

On Flags and Maximal Chains of Lower Modular Subalgebras of Lie Algebras

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Abstract. In this paper we study the class \mathcal{F} of Lie algebras having a flag of subalgebras, and the class \mathcal{Ch}_{lm} of Lie algebras having a maximal chain of lower modular subalgebras. We show that $\mathcal{F} \subseteq \mathcal{Ch}_{\text{lm}}$ and that both are extensible formations that are subalgebra closed. We derive a number of properties relating to these two classes, including a classification of the algebras in each class over a field of characteristic zero.

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1. Introduction

This paper is a further contribution to the study by a number of authors of the relationship between the structure of a Lie algebra and that of its lattice of subalgebras. We say that a Lie algebra L has a *flag* of subalgebras whenever there is a chain

$$0 = L_n < L_{n-1} < \cdots < L_0 = L$$

where L_i is an $(n - i)$ -dimensional subalgebra of L for $0 \leq i \leq n$. Clearly, every solvable Lie algebra has a flag of subalgebras. In section 2 we seek a characterisation of those Lie algebras possessing a flag of subalgebras. In characteristic zero such a Lie algebra must be either solvable or else a direct sum of copies of $\mathfrak{sl}(2)$ modulo its radical. For more general fields we have that L possesses a composition series in which the composition factors are either one dimensional or else simple algebras with a subalgebra of codimension one, as described by Amayo in [2].

A subalgebra Q of L is called a *quasi-ideal* of L if $[Q, V] \subseteq Q + V$ for every subspace V of L . We close section 2 by showing that L has a flag of quasi-ideals if and only if L is supersolvable or a certain algebra over a field of characteristic two.

A chain of subalgebras of L

$$0 = L_n < L_{n-1} < \cdots < L_0 = L$$

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is called a *maximal chain* of L if L_i is maximal in L_{i-1} for every $i \geq 1$.

A subalgebra U of a Lie algebra L is called

- *modular* in L if it is a modular element in the lattice of subalgebras of L ; that is, if

$$\langle U, B \rangle \cap C = \langle B, U \cap C \rangle \quad \text{for all subalgebras } B \subseteq C,$$

and

$$\langle U, B \rangle \cap C = \langle B \cap C, U \rangle \quad \text{for all subalgebras } U \subseteq C,$$

(where, $\langle U, B \rangle$ denotes the subalgebra of L generated by U and B)

- *upper modular* in L (um in L) if, whenever B is a subalgebra of L which covers $U \cap B$ (that is, such that $U \cap B$ is a maximal subalgebra of B), then $\langle U, B \rangle$ covers U ;
- *lower modular* in L (lm in L) if, whenever B is a subalgebra of L such that $\langle U, B \rangle$ covers U , then B covers $U \cap B$;
- *semi-modular* in L (sm in L) if it is both um and lm in L .
- *strongly lm* in L if U has codimension one in every subalgebra of L that covers U .

In section 3 we go on to look at maximal chains in which each subalgebra is lower modular in the next. We see that every flag of subalgebras is a maximal chain of lower modular subalgebras, and that all such maximal chains have the same length. Every Lie algebra L contains a smallest ideal for which the corresponding factor algebra has a flag of subalgebras to L ; also, if there is a flag of subalgebras from S to L then every subalgebra of L not contained in S has a subalgebra of codimension one. Likewise every Lie algebra L contains a smallest ideal for which the corresponding factor algebra has a maximal chain of lm subalgebras; also, if there is a chain of lower modular subalgebras from T to L then every subalgebra of L not contained in T has a maximal and modular subalgebra. A consequence is that our classification of Lie algebras having a flag of quasi-ideals from section 2 applies equally to those having a flag of upper modular subalgebras, of semimodular subalgebras, or of modular subalgebras. Moreover, in every Lie algebra, every descending flag terminates in the same subalgebra, and, likewise, every descending maximal chain of lower modular subalgebras terminates in the same subalgebra. In many situations we find that a maximal chain of lower modular subalgebras is a flag. Finally we give a classification of the Lie algebras over a field of characteristic zero which have a maximal chain of lower modular subalgebras.

Throughout L will denote a finite-dimensional Lie algebra over a field F .

2. Flags.

First we collect together some elementary properties of flags.

Lemma 2.1. *If a Lie algebra L has a flag of subalgebras, then every subalgebra of L also has a flag of subalgebras.*

Proof. Let L be a minimal counter-example. Then L has a flag of subalgebras but L has a subalgebra S which does not have a flag of subalgebras. Clearly $\dim L > 2$. Also, we see that L has a maximal subalgebra M of codimension one in L which has a flag of subalgebras. The minimality of L implies that every subalgebra of M has a flag of subalgebras, whence $M \cap S$ has a flag of subalgebras. On the other hand, since $S \neq M$ and $\dim L/M = 1$, we have that $\dim S/S \cap M = 1$. It follows that S does have a flag of subalgebras, which is a contradiction. ■

Lemma 2.2. *If a Lie algebra L has a flag of subalgebras, then every epimorphic image of L also has a flag of subalgebras.*

Proof. Let L be a minimal counter-example. We have that L has a flag of subalgebras of L , $0 = L_n < L_{n-1} < \dots < L_0 = L$. Moreover, L has a proper ideal N such that L/N does not have a flag of subalgebras. Suppose first that $N \not\subseteq L_1$. Then we have $L = L_1 + N$, whence $L/N \cong L_1/L_1 \cap N$. As L_1 has a flag of subalgebras, it follows from the minimality of L that $L_1/L_1 \cap N$, and hence L/N , has a flag of subalgebras, which is a contradiction. Thus $N \subseteq L_1$. Using the minimality of L again, we obtain that L_1/N has a flag of subalgebras. Since $\dim L/L_1 = 1$, it follows that L/N has a flag of subalgebras. This contradiction completes the proof. ■

Lemma 2.3. *Let L be a Lie algebra and let N be an ideal of L . If N and L/N have flags of subalgebras, then L has a flag of subalgebras.*

We shall denote the *core* of S in L (that is, the largest ideal of L contained in S) by S_L . Next we seek the simple Lie algebras containing a flag of subalgebras. We recall the algebras $L_m(\Gamma)$ constructed by Amayo in [2, page 46] as examples of Lie algebras L containing a core-free subalgebra of codimension one in L .

Proposition 2.4. *Let L be a simple Lie algebra (over any field). Then the following are equivalent*

- (i) L has a flag of subalgebras;
- (ii) L has a core-free maximal subalgebra of codimension one in L ; and
- (iii) $L \cong L_m(\Gamma)$, where m is odd.

Proof. (i) \Rightarrow (ii): This is clear.

(ii) \Rightarrow (iii): Let L have a core-free maximal subalgebra of codimension one in L . By Theorem 3.1 of [2] we have that $L \cong L_m(\Gamma)$. Moreover, since L is simple, it follows from Theorem 3.2(b),(c) of [2] that m is odd.

(iii) \Rightarrow (i): Suppose now that $L \cong L_m(\Gamma)$. Then $L = Fv_{-1} + Fv_0 + \dots + Fv_m$ and $H_{m,i} = Fv_i + \dots + Fv_m$ for $i \geq 0$ is a subalgebra of L_m (see [2, page 46]). Thus L_m has a flag of subalgebras. ■

Corollary 2.5. *Let L be a simple Lie algebra over a perfect field of characteristic different from two. Then L has a flag of subalgebras if and only if either L is three-dimensional split simple or else a Zassenhaus algebra.*

Proof. This follows from Proposition 2.4 and Corollary 2.3 of [8]. ■

We can now classify the Lie algebras over a field of characteristic zero which have a flag of subalgebras.

Proposition 2.6. *Let F have characteristic zero. Then a semisimple Lie algebra L has a flag of subalgebras if and only if it is a direct sum of copies of $\mathfrak{sl}(2)$.*

Proof. Let $L = S_1 \oplus \cdots \oplus S_r$ be the decomposition of L into its simple components, and suppose that L has a flag of subalgebras. By Lemma 2.1, each S_i has a flag of subalgebras and so is isomorphic to $\mathfrak{sl}(2)$ by [2]. The converse is clear. ■

We shall denote the solvable radical of L by $R(L)$.

Theorem 2.7. *Let F have characteristic zero. Then a Lie algebra L has a flag of subalgebras if and only if either L is solvable or else $L/R(L)$ is a direct sum of copies of $\mathfrak{sl}(2)$.*

Proof. Let L have a flag of subalgebras and suppose that L is not solvable. Then $L/R(L)$ is semisimple and has a flag of subalgebras by Lemma 2.2. It follows from Proposition 2.6 that $L/R(L)$ is a direct sum of copies of $\mathfrak{sl}(2)$. The converse follows from the fact that $\mathfrak{sl}(2)$ has a flag of subalgebras and Lemma 2.3. ■

The best that we can do over a general field is the following.

Theorem 2.8. *Let L be over any field F . Then L has a flag of subalgebras if and only if L has a composition series $0 = I_k < I_{k-1} < \cdots < I_1 = L$ where I_j/I_{j+1} is one dimensional or else isomorphic to $L_m(\Gamma)$ where m is odd, for $1 \leq j \leq k-1$. (Note that different factors may be isomorphic to different $L_m(\Gamma)$).*

Proof. Suppose that $0 = S_n < \cdots < S_0 = L$ is a flag of subalgebras for L . Choose the smallest j with $1 \leq j \leq n$ such that S_{j+1} is NOT an ideal of S_j . If no such j exists then our flag is a composition series in which all of the factors are one dimensional. Put $K_{j+1} = (S_{j+1})_{S_j}$. Clearly K_{j+1} is contained in but not equal to S_{j+1} . Now by Lemmas 2.1, 2.2 we have that K_{j+1} , S_j/K_{j+1} both have flags of subalgebras. Moreover S_j/K_{j+1} contains a core-free maximal subalgebra of codimension one and $\dim S_j/K_{j+1} \geq 2$. Thus, by Theorem 3.1 of [2], either S_j/K_{j+1} is the two-dimensional non-abelian algebra or else it is equal to some $L_m(\Gamma)$. In the former case we can form a composition series for S_j/K_{j+1} where each factor is one dimensional. So suppose that the latter case holds.

If m is odd then $L_m(\Gamma)$ is simple, by Theorem 3.2(b) of [2], and $K_{j+1} < S_j$ can be included in the required composition series. If m is even then a series of

subalgebras for $L_m(\Gamma)$ is $0 < (L_m(\Gamma))^2 < L_m(\Gamma)$. This is a composition series, since the derived algebra has codimension one in $L_m(\Gamma)$ and is a simple ideal of $L_m(\Gamma)$, by Theorem 3.2(c) of [2]. Using Lemma 2.1 again, we have that the derived algebra has a flag of subalgebras, and so $(L_m(\Gamma))^2 \cong L_{m'}(\Gamma)$, by Proposition 2.4. Since m is even we have that $\dim(L_m(\Gamma))^2 \geq 3$ and odd, whence m' is odd, by Theorem 3.2(c) of [2]. Hence $K_{j+1} < S_j^2 < S_j$ can be included in the required composition series. Continuing in this manner, by the finite dimensionality of L , we get the required composition series for L .

Conversely suppose that L has such a composition series. Then the result follows from Proposition 2.4 and the proof is now complete. ■

Finally, when the subalgebras in a flag are quasi-ideals we have the following result.

Theorem 2.9. *Let L be a Lie algebra over any field F . Then L has a flag of quasi-ideals of L if and only if either*

- (i) L is supersolvable, or
- (ii) F has characteristic two and $L = U \oplus K$, where U is a supersolvable ideal of L and K is a three-dimensional split simple ideal of L .

Proof. Let $0 = L_n < L_{n-1} < \dots < L_0 = L$ be a flag of subalgebras of L such that L_i is a quasi-ideal of L for every $0 \leq i \leq n$. Assume that L is not supersolvable. As $\dim(L/L_3) = 3$ it follows from Theorem 3.6 of [1] that L_3 is an ideal of L . But then L_j is an ideal of L for every $3 \leq j \leq n$, by Lemma 2.2 of [6] for instance, and so L_3 is supersolvable. Moreover, L_2 is not an ideal of L because L is not supersolvable. This implies that F has characteristic two and L/L_3 is three-dimensional split simple, by [1] again. Now, by using Lemma 1.4 of [9] we obtain that $L = L_3 \oplus L^{(\infty)}$. This completes the proof in one direction. The converse is clear. ■

3. Maximal chains of lower modular subalgebras.

Lemma 3.1. *Let $U \leq B \leq L$.*

- (i) *If U is lm in B and B is sm in L , then U is lm in L .*
- (ii) *If U is strongly lm in B and B is um and strongly lm in L , then U is strongly lm in L .*

Proof. (i) Suppose that U is lm in B and B is sm in L . Let $S \leq L$ such that U is maximal in $\langle U, S \rangle$. We need to show that $U \cap S$ is maximal in S . If $S \leq B$ this is clear, as U is lm in B , so assume that $S \not\leq B$. Then $B \cap \langle U, S \rangle = U$. It follows that B is maximal in $\langle B, U, S \rangle = \langle B, S \rangle$, because B is um in L . This yields that $B \cap S$ is maximal in S , since B is lm in L . But now $B \cap S \leq B \cap \langle U, S \rangle = U$ and so $B \cap S = U \cap S$, completing the proof of our first claim.

(ii) Now suppose that U is strongly lm in B and that B is um and strongly lm in L . Let C be a subalgebra of L covering U . If $C \leq B$, then $\dim(C/U) = 1$, because U is strongly lm in B , so suppose that $C \not\leq B$. Then $U = C \cap B$, so B is covered by $\langle C, B \rangle$, since B is um in L . It follows from the fact that B is strongly lm in L that B has codimension one in $\langle C, B \rangle = C + B$. But now $\dim(C/U) = \dim(C/C \cap B) = \dim((C + B)/B) = 1$, as required. ■

Notice that the above lemma shows that every subideal of L is lm in L .

Proposition 3.2. *For a maximal chain $0 = L_n < L_{n-1} < \cdots < L_0 = L$ of L the following are equivalent:*

(i) L_i is lm in L_{i-1} for all $i \geq 1$; and

(ii) L_i is lm in L for all $i \geq 1$.

Proof. Simply note that each member of such a chain is maximal, and so um, in the next. ■

A maximal chain satisfying conditions (i) and (ii) will be called a *maximal chain of lm subalgebras* of L . The above result has a number of consequences.

Corollary 3.3. *Every flag of subalgebras of L is a maximal chain of lm subalgebras of L . Furthermore, whenever L has a flag of subalgebras, every lm subalgebra of L is strongly lm in L .*

Proof. Every flag is a maximal chain and each term has codimension one in the next, and so is lm in the next. The first assertion therefore follows from Proposition 3.2.

Suppose now that L has a flag of subalgebras $0 = L_n < L_{n-1} < \cdots < L_0 = L$, and let U be a lm subalgebra of L . To prove the second assertion it suffices to show that if U is maximal in L then it has codimension one in L . So suppose that U is maximal in L . There is a k with $n \geq k \geq 1$ such that $L_k \leq U$ but $L_{k-1} \not\leq U$. Suppose that $U \cap L_i \leq L_{i+1}$ for some $0 \leq i \leq k-2$. If $U \leq L_i$, then $U = L_1$ and we have finished. Otherwise, U is covered by $\langle U, L_i \rangle = L$ and so $U \cap L_i$ is covered by L_i , because U is lm in L . But then $U \cap L_i = L_{i+1}$, whence $L_{i+1} \leq U$, a contradiction. Hence $U \cap L_i \not\leq L_{i+1}$ for all $0 \leq i \leq k-2$. This yields that $L_i = L_{i+1} + U \cap L_i$, from which $1 = \dim(L_i/L_{i+1}) = \dim((U \cap L_i)/(U \cap L_{i+1}))$ for all $0 \leq i \leq k-2$. But then

$$\begin{aligned} k+1 &= \dim(L/L_k) = \dim(L/U) + \sum_{i=-1}^{k-1} \dim((U \cap L_i)/(U \cap L_{i+1})) \\ &= \dim(L/U) + k, \end{aligned}$$

which gives $\dim(L/U) = 1$, as required. ■

Corollary 3.4. *If L has a flag of subalgebras then every maximal chain of lm subalgebras is a flag of L .*

We shall denote by \mathcal{X} the class of Lie algebras L in which every maximal subalgebra has codimension one in L , and by $s\mathcal{X}$ the class of all subalgebras of \mathcal{X} -algebras. The Lie algebra L is called *completely lower modular* if every subalgebra is lm in L .

Corollary 3.5. (i) *Every subalgebra of L lies in a flag of subalgebras of L if and only if $L \in s\mathcal{X}$.*

(ii) *Every subalgebra of L lies in a maximal chain of lm subalgebras if and only if L is completely lower modular.*

Lemma 3.6. *If a Lie algebra L has a maximal chain of lm subalgebras, then every subalgebra of L also has a maximal chain of lm subalgebras.*

Proof. Let L be a minimal counter-example. Then L has a maximal chain of lm subalgebras but L has a subalgebra S which does not have such a maximal chain. Clearly $\dim L > 2$. Also, we see that L has a maximal subalgebra M which is lm in L and which has a maximal chain of lm subalgebras. The minimality of L implies that every subalgebra of M has a maximal chain of lm subalgebras, whence $M \cap S$ has such a maximal chain. Now $S \neq M$ so $\langle M, S \rangle = L$ which covers M , so S covers $M \cap S$ by the lower modularity of M in L . We show that $M \cap S$ is lm in S by using Theorem 2.3 of [7]. Let B be a subalgebra of S such that $B \not\subseteq M \cap S$. Then $M \cap S \cap B = M \cap B$. Since $B \not\subseteq M$ and M is lm in L , we have that $M \cap B$ is covered by B . It follows that $M \cap S$ is lm in S and hence that S does have a maximal chain of lm subalgebras, which is a contradiction. ■

Lemma 3.7. *If a Lie algebra L has a maximal chain of lm subalgebras, then every epimorphic image of L also has a maximal chain of lm subalgebras.*

Proof. Let L be a minimal counter-example. We have that L has a maximal chain of lm subalgebras of L , $0 = L_n < L_{n-1} < \cdots < L_0 = L$. Moreover, L has a proper ideal N such that L/N does not have a maximal chain of lm subalgebras. Suppose first that $N \not\subseteq L_1$. Then we have $L = L_1 + N$, whence $L/N \cong L_1/L_1 \cap N$. As L_1 has a maximal chain of lm subalgebras, it follows from the minimality of L that $L_1/L_1 \cap N$, and hence L/N , has a maximal chain of lm subalgebras, which is a contradiction. Thus $N \subseteq L_1$. Using the minimality of L again, we obtain that L_1/N has a maximal chain of lm subalgebras. Since L_1 is maximal and lm in L , it follows from Lemma 2.2 of [7] that L/N has a maximal chain of lm subalgebras. This contradiction completes the proof. ■

Lemma 3.8. *Let L be a Lie algebra and let N be an ideal of L . If N and L/N have maximal chains of lm subalgebras, then L has a maximal chain of lm subalgebras.*

We shall show next that every maximal chain of lm subalgebras has the same length.

Lemma 3.9. *Let H, K be distinct lm maximal subalgebras of L . Then $H \cap K$ is a lm maximal subalgebra of both H and K .*

Proof. It suffices to show that $H \cap K$ is a lm maximal subalgebra of K . We have that $\langle H, K \rangle = L$, so H is maximal in $\langle H, K \rangle$. The lower modularity of H in L implies that $H \cap K$ is maximal in K .

Now let B be a subalgebra of K with $H \cap K$ maximal in $\langle H \cap K, B \rangle = K$. Clearly $B \not\subseteq H$, and so $\langle H, B \rangle = L$. From the lower modularity of H in L we now have that $H \cap B$ is maximal in B . But $H \cap B = (H \cap K) \cap B$, giving that $H \cap K$ is lm in K . ■

Theorem 3.10. *If A, B are subalgebras of the Lie algebra L , and there is a maximal chain of lm subalgebras of length n from A to B , then all such maximal chains have length n .*

Proof. We use induction on n . The result is clear if $n = 1$. So assume the result holds for $n - 1$. Let

$$A = A_0 < \cdots < A_n = B,$$

$$A = B_0 < \cdots < B_m = B$$

be maximal chains of lm subalgebras from A to B . If $A_{n-1} = B_{m-1}$, then $n - 1 = m - 1$ by the inductive hypothesis and we are done.

So assume that $A_{n-1} \neq B_{m-1}$ and let $C = A_{n-1} \cap B_{m-1}$. Then, by Lemma 3.9, C is lm and maximal in both A_{n-1} and B_{m-1} . Now we can create a maximal chain of lm subalgebras from A to C (by considering $C \cap A_i$ for $0 \leq i \leq n - 2$ for instance). By the inductive hypothesis, this chain must have length $n - 2$:

$$A = C_0 < \cdots < C_{n-2} = C,$$

say, as it can be extended to a chain from A to A_{n-1} . But then

$$A = C_0 < \cdots < C_{n-2} < B_{m-1}$$

is a chain of length $n - 1$ from A to B_{m-1} . It follows that $n - 1 = m - 1$ and the result holds. ■

Let \mathcal{F} denote the class of Lie algebras having a flag of subalgebras, and let \mathcal{Ch}_{lm} be the class of Lie algebras having a maximal chain of lm subalgebras. Recall that a class \mathcal{H} of Lie algebras is called a *homomorph* if it contains, along with an algebra L , all epimorphic images of L . A homomorph \mathcal{H} is called a *formation* if $L/M, L/N \in \mathcal{H}$, where M, N are ideals of L , implies that $L/M \cap N \in \mathcal{H}$. We say that the formation \mathcal{H} is *extensible* if it is closed under extensions; that is if $L \in \mathcal{H}$ whenever $M, L/M \in \mathcal{H}$. Then we have

Proposition 3.11. *The classes \mathcal{F} and \mathcal{Ch}_{lm} are extensible formations and are subalgebra-closed.*

Proof. This follows from Lemmas 2.1, 2.2, 2.3, 3.6, 3.7 and 3.8. ■

As \mathcal{F} and \mathcal{Ch}_{lm} are formations, every Lie algebra L contains smallest ideals $L^{\mathcal{F}}$ and $L^{\mathcal{Ch}_{\text{lm}}}$ such that $L/L^{\mathcal{F}} \in \mathcal{F}$ and $L/L^{\mathcal{Ch}_{\text{lm}}} \in \mathcal{Ch}_{\text{lm}}$. We have that from $L^{\mathcal{F}}$ to L there is a “flag of subalgebras”, and that from $L^{\mathcal{Ch}_{\text{lm}}}$ to L there is a “maximal chain of lm subalgebras”.

Proposition 3.12. *Let $S \leq L$ be such that there is a “flag of subalgebras” from S to L , and let $T \leq L$ be such that there is a “maximal chain of lm subalgebras” from T to L . Then*

- (i) every subalgebra of L not contained in S has a subalgebra of codimension one; and
- (ii) every subalgebra of L not contained in T has a maximal and modular subalgebra.

Proof. (i) Let $S = S_n < S_{n-1} < \dots < S_0 = L$ be a flag from S to L and let U be a subalgebra of L such that $U \not\leq S$. There is a k with $1 \leq k \leq n$ such that $U \leq S_{k-1}$ but $U \not\leq S_k$. Then we have $S_{k-1} = U + S_k$ and so $\dim(U/U \cap S_k) = \dim(S_{k-1}/S_k) = 1$ and so $U \cap S_k$ has codimension one in U .

(ii) Let $T = T_n < T_{n-1} < \dots < T_0 = L$ be a maximal chain of lm subalgebras from T to L and let U be a subalgebra of L such that $U \not\leq T$. There is a k with $1 \leq k \leq n$ such that $U \leq T_{k-1}$ but $U \not\leq T_k$. Then $\langle U, T_k \rangle = T_{k-1}$, so T_k is covered by $\langle U, T_k \rangle$. As T_k is lm in T_{k-1} it follows that $U \cap T_k$ is maximal in U . Similarly, if $W \leq U$ with $W \not\leq U \cap T_k$, then $W \cap T_k$ is maximal in W , from which it follows that $U \cap T_k$ is modular in U . ■

Corollary 3.13. *If U is a um subalgebra of L and there is a flag of subalgebras from U to L , then U is a quasi-ideal of L .*

Proof. Let $x \notin U$ and put $C = \langle U, x \rangle$. Then, as in the above result, C has a subalgebra $M = U_k \cap C$ of codimension one in C . Now, U is covered by C , since U is um in L , and $U \leq M < C$, so $U = M$. It follows that $C = U + Fx$ and U is a quasi-ideal of L . ■

Corollary 3.14. *Let F be a flag of subalgebras of L . Then the following are equivalent:*

- (i) F is a flag of um subalgebras of L ;
- (ii) F is a flag of sm subalgebras of L ;
- (iii) F is a flag of modular subalgebras of L ; and
- (iv) F is a flag of quasi-ideals of L .

Corollary 3.15. (i) *All descending flags from L stop in the same subalgebra of L .*

(ii) *All descending maximal chains of lm subalgebras of L stop in the same subalgebra of L .*

We will denote by $\hat{L}^{\mathcal{F}}$ and by $\hat{L}^{\text{Ch}_{\text{lm}}}$ the subalgebras referred to in (i) and (ii) of Corollary 3.15 above. Then we have

$$\hat{L}^{\text{Ch}_{\text{lm}}} \leq L^{\text{Ch}_{\text{lm}}} \cap \hat{L}^{\mathcal{F}} \text{ and } L^{\text{Ch}_{\text{lm}}} + \hat{L}^{\mathcal{F}} \leq L^{\mathcal{F}}.$$

Moreover,

$$\hat{L}^{\text{Ch}_{\text{lm}}} \leq \bigcap \{M \leq L : M \text{ maximal and modular in } L\}$$

$$\text{and } \hat{L}^{\mathcal{F}} \leq \bigcap \{M \leq L : \dim(L/M) = 1\}.$$

For fields of characteristic zero we have that $\hat{L}^{\text{Ch}_{\text{lm}}}$ and $\hat{L}^{\mathcal{F}}$ are ideals of L , by Corollary 3.3 of [5], and hence that $L^{\text{Ch}_{\text{lm}}} = \hat{L}^{\text{Ch}_{\text{lm}}}$ and $L^{\mathcal{F}} = \hat{L}^{\mathcal{F}}$.

Corollary 3.16. *Let L be a Lie algebra in which every modular subalgebra of L is a quasi-ideal of L . Then every maximal chain of lm subalgebras of L is a flag in L .*

Proof. Let $0 = L_n < L_{n-1} < \cdots < L_0 = L$ be a maximal chain of lm subalgebras of L . Putting $U = L_{k-1}$, $T = 0$, $T_i = L_i$ ($0 \leq i \leq n$) in the proof of Proposition 3.12 shows that L_k is maximal and modular in L_{k-1} for $1 \leq k \leq n$. It follows that L_k is a quasi-ideal of L_{k-1} , and hence has codimension one in L_{k-1} for $1 \leq k \leq n$, yielding that our chain is a flag. ■

Corollary 3.17. *Let L be a Lie algebra over an algebraically closed field of characteristic zero or a restricted Lie algebra over an algebraically closed field of characteristic $p > 7$. Then every maximal chain of lm subalgebras of L is a flag in L .*

Proof. This follows from Corollary 3.16, Corollary 2.6 of [3] and Theorem 2.2 of [10]. ■

Few examples of modular subalgebras that are not quasi-ideals are known. The obvious ones are the one-dimensional subalgebras of a three-dimensional non-split simple Lie algebra. Apart from these the only known example is the standard maximal subalgebra \mathcal{H}_0 in the non-restricted simple Lie algebra of Cartan type $\mathcal{H} = \mathcal{H}(2 : \underline{1} : \Phi(\gamma))^{(1)}$ over an algebraically closed field of characteristic $p > 7$ (see section 3 of [10]). Also, $\mathcal{H}_0/\mathcal{H}_1 \cong \text{sl}(2)$ and \mathcal{H}_1 is nilpotent (see section 2 of [4]), so \mathcal{H} has a maximal chain of lower modular subalgebras. However, it has no flag of subalgebras by Corollary 2.5.

Proposition 3.18. (i) From $\cap\{M \leq L : M \text{ maximal and modular in } L\}$ to L there is a maximal series of lm subalgebras of L .

(ii) $\cap\{M \leq L : M \text{ maximal and modular in } L\}$ is lm in L .

(iii) From $\cap\{M \leq L : \dim(L/M) = 1\}$ to L there is a flag of subalgebras to L whenever $\cap\{M \leq L : \dim(L/M) = 1\} \neq L$.

(iv) $\cap\{M \leq L : \dim(L/M) = 1\}$ is strongly lm in L .

Proof. (i) Pick a maximal and modular subalgebra M of L . The result is clear if $M = \cap\{M \leq L : M \text{ maximal and modular in } L\}$, so assume that $M < \cap\{M \leq L : M \text{ maximal and modular in } L\}$. Then there is a modular maximal subalgebra K of L with $K \neq L$. The subalgebra $M \cap K$ is modular and maximal in M . Continuing in this way we get a maximal chain of lm subalgebras of L from $\cap\{M \leq L : M \text{ maximal and modular in } L\}$ to L .

(ii) This follows from (i) and Lemma 3.1(i).

(iii) This is similar to (i).

(iv) This follows from (iii) and Lemma 3.1(ii). ■

Over fields of characteristic zero, each of the above intersections is an ideal of L , by Corollary 3.2 of [5], but this is not necessarily the case over more general fields. Finally we can classify the algebras in \mathcal{Ch}_{lm} over a field of characteristic zero.

Theorem 3.19. Let L be a Lie algebra over a field F of characteristic zero. Then L has a maximal chain of lm subalgebras if and only if either L is solvable, or $L/R(L) \cong P_1 \oplus \cdots \oplus P_r$ where P_i is three-dimensional simple for each $1 \leq i \leq r$.

Proof. Suppose that L has a maximal chain of lm subalgebras and that L is not solvable. By Proposition 3.11, $L/R(L)$ has a maximal chain of lm subalgebras. Decompose $L/R(L)$ as $P_1 \oplus \cdots \oplus P_r$ where P_i is simple for each $1 \leq i \leq r$. It follows from Proposition 3.12 that each P_i has a maximal and modular subalgebra, and so is three dimensional, by [3].

The converse is straightforward. ■

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