. ∎

4.5 Remark. Note that the differentiability conditions of the curve c in the current section are best possible: In the case when the symmetric group S_n is acting in $\mathbb{R}^n$ by permuting the coordinates, and $\sigma_1,\dots,\sigma_n$ are the elementary symmetric polynomials with degrees $1,\dots,n$, there need not exist a differentiable lift if the differentiability assumptions made on c are weakened, see [1] 2.3. first example.

References

[1] Alekseevsky, D., A. Kriegl, M. Losik, P. W. Michor, *Choosing roots of polynomials smoothly*, Israel J. Math., **105** (1998), 203–233. arXiv:math.CA/9801026.

[2] —, *Lifting smooth curves over invariants for representations of compact Lie groups*, Transformation Groups **5** (2000), 103–110. arXiv:math.AG/0312030.

[3] Bredon, G. E., *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[4] Bronshtein, M. D., *Smoothness of polynomials depending on parameters*, Sib. Mat. Zh. **20** (1979), 493–501 (Russian); English transl. in Siberian Math. J **20** (1980), 347–352.

[5] Kriegl, A., M. Losik, P. W. Michor, *Choosing roots of polynomials smoothly, II*, Israel J. Math. **139** (2004), 183–188. arXiv:math.CA/0208228.

[6] Kriegl, A., and P. W. Michor, *Differentiable perturbation of unbounded operators*, Math. Ann. **327** (2003), 191–201. arXiv:math.FA/0204060.

[7] Montgomery, D., and C. T. Yang, *The existence of a slice*, Ann. of Math., **65** (1957), 108–116.

[8] Luna, D., *Sur certaines opérations différentiables des groupes de Lie*, Amer. J. Math. **97** (1975), 172–181.

[9] Procesi, C., and G. Schwarz, *Inequalities defining orbit spaces*, Invent. Math. **81** (1985), 539–554.

[10] Schwarz, G. W., *Lifting smooth homotopies of orbit spaces*, Publ. Math. IHES **51** (1980), 37–136.

[11] —, *Smooth functions invariant under the action of a compact Lie group*, Topology **14** (1975), 63–68.

[12] Wakabayashi, S., *Remarks on hyperbolic polynomials*, Tsukuba J. Math. **10** (1986), 17–28.

A. Kriegl
Fakultät für Mathematik
Universität Wien
Nordbergstrasse 15
A-1090 Wien, Austria
Andreas.Kriegl@univie.ac.at

P. W. Michor
Fakultät für Mathematik
Universität Wien
Nordbergstrasse 15
A-1090 Wien, Austria
and: Erwin Schrödinger Institut für Mathematische Physik
Boltzmanngasse 9
A-1090 Wien, Austria
Peter.Michor@esi.ac.at

M. Losik
Saratov State University
ul. Astrakhanskaya, 83
410026 Saratov, Russia
losikMV@info.sgu.ru

A. Rainer
Fakultät für Mathematik
Universität Wien
Nordbergstrasse 15
A-1090 Wien, Austria
armin_rainer@gmx.net

Received August 4, 2004
and in final form August 25, 2004