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Branching of some Holomorphic Representations of SO(2,n)

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Abstract. In this paper we consider the analytic continuation of the weighted Bergman spaces on the Lie ball $\mathscr{D} = SO(2,n)/S(O(2) \times O(n))$ and the corresponding holomorphic unitary (projective) representations of $SO(2,n)$ on these spaces. These representations are known to be irreducible. Our aim is to decompose them under the subgroup $SO(1,n)$ which acts as the isometry group of a totally real submanifold $\mathscr{X}$ of $\mathscr{D}$. We give a proof of a general decomposition theorem for certain unitary representations of semisimple Lie groups. In the particular case we are concerned with, we find an explicit formula for the Plancherel measure of the decomposition as the orthogonalising measure for certain hypergeometric polynomials. Moreover, we construct an explicit generalised Fourier transform that plays the role of the intertwining operator for the decomposition. We prove an inversion formula and a Plancherel formula for this transform. Finally we construct explicit realisations of the discrete part appearing in the decomposition and also for the minimal representation in this family.
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Introduction

One of the main problems in the representation theory of Lie groups and harmonic analysis on Lie groups is to decompose some interesting representations of a Lie group G under a subgroup $H \subset G$. This decomposition is also called the *branching rule*. Among other things, this has led to the discovery of new interesting representations. An exposition of the general theory for compact connected Lie groups, including the classical results for $U(n)$ and $SO(n)$ (by Weyl and Murnaghan respectively), can be found in [12].

Since the work by R. Howe [7] and M. Kashiwara and M. Vergne (cf [10]), it has turned out to be fruitful to study the branching of singular and minimal holomorphic representations of a Lie group acting on a function space of holomorphic functions on a bounded symmetric domain. In [9], Jakobsen and Vergne

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study the restriction of the tensor product of two holomorphic representations to the diagonal subgroup.

In this paper we will study the branching of the analytic continuation of the scalar holomorphic discrete series of $SO(2,n)$ under the subgroup $H = SO_0(1,n)$. The subgroup H here is realised as the isometry group of a totally real submanifold of the Lie ball $SO(2,n)/S(O(2)\times O(n))$. The branching for a general Lie group G of Hermitian type under a symmetric subgroup H has been studied recently by Neretin ([19], [18]), Zhang ([28],[30],[29]) and by van Dijk and Pevzner [25]. In [14], Kobayashi and Ørsted studied the branching for some minimal representations. The branching rule for regular parameter and for some minimal representations is now well understood. However, the problem of finding the branching rule for non-discrete, non-regular parameter is a difficult one, and there is still no complete theory for the general case.

We find the branching rule for arbitrary scalar parameter ν in the Wallach set of $SO(2,n)$. It turns out that for small parameters ν there appears a discrete part in the decomposition. We discover here an intertwining operator realising the corresponding representation. It should be mentioned that for large parameter (in this case $\nu > n-1$) the corresponding branching problem has been solved by Zhang in [28] for arbitrary bounded symmetric domains.

The paper is organised as follows. In Section 1 we describe the geometry of the Lie ball. In Section 2 we recall some facts about general bounded symmetric domains and Jordan triple systems. In Section 3 we establish some facts about the real part of the Lie ball. In Section 4 we consider a family of function spaces and corresponding unitary representations. Section 5 is devoted to branching theorems and to finding the Plancherel measure. In Sections 6 and 7 we find realisations of the representations corresponding to the discrete part in the decomposition and to the minimal point in the Wallach set respectively.

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1. The Lie ball as a symmetric space $SO_0(2,n)/SO(2)\times SO(n)$

In this paper we study representations on function spaces on the domain

$$\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}. \tag{1}$$

We will only be concerned with the case $n > 2$. (If $n = 1$ it is the unit disk, U, and if $n = 2$, $\mathscr{D} \cong U \times U$). In this section we describe $\mathscr{D}$ as the quotient of $SO_0(2,n)$ by $SO(2)\times SO(n))$ by studying a holomorphically equivalent model on which we have a natural group action induced by the linear action on a submanifold of a Grassmanian manifold. Consider $\mathbb{R}^{n+2} \cong \mathbb{R}^2 \oplus \mathbb{R}^n$ equipped with the non-degenerate bilinear form

$$(x|y) := x_1y_1 + x_2y_2 - x_3y_3 - \ldots - x_{n+2}y_{n+2},$$

where the coordinates are with respect to the standard basis $e_1, \cdots, e_{n+2}$. Let $SO(2,n)$ be the group of all linear transformations on $\mathbb{R}^{n+2}$ that preserve this form and have determinant 1, i.e.,

$$SO(2,n) = \{g \in GL(2+n,\mathbb{R}) | (gx|gy) = (x|y), x, y \in \mathbb{R}^{2+n}, \det g = 1\} \tag{2}$$

Let $\mathcal{G}^+_{(2,n)}$ denote the set of all two-dimensional subspaces of $\mathbb{R}^2 \oplus \mathbb{R}^n$ on which $(\cdot|\cdot)$ is positive definite. Clearly $\mathbb{R}^2 \oplus \{0\}$ is one of these subspaces. It will be the reference point in $\mathcal{G}^+_{(2,n)}$ and we will denote it by V_0. The group $SO(2,n)$ acts naturally on this set and the action is transitive. In fact, the connected component of the identity, $SO_0(2,n)$ acts transitively. We will let G denote this group.

We denote by K the stabilizer subroup of V_0, i.e.,

$$K = \{g \in G | g(V_0) = V_0\}. \tag{3}$$

Any element $g \in G$ can be identified with a $(2+n) \times (2+n)$- matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4}$$

where A is a 2×2-matrix. With this identification, K clearly corresponds to the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix},$$

where A and D are orthogonal 2×2- and $n \times n$-matrices with deerminant one respectively, i.e., $K \cong SO(2) \times SO(n)$. The space $\mathcal{G}^+_{(2,n)}$ can be realised as the unit ball in $M_{n2}(\mathbb{R})$ with the operator norm. Indeed, let $V \in \mathcal{G}^+_{(2,n)}$. If $v = v_1 + v_2 \in V$, then $v_1 = 0$ implies that $v_2 = 0$, i.e., the projection $v \mapsto v_1$ is an injective mapping. This means that there is a real $n \times 2$ matrix Z with $Z^t Z < I_2$, such that

$$V = \{(v \oplus Zv) | v \in \mathbb{R}^2\}. \tag{5}$$

Conversely, if $Z \in M_{n2}(\mathbb{R})$ satisfies $Z^t Z < I_2$, then (5) defines an element in $\mathcal{G}^+_{(2,n)}$.

Using (4) to identify g with a matrix and letting V correspond to the matrix Z, then clearly

$$\begin{aligned} gV &= \{(Av + BZv \oplus Cv + DZv) | v \in \mathbb{R}^2)\} \\ &= \{v \oplus (C + DZ)(A + BZ)^{-1}v) | v \in \mathbb{R}^2)\}. \end{aligned}$$

In other words, we have a G-action on the set

$$M = \{Z \in M_{n2}(\mathbb{R}) | Z^t Z < I_2\}$$

given by

$$Z \mapsto (C + DZ)(A + BZ)^{-1}.$$

This exhibits M as a symmetric space.

$$M \cong G/K.$$

Moreover, we identify the matrix $Z = (XY)$ with the vector $X + iY$ in $\mathbb{C}^n$ in order to obtain an almost complex structure on M. With respect to this almost complex structure, the action of G is in fact holomorphic. Moreover we have the following result by Hua (see [8]).

Theorem 1. *The mapping*

$$\mathcal{H} : z \mapsto Z = 2\left(\begin{pmatrix} zz^t + 1 & i(zz^t - 1) \\ \overline{zz}^t + 1 & -i(\overline{zz}^t - 1) \end{pmatrix}^{-1} \begin{pmatrix} z \\ \overline{z} \end{pmatrix}\right)^t,$$

where $zz^t = z_1^2 + \cdots + z_n^2$, is a holomorphic diffeomorphism of the bounded domain

$$\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}$$

onto M.

We will call this mapping the *Hua transform.* It allows us to describe $\mathscr{D}$ as a symmetric space

$$\mathscr{D} \cong M \cong G/K.$$

2. Bounded symmetric domains and Jordan pairs

In this section we review briefly some general theory on bounded symmetric domains and Jordan pairs. All proofs are omitted. For a more detailed account we refer to Loos ([15]) and to Faraut-Koranyi ([2]).

Let $\mathcal{D}$ be a bounded open domain in $\mathbb{C}^n$ and $\mathcal{H}^2(\mathcal{D})$ be the Hilbert space of all square integrable holomorphic functions on $\mathcal{D}$,

$$\mathcal{H}^2(\mathcal{D}) = \{f, f \text{ holomorphic on } \mathcal{D} \,|\, \int_{\mathcal{D}} |f(z)|^2 dm(z) < \infty\},$$

where m is the $2n$-dimensional Lebesgue measure. It is a closed subspace of $L^2(\mathcal{D})$. For every $w \in \mathcal{D}$, the evaluation functional $f \mapsto f(w)$ is continuous, hence $\mathcal{H}^2(\mathcal{D})$ has a reproducing kernel $K(z, w)$, holomorphic in z and antiholomorphic in w such that

$$f(w) = \int_{\mathcal{D}} f(z)\overline{K(z, w)} dm(z).$$

$K(z, w)$ is called the Bergman kernel. It has the transformation property

$$K(\varphi(z), \varphi(w)) = J_\varphi(z)^{-1} K(z, w) \overline{J_\varphi(w)}^{-1}, \tag{6}$$

for any biholomorphic mapping φ on $\mathcal{D}$ with complex Jacobian $J_\varphi(z) = \det d\varphi(z)$. Hereafter biholomorphic mappings will be referred to as automorphisms. The formula

$$h_z(u, v) = \partial_u \partial_{\overline{v}} \log K(z, z) \tag{7}$$

defines a Hermitian metric, called the Bergman metric. It is invariant under automorphisms and its real part is a Riemannian metric on $\mathcal{D}$.

A bounded domain $\mathcal{D}$ is called *symmetric* if, for each $z \in \mathcal{D}$ there is an involutive automorphism s_z with z as an isolated fixed point. Since the group of automorphisms, $Aut(\mathcal{D})$ preserves the Bergman metric, s_z coincides with the local geodesic symmetry around z. Hence $\mathcal{D}$ is a Hermitian symmetric space.

A domain $\mathcal{D}$ is called *circled* (with respect to 0) if $0 \in \mathcal{D}$ and $e^{it}z \in \mathcal{D}$ for every $z \in \mathcal{D}$ and real t.

Every bounded symmetric domain is holomorphically isomorphic with a bounded symmetric and circled domain. It is unique up to linear isomorphisms.

From now on $\mathcal{D}$ denotes a circled bounded symmetric domain. G is the identity component of $Aut(\mathcal{D})$, K is the isotropy group of 0 in G. The Lie algebra $\mathfrak{g}$ will be considered as a Lie algebra of holomorphic vector fields on $\mathcal{D}$, i.e., vector fields X on $\mathcal{D}$ such that Xf is holomorphic if f is. The symmetry $s, z \mapsto -z$ around the origin induces an invoulution on G by $g \mapsto sgs^{-1}$ and, by differentiating, an involution $Ad(s)$ of $\mathfrak{g}$. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into the ± 1-eigenspaces.

For every $v \in \mathbb{C}^n$, let ξ_v be the unique vector field in $\mathfrak{p}$ that takes the value v at the origin. Then

$$\xi_v(z) = v - Q(z)\overline{v} \tag{8}$$

where $Q(z) : \overline{V} \to V$ is a complex linear mapping and $Q : V \to \mathrm{Hom}(\overline{V}, V)$ is a homogeneous quadratic polynomial. Hence $Q(x,z) = Q(x+z) - Q(x) - Q(z) : \overline{V} \to V$ is bilinear and symmetric in x and z. For $x, y, z \in V$, we define

$$\{x\overline{y}z\} = D(x, \overline{y})z = Q(x,z)\overline{y} \tag{9}$$

Thus $\{x\overline{y}z\}$ is complex bilinear and symmetric in x and z and complex antilinear in y, and $D(x,\overline{y})$ is the endomorphism $z \mapsto \{x\overline{y}z\}$ of V.

The pair $(V, \{\ \})$ is called a *Jordan triple system.* This Jordan triple system is positive in the sense that if $v \in V, v \neq 0$ and $Q(v)\overline{v} = \lambda v$ for some $\lambda \in \mathbb{C}$, then λ is positive. We introduce the endomorphisms

$$B(x,y) = I - D(x,\overline{y}) + Q(x)\overline{Q}(\overline{y}) \tag{10}$$

of V for $x, y \in V$, where $\overline{Q}(\overline{y})x = \overline{Q(y)\overline{x}}$. We summarise some results in the following proposition.

Proposition 2. *a) The Lie algebra $\mathfrak{g}$ satisfies the relations*

$$[\xi_u, \xi_v] = D(u,\overline{v}) - D(v,\overline{u}) \tag{11}$$

$$[l, \xi_u] = \xi_{lu} \tag{12}$$

for $u, v \in V$ and $l \in \mathfrak{k}$

b) The Bergman kernel $k(x,y)$ of $\mathcal{D}$ is

$$m(\mathcal{D})^{-1} \det B(x,y)^{-1} \tag{13}$$

c) The Bergman metric at 0 is

$$h_0(u,v) = trD(u,\overline{v}), \tag{14}$$

and at an arbitrary point $z \in \mathcal{D}$

$$h_z(u,v) = h_0(B(z,z)^{-1}u, v) \tag{15}$$

d) The triple product $\{\ \}$ is given by

$$h_0(\{u\overline{v}w\}, y) = \partial_u \partial_{\overline{v}} \partial_x \partial_{\overline{y}} \log K(z,z)|_{z=0} \tag{16}$$

We define odd powers of an element $x \in V$ by

$$x^1 = x,\ x^3 = Q(x)\overline{x}, \cdots, x^{2n+1} = Q(x)\overline{x^{2n-1}}.$$

An element $x \in V$ is said to be *tripotent* if $x^3 = x$, i.e., if $\{x\overline{x}x\} = 2x$. Two tripotents c and e are called orthogonal if $D(c, \overline{e}) = 0$. In this case $D(c, \overline{c})$ and $D(e, \overline{e})$ commute and $e + c$ is a tripotent.

Every $x \in V$ can be written uniquely

$$x = \lambda_1 c_1 + \cdots + \lambda_n c_n,$$

where the c_i are pairwise orthogonal nonzero tripotents which are real linear combinations of odd powers of x, and the λ_i satisfy

$$0 < \lambda_1 < \cdots < \lambda_n.$$

This expression for x is called its *spectral decomposition* and the λ_i the eigenvalues of x. Moreover, the domain $\mathcal{D}$ can be realised as the unit ball in V with the spectral norm

$$\|x\| = \max |\lambda_i|,$$

where the λ_i are the eigenvalues of x, i.e.,

$$\mathcal{D} = \{x \in V | \|x\| < 1\}.$$

Let $f(t)$ be an odd complex valued function of the real variable t, defined for $|t| < \rho$. For every $x \in V$ with $|x| < \rho$ we define $f(x) \in V$ by

$$f(x) = f(\lambda_1)c_1 + \cdots + f(\lambda_n)c_n, \tag{17}$$

where $x = \lambda_1 c_1 + \cdots + \lambda_n c_n$ is the spectral resolution of x. This functional calculus is used in expressing the action on $\mathcal{D}$ of the elements $\exp \xi_v$ in G:

$$\exp \xi_v(z) = u + B(u,u)^{1/2} B(z, -u)^{-1}(z + Q(z)\overline{u}) \tag{18}$$

and

$$d(\exp \xi_v)(z) = B(u,u)^{1/2} B(z, -u)^{-1}, \tag{19}$$

where $u = \tanh v$, for $v \in \mathbb{C}^n$ and $z \in \mathcal{D}$.

3. The real part of the Lie ball

We consider the non-degenerate quadratic form

$$q(z) = z_1^2 + \cdots + z_n^2 \tag{20}$$

on $V = \mathbb{C}^n$. In the following we will often denote $q(z, w)$ by (z, w). Defining $Q(x)y = q(x, y)x - q(x)y$, where $q(x, y) = q(x + y) - q(x) - q(y)$, we get a Jordan triple system. The Lie ball $\mathscr{D} = \{z \in \mathbb{C}^n | 1 - 2\langle z, z\rangle + |zz^t|^2 > 0, |z| < 1\}$ is the open unit ball in this Jordan triple system. An easy computation shows the following identity.

$$D(x,\overline{y})z = 2(\sum_{k=1}^{n} x_k\overline{y_k})z + 2(\sum_{k=1}^{n} z_k\overline{y_k})x - 2(\sum_{k=1}^{n} x_k z_k)\overline{y}$$

Recalling that $B(x,y) = I - D(x,\overline{y}) + Q(x)\overline{Q}(\overline{y})$. The Bergman kernel of $\mathscr{D}$ is

$$K(z,w) = (1 - 2\langle z,w\rangle + (zz^t)\overline{(ww^t)})^{-n}. \tag{21}$$

We will hereafter denote it by $h(z,w)^{-n}$. Consider the real form $\mathbb{R}^n$ in $\mathbb{C}^n$. Observe that

$$\mathscr{X} := \mathscr{D}\bigcap\mathbb{R}^n$$

is the unit ball of $\mathbb{R}^n$. On $\mathscr{X}$ we have a simple expression for the Bergman metric:

$$B(x,x) = (1-|x|^2)^{-2}I, x \in \mathscr{X}. \tag{22}$$

The submanifold $\mathscr{X}$ is a totally real form of $\mathscr{D}$ in the sense that

$$T_x(\mathscr{X}) + iT_x(\mathscr{X}) = T_x(\mathscr{D}),\ T_x(\mathscr{X})\bigcap iT_x(\mathscr{X}) = \{0\}$$

This implies that every holomorphic function on $\mathscr{D}$ that vanishes on $\mathscr{X}$ is identically zero. We define the subgroup H as the identity component of

$$\{h \in G | h(x) \in \mathscr{X} \text{ if } x \in \mathscr{X}\}$$

We will denote $H\bigcap K$ by L.

Using the fact that the real form $\mathbb{R}^n$ is a sub-triple system of $\mathbb{C}^n$, one can show that $\mathscr{X}$ is a totally geodesic submanifold of $\mathscr{D}$ (cf Loos [15]). Hence we can describe $\mathscr{X}$ as a symmetric space

$$\mathscr{X} \cong H/L.$$

We now study the image of $\mathscr{X}$ in the $M_{n2}(\mathbb{R})$- model of the Lie ball. For computational convenience, we now work with the transposes of these matrices. The defining equation of the Hua-transform can be written as

$$\frac{1}{2}\begin{pmatrix} zz^t+1 & i(zz^t-1) \\ \overline{zz}^t+1 & -i(\overline{zz}^t-1) \end{pmatrix} Z = \begin{pmatrix} z \\ \overline{z} \end{pmatrix} \tag{23}$$

In the coordinates $(z_1,\ldots,z_n)$ of z, this identity takes the form

$$z_k = \frac{1}{2}(\ (zz^t+1)x_k + i(zz^t-1)y_k). \tag{24}$$

This gives

$$4zz^t = (zz^t)^2(X+iY)(X+iY)^t + 2(XX^t+YX^t)zz^t \tag{25}$$

$$+(X-iY)(X-iY)^t, \tag{26}$$

which is a quadratic equation in zz^t with unique solution

$$zz^t = \frac{2-(XX^t+YY^t)-2\sqrt{(1-XX^t)(1-YY^t)-(YX^t)^2}}{(X+iY)(X+iY)^t}. \tag{27}$$

From (24) we see that if z is real, then $y_k = 0$ for all k. On the other hand, if $Y=0$, then (27) shows that zz^t is real and therefore z is real by (24). Hence the image of the real part $\mathscr{X} \subset \mathscr{D}$ under the Hua-transform is the set

$$\mathcal{H}(\mathscr{X}) = \{Z = (X\,0)\,|\,X \in M_{n1}(\mathbb{R}), |X| < 1\}, \tag{28}$$

since for an element $Z = (X\,0)$, the condition that $Z^tZ < I_2$ is clearly equivalent with $|X| < 1$.

Recall that the real n-dimensional unit ball can be described as a symmetric space $SO_0(1,n)/SO(n)$ by a procedure analogous to the one in the first section. One first considers all lines in $\mathbb{R}^{1+n}$ on which the quadratic form $x_1^2 - x_2^2 - \cdots - x_{n+1}^2$ is positive definite and identifies these lines with all real $n\times 1$-matrices with norm less than one. If we write elements $g \in SO(1,n)$ as matrices of the form

$$g = \begin{pmatrix} a & - & b & - \\ | & & & \\ c & & D & \\ | & & & \end{pmatrix}, \tag{29}$$

the action is given by

$$X \mapsto (c+DX)(a+bX)^{-1}. \tag{30}$$

The group $SO(1,n)$ can be embedded into $SO(2,n)$. Indeed, the equality

$$\begin{pmatrix} a & 0 & - & b & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ c & 0 & & D & \\ | & | & & & \end{pmatrix} \begin{pmatrix} a' & 0 & - & b' & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ c' & 0 & & D' & \\ | & | & & & \end{pmatrix}$$

$$= \begin{pmatrix} aa'+bc' & 0 & - & ab'+bD' & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ ca'+Dc' & 0 & & cb'+DD' & \\ | & | & & & \end{pmatrix}$$

shows that we can define an injective homomorphism $\theta : SO(1,n) \to SO(2,n)$ by

$$\theta : \begin{pmatrix} a & - & b & - \\ | & & & \\ c & & D & \\ | & & & \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & - & b & - \\ 0 & 1 & - & 0 & - \\ | & | & & & \\ c & 0 & & D & \\ | & | & & & \end{pmatrix}.$$

This subgroup acts on $\mathcal{H}(\mathscr{X})$ as

$$(X\,0) \mapsto \big((c+DX)(a+bX)^{-1}\;0\big)$$

and the action is transitive. Suppose now that $h \in SO(2,n)$ preserves $H(\mathscr{X})$. Let $p = h(0)$. We can choose a $g \in SO_0(1,n)$ such that $g(0) = p$ (here we identify g with $\theta(g)$). Then $g^{-1}h(0) = 0$ and hence we can write it in block form as

$$g^{-1}h = \begin{pmatrix} I_2 & 0 \\ 0 & D \end{pmatrix},$$

with $D \in SO(n)$. This is an element in $\theta(SO(1,n))$ and hence $h \in \theta(SO(1,n))$. We have now proved the following theorem.

Theorem 3. *The Hua transform $\mathcal{H} : \mathscr{D} \to M$ maps the real part $\mathscr{X}$ diffeomorphically onto*

$$\mathcal{H}(\mathscr{X}) = \{Z = (X\,0)\,|\,X \in M_{n1}(\mathbb{R}), |X| < 1\} \tag{31}$$

by $x \mapsto \frac{2x}{1+|x|^2}$. Moreover, the induced group homomorphism $h \mapsto \mathcal{H}\,h\,\mathcal{H}^{-1}$ is an isomorphism between the groups H and $SO_0(1,n)$

Remark 3.1. The model $\mathcal{H}(\mathscr{X})$ of $SO_0(1,n)/SO(n)$ is the real part of the complex n-dimensional unit ball $SU(1,n)/SU(n)$ with fractional- linear group action. It is therefore equipped with a Riemannian metric given by the restriction of the Bergman metric of the complex unit ball. If $x \in \mathcal{H}(\mathscr{X}), x \neq 0$, we decompose $\mathbb{R}^n = \mathbb{R}x \oplus (\mathbb{R}x)^{\perp}$. We let $v = v_x + v_{x^\perp}$ be the corresponding decomposition of a tangent vector v at x. In this model, the Riemannian metric at x is (cf [21])

$$g_x(v,v) = \frac{|v_x|^2}{(1-|x|^2)^2} + \frac{|v_{x^\perp}|^2}{(1-|x|^2)}.$$

We recall from equation (22) that if $x \in X$, then the Riemannian metric at x is

$$h_x(v,v) = \frac{1}{2n}\frac{|v|^2}{(1-|x|^2)^2}.$$

The Hua transform thus induces an isometry (up to a constant) of the real n-dimensional unit ball equipped with two different Riemannian structures.

3.1. Iwasawa decomposition of $\mathfrak{h}$. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ induces a decomposition $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{q}$. We let

$$\mathfrak{a} = \mathbb{R}\xi_e$$

be the one-dimensional subspace of $\mathfrak{q}$, where $e = e_1$ denotes the first standard basis vector and the corresponding vector field ξ_e is defined in (8).

Proposition 4. *The Lie algebra $\mathfrak{h}$ has rank one, and the roots with respect to the abelian subalgebra $\mathfrak{a}$ of $\mathfrak{q}$ are $\{\alpha, -\alpha\}$, where $\alpha(\xi_e) = 2$. The corresponding posive root space is*

$$\mathfrak{q}_\alpha = \{\xi_v + \frac{1}{2}(D(e,v) - D(v,e)) | v \in \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_n\}$$

Proof. This is known in a general context, but we give here an elementary proof.

Take u and v in $\mathbb{R}^n$ and assume that $[\xi_u, \xi_v] = 0$. Then, for any $x \in \mathbb{R}^n$ we have

$$D(u,v)x = D(v,u)x.$$

A simple calculation shows that this amounts to

$$(u,x)v = (v,x)u,$$

which can only hold for all real x if $u = v$.

Thus $\mathfrak{a}$ is a maximal abelian subalgebra in $\mathfrak{q}$. The vector e is a maximal tripotent in the Jordan triple system corresponding to $\mathscr{D}$. Suppose that $[\xi_e, \xi_v + l] = \alpha(\xi_e)(\xi_v + l)$. Identifying the q- and l-components yields

$$D(e,v) - D(v,e) = \alpha(\xi_e)l \tag{32}$$

$$-\xi_{le} = \alpha(\xi_e)\xi_v \tag{33}$$

From (33) it follows that $le = -\alpha(\xi_e)v$ and, thus, applying both sides of (32) to e gives

$$D(e,v)e - D(v,e)e = -\alpha(\xi_e)^2 v,$$

i.e.,

$$D(e,e)v - D(e,v)e = \alpha(\xi_e)^2 v,$$

An easy computation gives

$$4v - 4(e,v)e = \alpha(\xi_e)^2 v.$$

Hence e is orthogonal to v and $\alpha(\xi_e)^2 = 4$. The rest follows immediately. ■

We shall fix the positive root α. Elements in $\mathfrak{a}_{\mathbb{C}}^*$ are of the form $\lambda\alpha$ and will hereafter be identified with the complex numbers λ. In particular, the half sum of the positive roots (with multiplicities), ρ, will be identified with the number $(n-1)/2$.

3.2. The Cayley transform. The Cayley transform is a biholomorphic mapping from a bounded symmetric domain onto a *Siegel domain*. We describe it for the domain $\mathscr{D}$ and use it to express the spherical functions on $\mathscr{X}$ in terms of the spherical functions on the unbounded domain. We fix the maximal tripotent e. Then $\mathbb{C}^n$ equipped with the bilininear mapping

$$(z,w) \mapsto z \circ w = \frac{1}{2}\{zew\} \tag{34}$$

is a complex Jordan algebra. Observe that since e is a tripotent, it is a unity for this multiplication. The Cayley transform is the mapping $c : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$c(z) = (e+z) \circ (e-z)^{-1}, \tag{35}$$

where $(e-z)^{-1}$ denotes the inverse of $(e-z)$ with respect to the Jordan product.

Proposition 5. *The Cayley transform is given by the formula*

$$c(z) = \frac{1 - zz^t}{1 - 2z_1 + (zz^t)^2} e + \frac{2z'}{1 - 2z_1 + (zz^t)^2}, \tag{36}$$

for $z = (z_1, z') = z_1 e + z' \in \mathscr{D}$*. Moreover, it maps* $\mathscr{X}$ *onto the halfspace*

$$\{(x_1, \ldots, x_n\} \in \mathbb{R}^n | x_1 > 0\}.$$

Proof. We first find the inverse for an element x. Suppose therefore that $e = \frac{1}{2}\{xez\} = \frac{1}{2}D(x,e)z$, i.e.,

$$e = (x,e)z + (z,e)x - (x,z)e = x_1 z + z_1 x - (x,z)e$$

Identifying coordinates gives

$$\begin{aligned} 1 &= 2x_1 z_1 - (x,z) \\ 0 &= x_1 z' + z_1 x' \end{aligned}$$

These equations have the solution

$$\begin{aligned} z_1 &= x_1/(x,x) \\ z' &= -x'/(x,x). \end{aligned}$$

If we apply this to the expression $(e-z)^{-1}$ in the definition of c, we get

$$(e-z)^{-1} = \frac{1 - z_1}{(1-z_1)^2 + (z', z')} e + \frac{z'}{(1-z_1)^2 + (z', z')}.$$

Now the formula (36) follows by an easy computation. Moreover, we observe that the inverse transform is given by

$$w \mapsto (w - e) \circ (w + e)^{-1} = -c(-w).$$

Hence both c and c^{-1} preserve $\mathbb{R}^n$ and therefore

$$c(\mathscr{X}) = c(\mathscr{D}) \bigcap \mathbb{R}^n.$$

We now determine $c(\mathscr{X})$.

From ([15]) we know that (since e is a maximal tripotent)

$$c(\mathscr{D}) = \{u + iv | u \in A^+, v \in A\}, \tag{37}$$

where A is the real Jordan algebra

$$\{z \in V | Q(e)\overline{z} = z\}$$

and A^+ is the positive cone $\{z \circ z | z \in A\}$ in A. By a simple computation we see that

$$A = \mathbb{R}e \oplus \mathbb{R}ie_2 \oplus \cdots \oplus \mathbb{R}ie_n.$$

Since we have the identities

$$z + Q(e)\overline{z} = 2u,$$

$$z - Q(e)\overline{z} = 2iv$$

and

$$Q(e)\overline{z} = 2\overline{z_1} - \overline{z},$$

we get expressions for u and v:

$$2u = (z_1 + \overline{z_1}, z_2 - \overline{z_2}, \ldots, z_n - \overline{z_n})$$

$$2iv = (z_1 - \overline{z_1}, z_2 + \overline{z_2}, \ldots, z_n + \overline{z_n})$$

The condition that $x = u + iv$ be in the image of $\mathscr{X}$ thus implies that

$$u = (x_1, 0, \ldots, 0),$$

$$iv = (0, x_2, \ldots, x_n).$$

Moreover we require that

$$u = w \circ w = 2w_1 w - (w, w)e,$$

for some

$$w = c_1 e + c_2 i e_2 + \cdots + c_n i e_n.$$

This yields

$$(x_1, \ldots, 0) = (c_1^2 + \cdots + c_n^2, ic_1c_2, \ldots, ic_1c_n).$$

Hence

$$c_1^2 = x_1, c_2 = \cdots = c_n = 0,$$

and thus

$$u + iv = (c_1^2, x_2, \ldots, x_n).$$

This proves the claim. ∎

Recall the expression for the spherical functions on a symmetric space of noncompact type (cf [6] Thm 4.3)

$$\varphi_\lambda(h) = \int_L e^{(i\lambda+\rho)A(lh)} dl,$$

where $A(lh)$ is the (logarithm) of the A part of lh in the Iwasawa decomposition $H = NAL$. The integrand in this formula is called the *Harish-Chandra e-function.* For the above Siegel domain it has the form $e_\lambda(w) = (w_1)^{i\lambda+\rho}$ (cf [24]). Hence we have the following corollary.

Corollary 6. *The spherical function φ_λ on $\mathscr{X} = H/L$ is*

$$\varphi_\lambda(x) = \int_{S^{n-1}} \left(\frac{1 - |x|^2}{1 - 2(x, \zeta) + xx^t} \right)^{i\lambda+\rho} d\sigma(\zeta). \tag{38}$$

where σ is the $O(n)$-invariant probability measure on S^{n-1}.

4. A family of unitary representations of G

4.1. The function spaces $\mathscr{H}_\nu$. The Bergman space $\mathcal{H}^2(\mathscr{D})$ has the reproducing kernel $h(z,w)^{-n}$. This means in particular that the function $h(z,w)^{-n}$ is positive definite in the sense that

$$\sum_{i,j=1}^{m} \alpha_i \overline{\alpha_j} h(z_i, z_j)^{-n} \geq 0,$$

for all $z_1, \ldots, z_n \in \mathscr{D}$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. It has been proved by Wallach ([26]) and Rossi-Vergne ([20]) that $h(z,w)^{-\nu}$ is positive definite precisely when ν in the set

$$\{0, (n-2)/2\} \bigcup ((n-2)/2, \infty)$$

This set will also be referred as the *Wallach set* (cf [3]). For ν in the Wallach set above, $h(z,w)^{-\nu}$ is the reproducing kernel of a Hilbert space of holomorphic functions on $\mathscr{D}$. We will call this space $\mathscr{H}_\nu$ and the reproducing kernel $K_\nu(z,w)$. The mapping $g \mapsto \pi_\nu(g)$, where

$$\pi_\nu(g) f(z) = J_{g^{-1}}(z)^{\frac{\nu}{n}} f(g^{-1}z)$$

defines a unitary projective representation of G on $\mathscr{H}_\nu$. Indeed, comparison with the Bergman kernel shows that $h(z,w)^{-\nu}$ transforms under automorphisms according to the rule

$$h(gz, gw)^{-\nu} = J_g(z)^{-\frac{\nu}{n}} h(z,w)^{-\nu} \overline{J_g(w)}^{-\frac{\nu}{n}}. \tag{39}$$

Recall that for functions f_1 and f_2 of the form

$$f_1(z) = \sum_{k=1}^{l} \alpha_k K_\nu(z, w_k), \;\; f_2(z) = \sum_{k=1}^{m} \beta_k K_\nu(z, w'_k),$$

the inner product is defined as

$$\langle f_1, f_2 \rangle_\nu = \sum_{i,j} \alpha_i \overline{\beta_j} K_\nu(w_i, w'_j) \tag{40}$$

Equation (39) implies that

$$K_\nu(g^{-1}z, w) = J_{g^{-1}}(z)^{-\frac{\nu}{n}} K_\nu(z, gw) \overline{J_{g^{-1}}(w)}^{-\frac{\nu}{n}}. \tag{41}$$

Hence we have the following two equalities

$$\pi_\nu(g) f_1(z) = \sum_{k=1}^{l} \alpha_k \overline{J_{g^{-1}}(w_k)}^{-\frac{\nu}{n}} K_\nu(z, gw_k)$$
$$\pi_\nu(g) f_2(z) = \sum_{k=1}^{m} \beta_k \overline{J_{g^{-1}}(w'_k)}^{-\frac{\nu}{n}} K_\nu(z, gw'_k).$$

The unitarity

$$\langle \pi_\nu(g) f_1, \pi_\nu(g) f_2 \rangle_\nu = \langle f_1, f_2 \rangle_\nu$$

now follows by an application of the transformation rule (39) in the definition (40). Since functions of the form above are dense in $\mathscr{H}_\nu$, it follows that each $\pi_\nu(g)$ is a unitary operator and it is easy to see that $g \mapsto \pi_\nu(g)$ is a projective homomorphism of groups. In fact, π_ν is an irreducible projective representation, cf [2].

4.2. Fock-Fischer spaces. It can be shown that for $\nu > (n-2)/2$ all holomorphic polynomials are in $\mathscr{H}_\nu$ and that polynomials of different homogeneous degree are orthogonal. In this context, the spaces $\mathscr{H}_\nu$ are closely linked with the *Fock-Fischer space*, $\mathscr{F}$, which we will now describe. The basis vector e_1 is a maximal tripotent which is decomposed into minimal tripotents as $e_1 = \frac{1}{2}(1, i, 0, \ldots, 0) + \frac{1}{2}(1, -i, 0, \ldots, 0)$. (We omit the easy computations.) In order to expand the reproducing kernel K_ν into a power series consistent with the treatment in [2], we need to introduce a new norm on $\mathbb{C}^n$ so that the minimal tripotents have norm 1, i.e., the Euclidean norm multiplied with $\sqrt{2}$. Then

$$\{f_1, \ldots, f_n\} := \left\{ \frac{1}{\sqrt{2}} e_1, \ldots, \frac{1}{\sqrt{2}} e_n \right\}$$

is an orthonormal basis with respect to this new norm. We write points $z \in \mathscr{D}$ as $z = w_1 f_1 + \cdots + w_n f_n$. For polynomials $p(w) = \sum_\alpha a_\alpha w^\alpha$, we define

$$p^*(w) = \sum_\alpha \overline{a_\alpha} w^\alpha.$$

The Fock-Fischer inner product is now defined as

$$\langle p, q \rangle_{\mathscr{F}} = p(\partial)(q^*)|_{w=0},$$

where $p(\partial)$ is the differential operator $\sum_\alpha a_\alpha \frac{\partial^\alpha}{\partial w^\alpha}$, for p as above. The Fock-Fischer space, $\mathscr{F}$, is the completion of the space of polynomials. It is easy to see that polynomials of different homogeneous degree are orthogonal in $\mathscr{F}$. Moreover, the representation of $SO(n)$ on $\mathcal{P}^m$, the polynomials of homogeneous degree m, can be decomposed into irreducible subspaces as

$$\mathcal{P}^m = \bigoplus_{m-2k \geq 0} E_{m-2k} \otimes \mathbb{C}(ww^t)^k, \tag{42}$$

where E_i are the spherical harmonic polynomials of degree i (cf [23]). This is a special case of the general Hua-Schmid decomposition (cf [2]).The following relation holds between the Fock-Fischer norm and the $\mathscr{H}_\nu$-norm on the space $E_{m-2k} \otimes \mathbb{C}(ww^t)^k$ (cf [2]).

$$\|p\|_\nu^2 = \frac{\|p\|_{\mathscr{F}}^2}{(\nu)_{m-k}(\nu - \frac{n-2}{2})_k}, \tag{43}$$

for $p \in E_{m-2k} \otimes \mathbb{C}(ww^t)^k$. We have the following decomposition of $\mathscr{H}_\nu$ under K:

Proposition 7. *(Faraut-Korànyi, [2]) a) If $\nu > \frac{n-2}{2}$, then*

$$\mathscr{H}_\nu|_K = \bigoplus \sum_{m-2k\geq 0} E_{m-2k} \otimes \mathbb{C}(zz^t)^k, \tag{44}$$

where E_{m-2k} is the space of spherical harmonic polynomials of degree $m-2k$. Moreover, we have the following expansion of the kernel function:

$$h(z,w)^{-\nu} = \sum_{m-2k\geq 0} (\nu)_{m-k} \left(\nu - \frac{n-2}{2}\right)_k K_{(m-k,k)}(z,w), \tag{45}$$

where $K_{(m-k,k)}$ is the reproducing kernel for the subspace $E_{m-2k} \otimes \mathbb{C}(zz^t)^k$ with the Fock-Fischer norm. The series converges in norm and uniformly on compact sets of $\mathscr{D} \times \mathscr{D}$.
b) If $\nu = \frac{n-2}{2}$, then

$$\mathscr{H}_\nu|_K = \bigoplus \sum_m E_m \tag{46}$$

We will later need the norm of $(zz^t)^k$ in $\mathscr{H}_\nu$.

Proposition 8.

$$\|(zz^t)^k\|_\nu^2 = \frac{k!\left(\frac{n}{2}\right)_k}{(\nu)_k\left(\nu - \frac{n-2}{2}\right)_k} \tag{47}$$

Proof. A straightforward computation shows that

$$(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2})(z_1^2 + \cdots + z_n^2)^k = (2^2k(k-1) + n2k)(z_1^2 + \cdots + z_n^2)^{k-1}$$

Proceeding inductively, we obtain

$$\begin{aligned}\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)^k (z_1^2 + \cdots + z_n^2)^k &= \prod_{j=1}^k 2j(2(j-1)+n) \\ &= 4^k k! \left(\frac{n}{2}\right)_k\end{aligned}$$

The Fock-Fischer norm is computed in the w-coordinates $w_i = \sqrt{2} z_i$, so

$$(zz^t)^k = 2^{-k}(ww^t)^k$$

and

$$\left(\frac{\partial^2}{\partial z_1^2} + \cdots + \frac{\partial^2}{\partial z_n^2}\right)^k = 2^{-k}\left(\frac{\partial^2}{\partial w_1^2} + \cdots + \frac{\partial^2}{\partial w_n^2}\right)^k.$$

Hence

$$\|(zz^t)^k\|_{\mathscr{F}}^2 = k!\left(\frac{n}{2}\right)_k$$

and an application of Prop. 7 gives the result. ■

5. Branching of π_ν under the subgroup H

5.1. A decomposition theorem. Recall the irreducible (projective) representations π_ν from the previous section. Our main objective is to decompose these into irreducible representations under the subgroup H. The fact that $\mathscr{X}$ is a totally real form is reflected in the restrictions of the representations π_ν to H.

Proposition 9. *The constant function* 1 *is in* $\mathscr{H}_\nu$ *and is an* L*-invariant cyclic vector for the representation* $\pi_\nu : H \to \mathscr{U}(\mathscr{H}_\nu)$.

Proof. First note that

$$\begin{aligned} K_\nu(z, h0) &= J_h(h^{-1}z)^{-\nu/n} K_\nu(h^{-1}z, 0) \overline{J_h(0)^{-\nu/n}} \\ &= \overline{J_h(0)^{-\nu/n}} J_{h^{-1}}(z)^{\nu/n} K_\nu(h^{-1}z, 0) \\ &= \overline{J_h(0)^{-\nu/n}} \pi_\nu(h) 1(z) \end{aligned}$$

Suppose now that the function $f \in \mathscr{H}_\nu$ is orthogonal to the linear span of the elements $\pi_\nu(h)1, h \in H$. By the above identity we have

$$\begin{aligned} f(h0) &= \langle f, K_\nu(\cdot, h0) \rangle_\nu \\ &= 0. \end{aligned}$$

Since H acts transitively on $\mathscr{X}$, f is zero on $\mathscr{X}$. Hence it is identically zero. ∎

We want decompose the representation of H into a *direct integral* of irreducible representations. For the definition of a direct integral over a measurable field of Hilbert spaces we refer to Naimark ([17]). The following general decomposition theorem is stated in several references (e.g. [19]), but the author has not been able to find a proof of it in the literature. A proof for abelian groups can be found in [17]. The proof we present below is based on the Gelfand-Naimark representation theory for C^*-algebras.

Theorem 10. *Let* π *be a unitary representation of the semisimple Lie group* H *on a Hilbert space,* $\mathscr{H}$. *Suppose further that* L *is a maximal compact subgroup and that the representation has a cyclic* L*-invariant vector. Then* π *can be decomposed as a multiplicity-free direct integral of irreducible representations,*

$$\pi \cong \int_\Lambda \pi_\lambda \, d\mu(\lambda), \tag{48}$$

where Λ *is a subset of the set of positive definite spherical functions on* H *and for* $\lambda \in \Lambda$, π_λ *is the corresponding unitary spherical representation.*

Proof. We consider the Banach space $L^1(H)$. This is a Banach $*$-algebra with multiplication defined as the convolution

$$(f * g)(x) = \int_H f(y) g(y^{-1}x) dy$$

and involution defined by

$$f^*(x) = \overline{f(x^{-1})}.$$

Recall that the representation π extends to a representation of the Banach algebra $L^1(H)$ by

$$f \mapsto \int_H f(x)\pi(x)dx.$$

We will also denote this mapping of $L^1(H)$ into $\mathscr{B}(\mathscr{H})$ (the set of bounded linear operators on $\mathscr{H}$) by π. This representation will also be cyclic as the following argument shows. Denote by ξ the L-invariant cyclic unit vector for H. Vectors of the form

$$\pi(f_\epsilon)(\pi(h_1)\xi + \cdots + \pi(h_n)\xi),$$

where $\{f_\epsilon\}$ is an approximate identity on H, will then be dense in $\mathscr{H}$. Moreover the identity

$$\pi(f)(\pi(h_1)\xi + \cdots + \pi(h_n)\xi) = \pi((R_{h_1^{-1}} + \cdots + R_{h_n^{-1}})f)\xi,$$

holds for $f \in L^1(H)$ and $h_1, \ldots, h_n \in H$. (Here $R_h f$ denotes the right-translation of the argument of f; $f \mapsto f(\cdot\, h)$. We similarly define $L_h f$.) Hence vectors of the form $\pi(f)\xi$, where $f \in L^1(H)$, form a dense subset in $\mathscr{H}$.

The function Φ defined as

$$\Phi : \pi(f) \mapsto \langle \pi(f)\xi, \xi \rangle \tag{49}$$

extends to a state on the C^*-algebra $\mathscr{C}$ generated by $\pi(L^1(H))$ and the identity operator. It is a well-known fact from the theory of C^*-algebras that the norm-decreasing positive functionals form a convex and weak*-compact set (cf [16]). For a C^*-algebra with identity, the extreme points of this set are the pure states. Therefore, Φ can be expressed as

$$\Phi = \int_X \varphi_x d\mu, \tag{50}$$

where X is the set of pure states and μ is a regular Borel measure on X (cf [22], Thm. 3.28). We recall the Gelfand-Naimark-Segal construction of a cyclic representation of a C^*-algebra associated with a given state (cf [16]). In this duality, the irreducible representations correspond to the pure states. So each φ_x in (50) parametrises an irreducible representation of $\pi(L^1(H))$ on some Hilbert space H_x with a $\pi(L^1(H))$-cyclic unit vector ξ_x.

Herafter we will, by an abuse of notation, write $\Phi(f)$ for $\Phi(\pi(f))$ and correspondingly for the functionals φ_x.

We define a unitary operator $T : \mathscr{H} \to \int_X H_x d\mu$ that intertwines the actions of $\mathscr{C}$ by

$$T : \pi(f)\xi \mapsto \{\pi_x(f)\xi_x\}, f \in L^1(H). \tag{51}$$

To see that this is well-defined, suppose that $\pi(f)\xi = 0$. Then we have

$$\langle \pi(f)\xi, \pi(f)\xi \rangle = \langle \pi(f^* * f)\xi, \xi \rangle = 0 \tag{52}$$

i.e.,

$$\Phi(f^* * f) = 0 \tag{53}$$

By (50) we have

$$\Phi(f^* * f) = \int_H \langle \pi_x(f^* * f)\xi_x, \xi_x\rangle_x d\mu = 0. \tag{54}$$

Therefore $\pi_x(f)\xi_x = 0$ for almost every x and hence T is well defined on a dense set of vectors. Note that (54) also shows that T is isometric on this set and it therefore extends to an isometry of $\mathscr{H}$ into $\int_X H_x d\mu$.

Consider now the subalgebra, $L^1(H)^{\#}$, consisting of all L^1-functions that are left- and right L-invariant, i.e.,

$$L_l f = R_l f = f,$$

for all l in L. This is a commutative Banach *-algebra (cf [6], Ch. IV). We know that $\varphi_x \circ \pi : L^1(H)^{\#} \to \mathbb{C}$ is a homomorphism of algebras and is therefore of the form (cf [6], Ch. IV)

$$\varphi_x(f) = \int_H f(h)\phi_x(h)dh, f \in L^1(H)^{\#}, \tag{55}$$

where ϕ_x is a bounded spherical function. In fact, this formula holds for all L^1-functions on H, as the following argument shows.

Since ξ is L-invariant, the identity

$$\pi(f)\xi = \pi(R_l f)\xi$$

holds for all L^1-functions f and $l \in L$. Applying T to both sides of this equality (and using the fact that both $L^1(H)$ and L are separable), we see that

$$\pi_x(f)\xi_x = \pi_x(R_l f)\xi_x \tag{56}$$

holds for all $f \in L^1(H)$ and $l \in L$ outside some set of measure zero with respect to μ. We now choose an approximation of the identity $\{\eta_\epsilon\}$ on H, and by replacing it with $\{\int_L \eta_\epsilon(l \cdot l^{-1})dl\}$ if necessary, we may assume that it is invariant under the conjugate action of L.

Consider now Φ_ϵ defined by

$$\varphi_\epsilon(f) = \langle \pi(f)\pi(\eta_\epsilon)\xi, \pi(\eta_\epsilon)\xi\rangle.$$

We define the functionals $\varphi_{x,\epsilon}$ analogously for all $x \in X$. Clearly $\Phi_\epsilon(f) \to \Phi(f)$ as $\epsilon \to 0$ and therefore

$$\lim_{\epsilon\to 0} \varphi_{x,\epsilon}(f) = \varphi_x(f)$$

holds for all L^1-functions f outside some set of measure zero with respect to μ. (Again we use the separability of $L^1(H)$.) Using the L-conjugacy invariance of η_ϵ and (56), a simple calculation shows that

$$\varphi_{x,\epsilon}(f) = \varphi_{x,\epsilon}(f^{\#}),$$

where

$$f(h) = \int_L \int_L f(l_1 h l_2) dl_1 dl_2,$$

and by letting ϵ tend to zero we get

$$\varphi_x(f) = \varphi_x(f^{\#})$$

for almost every x. Hence

$$\varphi_x(f) = \int_H f(h)\phi_x(h)dh,$$

for $f \in L^1(H)$.

Since φ_x also preserves the involution $*$, it is a positive linear functional, i.e.,

$$\int_H f(h)\phi_x(h)dh \geq 0, \tag{57}$$

for every $f \in L^1(H)$, such that $f = g * g^*$, for some $g \in L^1(H)$. The proof of the following lemma can be found in [4], p.85.

Lemma 5.1. *Suppose that φ is a bounded spherical function such that $\int_H f(h)\varphi(h)dh \geq 0$ for all $f \in L^1(H)$ of the form $f = g * g^*$ for some $g \in L^1(H)$. Then φ is positive definite.*

Since every positive definite spherical function defines an irreducible, unitary, spherical representation of H, it also gives rise to a representation $L^1(H)$. Its restriction to the subspace of L-invariant vectors, E_x will be $L^1(H)^{\#}$-invariant and one-dimensional (cf [6], Ch. IV). If the state φ_x corresponds to the spherical function ϕ_x, we denote by (π_x, H_x) both the representations of H and of $L^1(H)$ that it induces. Corresponding to this cyclic representation of $L^1(H)$ with cyclic unit vector ϕ_x, we have that the state $f \mapsto \langle \pi_x(f)\phi_x, \phi_x\rangle_x$ is

$$\begin{aligned}
\langle \pi_x(f)\phi_x, \phi_x\rangle_x &= \int_H f(h)\langle \pi_x(h)\phi_x, \phi_x\rangle_x dh \\
&= \int_H f(h)\langle L_h\phi_x, \phi_x\rangle_x dh \\
&= \int_H f(h)\phi_x(h^{-1})dh \\
&= \int_H f(h)\overline{\phi_x(h)}dh.
\end{aligned}$$

Therefore this representation of $L^1(H)$ is unitarily equivalent to the one given by the Gelfand-Naimark-Segal correspondence, i.e., we can regard the representation as coming from a representation of the group H.

The operator T clearly intertwines the group representations π and $\int_X \pi_x d\mu$. The only thing that remains is to prove that T is surjective.

Suppose that $c = \{c_x\}$ is orthogonal to $T(\pi(L^1(H))$, i.e.,

$$\int_X \langle \pi_x(f)\xi_x, c_x\rangle_x d\mu = 0.$$

We observe that the restriction of T to the space $\mathscr{H}^L$ of L-invariant vectors intertwines the representations of $\pi(L^1(H)^\#)$ on $\mathscr{H}^L$ and $\int_x E_x d\mu$. The mapping

$$\pi(f) \mapsto (x \mapsto \varphi_x(f))$$

is the Gelfand transform that realises the commutative C^*-algebra generated by $\pi(L^1(H)^\#)$ and the identity operator as the algebra, $C(X)$, of continuous functions on X. Continuous functions of the form $\Psi(x) = \varphi_x(f^\Psi)$, where $f^\Psi \in L^1(H)^\#$ are dense in $C(X)$. For such Ψ we have

$$\begin{aligned}\int_X \langle \pi_x(f)\xi_x, c_x\rangle_x \Psi(x) d\mu &= \int_X \langle \pi_x(f * f^\Psi)\xi_x, c_x\rangle_x d\mu \\ &= 0\end{aligned}$$

From this we can conclude that (using once more the separability of $L^1(H)^\#$) for all x outside a set of μ-measure zero, the equality

$$\langle \pi_x(f)\xi_x, c_x\rangle_x = 0$$

holds for all $f \in L^1(H)^\#$. Since the vectors ξ_x are $L^1(H)^\#$-cyclic, we can conclude that $c = 0$ and this finishes the proof. ■

Remark 5.2. The measure μ in the above theorem is called the *Plancherel measure* for the representation π.

5.2. Extension and expansion of the spherical functions. Consider the mapping $R : \mathscr{H}_\nu \to C^\infty(\mathscr{X})$ defined by

$$(Rf)(x) = h(x,x)^{\nu/2} f(x), x \in \mathscr{X}$$

(see [28]). When $\nu > n-1$, R is in fact an H-intertwining operator onto a dense subspace of $L^2(\mathscr{X}, d\iota)$ (where $d\iota$ is the H-invariant measure on $\mathscr{X}$) and the principal series representation gives the desired decomposition of π_ν into irreducible spherical representations. This is a heuristic motivation for studying the functions $R^{-1}\varphi_\lambda$, where φ_λ is a spherical function on $\mathscr{X}$.

Theorem 11. *Let $\nu > (n-2)/2$. The function $R^{-1}\varphi_\lambda(z)$ is holomorphic on $\mathscr{D}$ and has the power series expansion*

$$R^{-1}\varphi_\lambda(z) = \sum_k p_k(\lambda) e_k(z),$$

where $e_k(z)$ is the normalisation of the function $z \mapsto (zz^t)^k$ in the $\mathscr{H}_\nu$-norm, and the coefficients $p_k(\lambda)$ are polynomials of degree $2k$ of λ and satisfy the orthogonality relation a) If $\nu \geq \frac{n-1}{2}$, then

$$\begin{aligned}&\frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}+i\lambda)\Gamma(\frac{n-1}{2}+i\lambda)\Gamma(\nu-\frac{n-1}{2}+i\lambda)}{\Gamma(2i\lambda)}\right|^2 p_{\nu,k}(\lambda)\overline{p_{\nu,l}(\lambda)} d\lambda \\ = \;& \Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu-\frac{n-2}{2}\right)\Gamma(\nu)\delta_{kl}\end{aligned}$$

b) If $\nu < \frac{n-1}{2}$, then

$$\begin{aligned}
& \frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma(\frac{1}{2}+i\lambda)\Gamma(\frac{n-1}{2}+i\lambda)\Gamma(\nu-\frac{n-1}{2}+i\lambda)}{\Gamma(2i\lambda)}\right|^2 p_{\nu,k}(\lambda)\overline{p_{\nu,l}(\lambda)}d\lambda \\
+ & \frac{\Gamma(\nu)\Gamma(\nu-\frac{n-2}{2})\Gamma(n-1-\nu)\Gamma(\frac{n}{2}-\nu)}{\Gamma(n-1-2\nu)} \\
& \qquad \times p_{\nu,k}\left(i\left(\nu-\frac{n-1}{2}\right)\right)\overline{p_{\nu,l}\left(i\left(\nu-\frac{n-1}{2}\right)\right)} \\
= & \Gamma\left(\frac{n}{2}\right)\Gamma\left(\nu-\frac{n-2}{2}\right)\Gamma(\nu)\delta_{kl}
\end{aligned}$$

Proof. Recall the root space decomposition for $\mathfrak{h}$. Let $\langle\,,\rangle$ denote the inner product on $\mathfrak{a}_{\mathbb{C}}$ that is dual to the restriction of the Killing form to $\mathfrak{a}$. Let α_0 denote $\alpha/\langle\alpha,\alpha\rangle$.

In this setting the spherical function φ_λ is determined by the formula (cf [6], Ch. IV, exercise 8)

$$\varphi_\lambda(\exp(t\xi_e)0) = {}_2F_1(a',b',c';-\sinh(\alpha(t\xi_e))^2), \tag{58}$$

where

$$\begin{aligned}
a' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha + m_{2\alpha} + \langle i\lambda,\alpha_0\rangle\right) = \frac{1}{2}\left(\frac{n-1}{2}+i\lambda\right), \\
b' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha + m_{2\alpha} - \langle i\lambda,\alpha_0\rangle\right) = \frac{1}{2}\left(\frac{n-1}{2}-i\lambda\right), \\
c' &= \frac{1}{2}\left(\frac{1}{2}m_\alpha + m_{2\alpha} + 1\right) = \frac{1}{2}\left(\frac{n+1}{2}\right).
\end{aligned}$$

Letting $x = \exp(t\xi_e)0 = \tanh t$, (58) takes the form

$$\varphi_\lambda(x) = {}_2F_1(a',b',c';\frac{xx^t}{1-xx^t}) \tag{59}$$

By Euler's formula (cf [5]) we have

$$\varphi_\lambda(x) = {}_2F_1(a',b',c';\frac{xx^t}{1-xx^t}) = (1-xx^t)^{a'}\,{}_2F_1(a',c'-b',c';xx^t) \tag{60}$$

For the function $R^{-1}\varphi_\lambda$ we thus get the expression

$$R^{-1}\varphi_\lambda(z) = (1-zz^t)^{-\nu+a'}\,{}_2F_1(a',c'-b',c;zz^t) \tag{61}$$

Expanding (61) into a power series yields

$$R^{-1}\varphi_\lambda(z) = \sum_{m=0}^\infty\sum_{l=0}^m \frac{(\nu-a')_{m-l}(a')_l(c'-b')_l}{(m-l)!l!(c')_l}(zz^t)^m, \tag{62}$$

noticing that $|zz^t|<1$ for $z\in\mathscr{D}$. Next, we use the following simple identities:

$$\begin{aligned}
(\nu-a')_{m-l} &= \frac{(\nu-a')_m}{(\nu-a'+(m-l))_l} = \frac{(\nu-a')_m}{(-1)^l(-(\nu-a'+m-1))_l} && (63) \\
(m-l)! &= \frac{m!}{(m-l+1)_l}. && (64)
\end{aligned}$$

Substitution of these in (62) yields

$$R^{-1}\varphi_\lambda(z) = \sum_{m=0}^{\infty} \frac{(\nu - a')_m}{m!} \sum_{l=0}^{m} \frac{(a')_l (c'-b')_l (-m)_l}{(c')_l (-(\nu - a' + m - 1))_l} (zz^t)^m. \tag{65}$$

The inner sum in (65) can be recognised as a hypergeometric function, i.e., we have

$$\sum_{l=0}^{m} \frac{(a')_l (c'-b')_l (-m)_l}{(c')_l (-(\nu - a' + m - 1))_l} = {}_3F_2(a', c'-b', -m; c', -(\nu - a' + m - 1); 1).$$

Now we use Thomae's transformation rule (cf [5]) for the function ${}_3F_2$:

$$\begin{aligned}
&{}_3F_2(a', c'-b', -m; c', -(\nu - a' + m - 1); 1) \\
&= \frac{(-(\nu - a' + m - 1) - (c' - b'))_m}{(-(\nu - a' + m - 1))_m} \\
&\times \ {}_3F_2(c' - a', c' - b', -m; 1 + (c' - b') + (\nu - a' + m - 1) - m; 1)
\end{aligned}$$

We finally obtain the following expression:

$$R^{-1}\varphi_\lambda(z) = \sum_{k=0}^{\infty} c_{n,\nu,k}(\lambda)(zz^t)^k,$$

where

$$c_{n,\nu,k}(\lambda) = \frac{(\nu - \frac{n-2}{2})_k}{k!} {}_3F_2(-k, \frac{1+i\lambda}{2}, \frac{1-i\lambda}{2}; \frac{n}{2}, \nu - \frac{n-2}{2}; 1)$$

Recall the continuous dual Hahn polynomials (cf [27])

$$\begin{aligned}
S_k(x^2; a, b, c) = (a+b)_k (a+c)_k \\
\times {}_3F_2(-k, a+ix, a-ix; a+b, a+c; 1)
\end{aligned} \tag{66}$$

We can thus write

$$\begin{aligned}
R^{-1}\varphi_\lambda(z) &= \sum_{k=0}^{\infty} \frac{(\nu - \frac{n-2}{2})_k}{(\frac{n}{2})_k (\nu - \frac{n-2}{2})_k k!} S_k\left((\frac{\lambda}{2})^2; \frac{1}{2}, \frac{n-1}{2}, \nu - \frac{n-2}{2} \right) (zz^t)^k \\
&= \sum_{k=0}^{\infty} p_{\nu,k}(\lambda) \frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}.
\end{aligned}$$

For the orthogonality relation in the claim, we refer to [27]. ■

5.3. Principal and complementary series representations. In this section we let μ ($=\mu_\nu$) be the finite measure on the real line that orthogonalises the coefficients $p_k(\lambda)$ in (58). Let Λ_ν be its support. As we saw above, μ can, depending on the value of ν, either be absolutely continuous with respect to Lebesgue measure or have a point mass at $\lambda = i(\nu - (n-1)/2)$, i.e., we either have

$$\Lambda_\nu = (0, \infty) \bigcup \{i(\nu - (n-1)/2)\},\ \nu \in ((n-2)/2, (n-1)/2)$$

or

$$\Lambda_\nu = (0, \infty),\ \nu \geq (n-1)/2.$$

We will now construct explicit realisations for the spherical representations π_λ corresponding to the points $\lambda \in \Lambda_\nu$ on Hilbert spaces H_λ. For λ in the continuous part in Λ, the underlying space H_λ will be $L^2(S^{n-1})$ and for the discrete point $i(\nu - (n-1)/2)$, H_λ will be a Sobolev space.
We will hereafter suppress the index ν and simply denote the support of μ by Λ.

Lemma 5.3. *If $g \in H$, then g transforms the surface measure, σ, on S^{n-1} as*

$$d\sigma(g\zeta) = J_g(\zeta)^{\frac{n-1}{n}} d\sigma(\eta).$$

Proof. Clearly it suffices to prove the statement for automorphisms of the form

$$g = \exp \xi_v, v \in \mathbb{R}^n.$$

Moreover we can assume that $\zeta = e_1$, since any $\zeta \in S^{n-1}$ can be written as le_1, where $l \in L$, and

$$\begin{aligned}\exp \xi_v(le_1) = (\exp \xi_v l)(e_1) = (ll^{-1} \exp \xi_v l)(e_1) &= (l\sigma_{l^{-1}}(\exp\ \xi_v))(e_1)\\ = l \exp\,(Ad(l^{-1})\xi_v)(e_1) &= l \exp \xi_{\,l^{-1}v}(e_1).\end{aligned}$$

Consider now the tangent space of $\mathbb{R}^n$ at e_1. We have an orthogonal decomposition

$$T_{e_1}(\mathbb{R}^n) = T_{e_1}(S^{n-1}) \oplus \mathbb{R}e_1.$$

At ge_1 we have the corresponding decomposition

$$T_{ge_1}(\mathbb{R}^n) = T_{ge_1}(S^{n-1}) \oplus \mathbb{R}ge_1.$$

Since H preserves S^{n-1},

$$dg(e_1)\, T_{e_1}(S^{n-1}) = T_{ge_1}(S^{n-1}),$$

and by completing e_1 and ge_1 to orthonormal bases for their respective tangent spaces, $dg(e_1)$ corresponds to a matrix of the form

$$\begin{pmatrix} c & & 0 & \\ | & * & * & * \\ v & * & * & * \\ | & * & * & * \end{pmatrix}$$

Hence

$$J_g(e_1) = cJ_{g|_{S^{n-1}}}(e_1), \tag{67}$$

where

$$c = (dg(e_1)e_1, ge_1). \tag{68}$$

We next determine this constant c.

We have

$$c = (dg(e_1)e_1, ge_1) = \lim_{r\to 1}(dg(re_1)re_1, gre_1).$$

For fixed $r < 1$ we have

$$\begin{aligned}\exp \xi_v(re_1) &= u + B(u,u)^{1/2}B(re_1,-u)^{-1}(re_1 + Q(re_1)u)\\ &= u + dg(re_1)(re_1 + Q(re_1)u),\end{aligned}$$

and

$$J_g(re_1) = \left(\frac{h(re_1,-u)}{h(u,u)^{1/2}}\right)^{-n}, \tag{69}$$

where $u = \tanh v$. Since $Q(re_1)u = 2(u, re_1)re_1 - u$, we get

$$\begin{aligned}&(dg(re_1)re_1, g(re_1)) \\ &= (1 + 2(u, re_1))|dg(re_1)re_1|^2 + (dg(re_1)re_1, u - dg(re_1)u)\end{aligned} \tag{70}$$

For any $z \in \mathscr{D} \bigcap \mathbb{R}^n$ and $v, w \in \mathbb{R}^n$, the identity

$$(dg(z)v, w) = \frac{h(gz, gz)}{h(z,z)}(v, dg(z)^{-1}w) \tag{71}$$

can be established using the transformation properties of the function h and the operator B. Applying (71) in the cases $z = re_1$, $v = re_1$, and $w = dg(re_1)re_1$ and
$w = u - dg(re_1)u$, repectively, yields

$$(dg(re_1)re_1, dg(re_1)re_1) = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}r^2 \tag{72}$$

and

$$(dg(re_1)re_1, u - dg(re_1)u) = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}(re_1, dg(re_1)^{-1}u - u). \tag{73}$$

The expressions above and an elementary computation shows that (70) can be written as

$$(dg(re_1)re_1, g(re_1)) = \frac{h(g(re_1), g(re_1))}{h(re_1, re_1)}\, r^2\, \frac{1 + 2(u, re_1) + |u|^2}{1 - |u|^2} \tag{74}$$

By the transformation rule for the Bergman kernel

$$h(g(re_1), g(re_1)) = |J_g(re_1)|^{2/n} h(re_1, re_1).$$

So,

$$\begin{aligned}c &= \lim_{r\to 1} |J_g(re_1)|^{2/n} r^2\, \frac{1 + 2(u, re_1) + |u|^2}{1 - |u|^2}\\ &= |J_g(e_1)|^{2/n} \frac{h(e_1, -u)}{h(u,u)^{1/2}}.\end{aligned}$$

Comparing with the expression (69), we have determined the constant

$$c = J_g(e_1)^{1/n},$$

and this finishes the proof. ∎

For λ in the continuous part of Λ, the corresponding representation is a principal series representation described by the following proposition. (We will hereafter follow Helgason and in this context denote S^{n-1} by B. The measure σ will be denoted by db.)

Proposition 12. *For any real number λ, the map $h \mapsto \tau_\lambda(h)$, where*

$$\tau_\lambda(h)f(b) = J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} f(h^{-1}b)$$

defines a unitary representation of H on $L^2(B)$.

Proof. We have

$$\begin{aligned}\int_B |J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}}|^2 |f(h^{-1}b)|^2 db &= \int_B J_{h^{-1}}(hb)^{\frac{2\rho}{n}} |f(b)|^2 d(hb) \\ &= \int_B J_h(b)^{-\frac{2\rho}{n}} |f(b)|^2 J_h(b)^{\frac{n-1}{n}} db \\ &= \int_B |f(b)|^2 db,\end{aligned}$$

where the last equality follows by lemma 5.3. ■

It is well known that the representations τ_λ above are unitarily equivalent to the canonical spherical representations associated with the corresponding functionals λ on $\mathfrak{a}_{\mathbb{C}}$ (cf [11], ch. 7).

In order to realise the representation τ_λ for $\lambda = i(\nu - (n-1)/2)$, we consider the following Hilbert spaces.

Definition 13. *For $\frac{n-2}{2n} \leq \alpha < \frac{n-1}{2n}$, let $\mathscr{C}_\alpha$ be the Hilbert space completion of the C^∞-functions on S^{n-1} with respect to the norm*

$$\|f\|_{\mathscr{C}_\alpha} = \int_{S^{n-1}} \int_{S^{n-1}} f(\zeta)\overline{f(\eta)} K(\zeta, \eta)^\alpha d\sigma(\zeta) d\sigma(\eta)$$

Using the action of H on S^{n-1}, we can define a unitary representation of H on $\mathscr{C}_\alpha$ of the form

$$\sigma_\alpha : f \mapsto J_{h^{-1}}(\cdot)^\beta f(h^{-1}\cdot), h \in H,$$

where $\beta = -\alpha + (n-1)/n$. The unitarity follows from

$$\begin{aligned}&\int_{S^{n-1}} \int_{S^{n-1}} J_{h^{-1}}(\zeta)^\beta f(h^{-1}\zeta) \overline{J_{h^{-1}}(\eta)^\beta f(h^{-1}\eta)} K(\zeta, \eta)^\alpha d\sigma(\zeta) d\sigma(\eta) \\ &= \int_{S^{n-1}} \int_{S^{n-1}} J_h(\zeta)^{-\beta} f(\zeta) \overline{J_h(\eta)^{-\beta} f(\eta)} K(h\zeta, h\eta)^\alpha J_h(\zeta)^{\frac{n-1}{n}} J_h(\eta)^{\frac{n-1}{n}} d\sigma(\zeta) d\sigma(\eta) \\ &= \int_{S^{n-1}} \int_{S^{n-1}} J_h(\zeta)^{-\beta-\alpha+\frac{n-1}{n}} J_h(\eta)^{-\beta-\alpha+\frac{n-1}{n}} f(\zeta)\overline{f(\eta)} K(\zeta, \eta)^\alpha d\sigma(\zeta) d\sigma(\eta).\end{aligned}$$

In fact, this representation is irreducible (cf [1]). We denote this representation by σ_α. One can prove that for $\alpha = \nu/n$ and $\lambda = \nu - (n-1)/2$, σ_α and τ_λ are unitarily equivalent.

Recall the expression in Cor. 6 for the spherical functions. In this setting we write it as

$$\varphi_\lambda(x) = \int_B e_{\lambda,b}(x)db,$$

where

$$e_{\lambda,b}(x) = \left(\frac{h(x,x)^{1/2}}{h(x,b)}\right)^{i\lambda+\rho}$$

by Cor. 6. For fixed $z \in \mathscr{D}$ and $\lambda \in \Lambda$, $R^{-1}e_{\lambda,b}(z)$ is a function in $L^2(B)$. Moreover, $\pi_\nu(H)$ makes sense as a group of mappings on $\mathcal{O}(\mathscr{D})$, the set of holomorphic functions on $\mathscr{D}$. We have a relationship between these representations.

Lemma 5.4. *For every $g \in H$ and $\lambda \in \Lambda$,*

$$\pi_\nu(g)\tau_\lambda(g)R^{-1}e_{\lambda,b}(z) = R^{-1}e_{\lambda,b}(z). \tag{75}$$

Correspondingly, for $X \in \mathfrak{h}$, we have the relation

$$\pi_\nu(X)R^{-1}e_{\lambda,b}(z) = -\tau_\lambda(X)R^{-1}e_{\lambda,b}(z). \tag{76}$$

The proof is straightforward by applying the transformation rules for the function $h(z,w)$.

5.4. The Fourier-Helgason transform. The purpose of this section is to construct an H-intertwining unitary operator between the Hilbert spaces $\mathscr{H}_\nu$ and $\int_\Lambda H_\lambda d\mu$.

Any holomorphic function, f, on $\mathscr{D}$ has a power series expansion

$$f(z) = \sum_\alpha f_\alpha z^\alpha, \tag{77}$$

where $f_\alpha = \frac{\partial^\alpha f}{\alpha!\partial z^\alpha}(0)$. We can collect the powers of equal homogeneous degree together and write

$$f(z) = \sum_k f_k(z), \tag{78}$$

where f_k is of homogeneous degree k. We now consider the mapping

$$(\cdot,\cdot)_\nu : \mathcal{P} \times \mathcal{O}(\mathscr{D}) \to \mathbb{C}$$

defined as

$$(f,g)_\nu = \sum_k \langle f, g_k\rangle_\nu. \tag{79}$$

Observe that the definition makes sense since every polynomial is orthogonal to all but finitely many g_k.

Definition 14. *If f is a polynomial in $\mathscr{H}_\nu$, its generalised Fourier-Helgason transform is the function $\tilde{f}$ on $\Lambda \times B$ defined by*

$$\tilde{f}(\lambda, b) = (f, R^{-1} e_{\lambda,b})_\nu \tag{80}$$

Proposition 15. *(i) If the polynomial f is in $\mathscr{H}_\nu^L$, then $\tilde{f}$ is L-invariant and*

$$\|f\|_\nu^2 = \int_\Lambda \|\tilde{f}\|_\lambda^2 d\mu,$$

where $\|\cdot\|_\lambda$ is the norm on H_λ, and the Fourier-Helgason transform extends to an isometry from $\mathscr{H}_\nu^L$ onto $L^2(\Lambda, d\mu)$.
(ii) The inversion formula for L-invariant polynomials

$$f(z) = \int_\Lambda \tilde{f}(\lambda) R^{-1} \varphi_\lambda(z) d\mu(\lambda) \tag{81}$$

holds. Moreover, the above formula holds for arbitrary L-invariant functions, when restricted to the submanifold $\mathscr{X}$.

Proof. Writing

$$R^{-1} e_{\lambda,b} = \sum_\alpha c_\alpha(\lambda, b) z^\alpha = \sum_k e_{\lambda,b,k}$$

and

$$R^{-1} \varphi_\lambda(z) = \sum_\alpha c_\alpha(\lambda) z^\alpha = \sum_k p_k(\lambda) e_k(z),$$

we see that the coefficients and polynomials of homogeneous degree k are related by

$$c_\alpha(\lambda) = \int_B c_\alpha(\lambda, b) db \tag{82}$$

and

$$p_k(\lambda) e_k(z) = \int_B e_{\lambda,b,k}(z) db \tag{83}$$

respectively. Therefore we have

$$\begin{aligned}
\tilde{f}(\lambda, b) &= \sum_k \langle f, e_{\lambda,b,k} \rangle_\nu \\
&= \sum_k \langle \int_L \pi_\nu(l) f dl, e_{\lambda,b,k} \rangle_\nu \\
&= \sum_k \langle f, \int_L \pi_\nu(l^{-1}) e_{\lambda,b,k} dl \rangle_\nu \\
&= \sum_k \langle f, \int_L \pi_\lambda(l) e_{\lambda,b,k} dl \rangle_\nu \\
&= (f, R^{-1} \varphi_\lambda)_\nu.
\end{aligned}$$

This proves the L-invariance. Moreover, we have

$$(f, R^{-1}\varphi_\lambda)_\nu = \sum_k \overline{p_k(\lambda)} \langle f, e_k \rangle_\nu.$$

Hence

$$\int_\Lambda \|\tilde{f}\|_\lambda^2 d\mu = \sum_k |\langle f, e_k \rangle_\nu|^2 = \|f\|_\nu^2.$$

This proves the first part of the claim.

To prove the inversion formula, we now let f be an L-invariant polynomial and x be a point in $\mathscr{D} \bigcap \mathbb{R}^n$. Since we have an estimate of the form

$$|R^{-1}\varphi_\lambda(x)| \leq (1 - |x|^2)^{-\frac{\nu}{2}} C(x), \tag{84}$$

where C is some function of x, independently of λ, the integral

$$\int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda)$$

makes sense for real x. We then have

$$\begin{aligned} \int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda) &= \sum_k \int_\Lambda \langle f, e_k \rangle_\nu \overline{p_k(\lambda)} R^{-1}\varphi_\lambda(x) d\mu(\lambda) \\ &= \sum_k \langle f, e_k \rangle_\nu \int_\Lambda \sum_j \overline{p_k(\lambda)} p_j(\lambda) e_j(x) d\mu(\lambda) \\ &= f(x). \end{aligned}$$

Now let $f \in \mathscr{H}_\nu^L$ be arbitrary. We choose a sequence of polynomials $f_n \in \mathscr{H}_\nu^L$ such that

$$f = \lim f_n.$$

Since the evaluation functionals are continuous, we have

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_\Lambda \tilde{f}_n(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda)$$

for every real point x. By Jensen's inequality and (84)

$$\left| \int_\Lambda (\tilde{f}(\lambda) - \tilde{f}_n(\lambda)) R^{-1}\varphi_\lambda(x) d\mu(\lambda) \right|^2 \leq \mu(\Lambda) \int_\Lambda |\tilde{f}(\lambda) - \tilde{f}_n(\lambda)|^2 C(x) (1 - |x|^2)^{-\nu} d\mu(\lambda).$$

Hence

$$f(x) = \int_\Lambda \tilde{f}(\lambda) R^{-1}\varphi_\lambda(x) d\mu(\lambda).$$

Thus the inversion formula holds for real points, x. To see that the formula holds for arbitrary points when f is a polynomial, we note that both the left hand- and the right hand side of the formula define holomorphic functions on $\mathscr{D}$. Since they agree on the totally real form $\mathscr{X}$, they are equal. ∎

Theorem 16. The Plancherel Theorem *For* $\nu > (n-2)/2$, *the Fourier-Helgason transform is a unitary isomorphism from the* H*-modules* $\mathscr{H}_\nu$ *onto the* H*-module* $\int_\Lambda H_\lambda d\mu$, *i.e.,*

$$(\pi_\nu(h)f)\tilde{}(\lambda, b) = \tau_\lambda(h)\widetilde{f}(\lambda, b),$$

for $h \in H$, *and*

$$\|f\|_\nu^2 = \int_\Lambda \|\tilde{f}\|_\lambda^2 d\mu.$$

Proof. We divide the proof into three steps:
(i) We prove that the Fourier-Helgason transform intertwines the action of the Lie algebra of H.
(ii) We use (i) to prove that the norm is preserved.
(iii) We conclude that the group actions are intertwined from (i) and (ii).

We will see that these properties actually imply that the Fourier-Helgason transform is surjective.

Consider now the corresponding representations of the Lie algebra $\mathfrak{h}$. These will also be denoted by π_ν and τ_λ respectively. Moreover they extend naturally to representations of the universal enveloping algebra, $\mathfrak{U}(\mathfrak{h})$, of $\mathfrak{h}$.
Let $X \in \mathfrak{h}$. If f is a polynomial in $\mathscr{H}_\nu$, then differentiation of the mapping

$$t \mapsto J_{\exp tX}(z)^{\nu/n} f((\exp tX)z)$$

at $t = 0$ shows that $\pi_\nu(X)f$ is also a polynomial, and

$$\begin{aligned}
\widetilde{\pi_\nu(X)f}(\lambda, b) &= \sum_k \langle \pi_\nu(X)f, e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, -\pi_\nu(X)e_{\lambda,b,k}\rangle_\nu \\
&= \sum_k \langle f, \tau_\lambda(X)e_{\lambda,b,k}\rangle_\nu \\
&= (f, \tau_\lambda(X)R^{-1}e_{\lambda,b})_\nu \\
&= \tau_\lambda(X)(f, R^{-1}e_{\lambda,b})_\nu,
\end{aligned}$$

which proves (i).

To prove the second step, we recall that the adjoint representation of L on $\mathfrak{h}$ extends to an action on $\mathfrak{U}(\mathfrak{h})$ as homomorphisms of an associative algebra. The L-invariant elements in $\mathfrak{U}(\mathfrak{h})$ form a subalgebra, $\mathfrak{U}(\mathfrak{h})^L$. We let p denote the projection

$$X \mapsto \int_L Ad(l)X dl$$

of $\mathfrak{U}(\mathfrak{h})$ onto $\mathfrak{U}(\mathfrak{h})^L$. This action of L connects the representations of H and $\mathfrak{U}(\mathfrak{h})$ according to the following identity:

$$\pi_\nu(l)\pi_\nu(X)\pi_\nu(l^{-1}) = \pi_\nu(Ad(l)X),$$

for $l \in L$ and $X \in \mathfrak{U}(\mathfrak{h})$.

Since the vector $1 \in \mathscr{H}_\nu$ is cyclic for the representation of H, it is also cyclic for the representation of $\mathfrak{U}(\mathfrak{h})$. Hence it suffices to prove that the norm is preserved for elements of the form $\pi_\nu(X)1$, where $X \in \mathfrak{U}(\mathfrak{h})$. In the following equalities, we temporarily let τ denote the direct integral of the representations τ_λ, and analogously we let $\langle , \rangle$ denote the direct integral of the corresponding inner products.

$$\begin{aligned}\langle \pi_\nu(X)1, \pi_\nu(X)1\rangle_\nu &= \langle \pi_\nu(X)^*\pi_\nu(X)1, 1\rangle_\nu \\ &= \langle -\pi_\nu(X^2)1, 1\rangle_\nu\end{aligned}$$

Since the vector 1 is L-invariant, the last expression equals $\langle -\pi_\nu(p(X^2))1, 1\rangle_\nu$, and by proposition (15), we have

$$\langle -\pi_\nu(p(X^2))1, 1\rangle_\nu = \langle -\pi_\nu \widetilde{(p(X^2)})1, \tilde{1}\rangle.$$

By (i), the expression on the right-hand side equals $\langle -\tau(p(X^2))\tilde{1}, \tilde{1}\rangle$, and since $\tilde{1}$ is L-invariant, we have

$$\langle -\tau(p(X^2))\tilde{1}, \tilde{1}\rangle = \langle -\tau(X^2)\tilde{1}, \tilde{1}\rangle.$$

Thus (ii) is proved.

To prove (iii), we recall the following equalities (on the respective dense spaces of analytic vectors):

$$\pi_\nu(\exp(X)) = e^{\pi_\nu(X)}$$
$$\tau_\lambda(\exp(X)) = e^{\tau_\lambda(X)}.$$

From this and the facts that H is connected and that the Fourier-Helgason transform is bounded operator, we immediately see that (iii) holds.

To see that the operator is surjective, note that by (ii) and (iii)

$$\langle \pi_\nu(f)1, 1\rangle_\nu = \int_\Lambda \langle \tau_\lambda \tilde{1}(\lambda, \cdot), \tilde{1}(\lambda, \cdot)\rangle_\lambda d\mu,$$

for $f \in L^1(H)^\#$, i.e., we can write the positive functional

$$f \mapsto \langle \pi_\nu(f)1, 1\rangle_\nu$$

as an integral of pure states with respect to some measure. By uniqueness, it is the measure in Theorem 10. Since the Fourier-Helgason transform intertwines the group action, it is the intertwining operator constructed in Theorem 10. Thus it is surjective. ■

Theorem 17. (The Inversion Formula) *If f is a polynomial in $\mathscr{H}_\nu$, then*

$$f(z) = \int_\Lambda \int_B \widetilde{f}(\lambda, b) R^{-1} e_{\lambda,b}(z) db d\mu(\lambda). \tag{85}$$

Proof. Take $h \in H$. Define

$$f_1(z) = \int_L \pi_\nu(l)\pi_\nu(h)f(z)dl$$

This is a radial function, and we have that

$$f_1(0) = J_{h^{-1}}(0)^{\frac{\nu}{n}} f(h^{-1}0). \tag{86}$$

Prop. 15 gives

$$f_1(0) = \int_\Lambda \tilde{f}_1(\lambda) R^{-1}\varphi_\lambda(z) d\mu(\lambda). \tag{87}$$

Moreover

$$\begin{aligned} \widetilde{f}_1(\lambda) = (f_1, R^{-1}\varphi_\lambda)_\nu &= (\int_L \pi_\nu(l)\pi_\nu(h) f dl, R^{-1}\varphi_\lambda)_\nu \\ &= (\pi_\nu(h) f, R^{-1}\varphi_\lambda)_\nu. \end{aligned} \tag{88}$$

By Thm. 16 we have

$$\begin{aligned} (\pi_\nu(h) f, R^{-1}\varphi_\lambda)_\nu &= (f, \pi_\nu(h^{-1}) R^{-1}\varphi_\lambda)_\nu \\ &= (f, \int_B \pi_\nu(h^{-1}) R^{-1} e_{\lambda,b}\, db)_\nu \\ &= (f, \int_B \pi_\nu(h^{-1}) R^{-1} e_{\lambda,b}\, db)_\nu \\ &= (f, \int_B \tau_\lambda(h) R^{-1} e_{\lambda,b}\, db)_\nu \\ &= (f, \int_B J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} R^{-1} e_{\lambda,h^{-1}b}\, db)_\nu. \end{aligned}$$

The integrand above has a power series expansion where the coefficients are functions of b. If we integrate, we obtain a holomorphic functions for which the coefficients in the power series expansion are obtained by integrating the aforementioned coefficients over B. Hence we can proceed as follows.

$$\begin{aligned} (f, \int_B J_{h^{-1}}(b)^{\frac{i\lambda+\rho}{n}} R^{-1} e_{\lambda,h^{-1}b}\, db)_\nu &= \int_B J_{h^{-1}}(b)^{\frac{-i\lambda+\rho}{n}} (f, R^{-1} e_{\lambda,h^{-1}b})_\nu db \\ &= \int_B J_{h^{-1}}(b)^{\frac{-i\lambda+\rho}{n}} \widetilde{f}(\lambda, h^{-1}b) db \\ &= \int_B J_{h^{-1}}(hb)^{\frac{-i\lambda+\rho}{n}} \widetilde{f}(\lambda, b) J_h(b)^{\frac{n-1}{n}} db \\ &= \int_B \widetilde{f}(\lambda, b) J_h(b)^{\frac{i\lambda+\rho}{n}} db. \end{aligned} \tag{89}$$

It is easy to see that

$$J_h(b)^{\frac{i\lambda+\rho}{n}} = J_{h^{-1}}(0)^{\frac{\nu}{n}} R^{-1} e_{\lambda,b}(h^{-1}0), \tag{90}$$

and so combining (86), (87) and (89) finally yields

$$f(h^{-1}0) = \int_\Lambda \int_B \widetilde{f}(\lambda, b) R^{-1} e_{\lambda,b}(h^{-1}0) db\, d\mu. \tag{91}$$

Thus the inversion formula holds for real points, hence for all points by the same argument as in the proof of Prop. 15. ∎

6. Realisation of the discrete part of the decomposition

We recall the earlier defined complementary series representations. The following theorem states that $\sigma_{\nu/n}$ is the representation corresponding to the singular point in the decomposition theorem.

Theorem 18. *The operator T_ν defined by the formula*

$$(T_\nu f)(z) = \int_{S^{n-1}} f(\zeta) K_\nu(z,\zeta) d\sigma(\zeta)$$

is a unitary H-intertwining operator from $\mathscr{C}_{\nu/n}$ onto an irreducible H-submodule of $\mathscr{H}_\nu$.

Proof. First of all we note that T_ν maps functions in $\mathscr{C}_{\nu/n}$ to holomorphic functions on $\mathscr{D}$ and thus π_ν has a meaning on the range of T_ν. We start by showing that T_ν is formally intertwining. We have

$$\begin{aligned} T_\nu(\sigma_{\nu/n})f(z) &= \int_{S^{n-1}} J_{h^{-1}}(\zeta)^{-\nu/n+\frac{n-1}{n}} f(h^{-1}\zeta) K_\nu(z,\zeta) d\sigma(\zeta) \\ &= \int_{S^{n-1}} J_h(\zeta)^{\nu/n-\frac{n-1}{n}} f(\zeta) K_\nu(z,h\zeta) J_h(\zeta)^{\frac{n-1}{n}} d\sigma(\zeta) \\ &= \int_{S^{n-1}} J_h(\zeta)^{\nu/n} f(\zeta) K_\nu(h^{-1}z,\zeta) J_h(h^{-1}z)^{-\frac{\nu}{n}} J_h(\zeta)^{-\frac{\nu}{n}} d\sigma(\zeta) \\ &= J_{h^{-1}}(z)^{\frac{\nu}{n}} \int_{S^{n-1}} f(\zeta) K_\nu(h^{-1}z,\zeta) d\sigma(\zeta), \end{aligned}$$

i.e.,

$$T_\nu \sigma_{\nu/n} = \pi_\nu T_\nu.$$

The next step is to prove that the constant function 1 is mapped into $\mathscr{H}_\nu$ and that its norm is preserved. Note that for $\alpha = \nu/n, K(z,\zeta)^\alpha = K_\nu(z,\zeta)$, and by Prop. 7 we have an expansion

$$K_\nu(\zeta, e_1) = \sum_{m-2k\geq 0} c_{m,k}(\nu) K_{(m-k,k)}(\zeta, e_1),$$

where the coefficients $c_{m,k}(\nu)$ are given explicitly. Now, since $K_\nu(\zeta, e_1)$ is $SO(n-1)$-invariant and the action of $SO(n-1)$ is linear, each $K_{(m,k)}(\zeta, e_1)$ must also be $SO(n-1)$-invariant. Hence, $K_{(m,k)}(\zeta, e_1)$ can be assumed to be $\phi_{m-2k}(\zeta)(\zeta\zeta^t)^k$, where ϕ_{m-2k} is the unique element in E_{m-2k} that assumes the value 1 in e_1. Therefore

$$\begin{aligned} \int_{S^{n-1}} K(\zeta,\eta)^\alpha d\sigma(\zeta) &= \int_L K_\nu(\zeta, le_1) dl \int_L K_\nu(l^{-1}\zeta, e_1) dl && (92) \\ &= \sum_{m-2k\geq 0} c_{m,k}(\nu) \int_L (l^{-1}\zeta(l^{-1}\zeta)^t)^k \phi_{m-2k}(l^{-1}\zeta) dl && (93) \\ &= \sum_{m-2k\geq 0} c_{m,k}(\nu)(\zeta\zeta^t)^k \int_L \phi_{m-2k}(l^{-1}\zeta) dl && (94) \end{aligned}$$

Since $SO(n)$ acts irreducibly on E_{m-2k} and the function $\int_L \phi_{m-2k}(l^{-1}z)dl$ is an $SO(n)$-invariant element in E_{m-2k} it must be identically zero unless $m-2k=0$. Since

$$\|1\|^2_{\mathscr{C}_{\frac{\nu}{n}}} = \int_{S^{n-1}} \int_{S^{n-1}} K_\nu(\zeta, \eta) d\sigma(\zeta) d\sigma(\eta),$$

the computation above implies that

$$\begin{aligned} \|1\|^2_{\mathscr{C}_{\frac{\nu}{n}}} &= \int_{S^{n-1}} \sum_{k=0}^{\infty} c_{2k,k}(\nu)(\zeta\zeta^t)^k d\sigma(\zeta) \\ &= \sum_{k=0}^{\infty} c_{2k,k}(\nu) = \sum_{k=0}^{\infty} \frac{(\nu)_k(\nu - \frac{n-2}{2})_k}{\|(zz^t)^k\|^2_{\mathscr{F}}} \\ &= \sum_{k=0}^{\infty} \frac{(\nu)_k(\nu - \frac{n-2}{2})_k}{k!(\frac{n}{2})_k} \end{aligned} \tag{95}$$

On the other hand, the equalities (92)-(94) also show that

$$\begin{aligned} T_\nu 1(z) &= \sum_{k=0}^{\infty} \frac{(\nu)_k(\nu - \frac{n-2}{2})_k}{k!(\frac{n}{2})_k}(zz^t)^k \\ &= \sum_{k=0}^{\infty} \frac{((\nu)_k(\nu - \frac{n-2}{2})_k)^{1/2}}{(k!(\frac{n}{2})_k)^{1/2}} \frac{(zz^t)^k}{\|(zz^t)^k\|_\nu}. \end{aligned} \tag{96}$$

If we compare (95) and (96), we see that $T_\nu 1 \in \mathscr{H}_\nu$ and that $\|1\|_{\mathscr{C}_{\nu/n}} = \|1\|_\nu$. Recall that

$$\mathscr{C}_{\nu/n} = \overline{\bigoplus_m E_m(S^{n-1})}$$

and that the representation of $\mathfrak{h}$ on the algebraic sum $\bigoplus_m E_m(S^{n-1})$ is irreducible. Hence

$$\bigoplus_m E_m(S^{n-1}) = \mathrm{Span}_{\mathbb{C}}\{\sigma_{\nu/n}(X_1)\ldots\sigma_{\nu/n}(X_k)1 | X_i \in \mathfrak{h}, 1 \le i \le k\}.$$

Since T_ν interwines the representations of $\mathfrak{h}$, we have that π_ν is an irreducible representation of $\mathfrak{h}$ on the space $T_\nu(\bigoplus_m E_m(S^{n-1})) \subseteq \mathscr{H}_\nu$. By Schur's lemma ([13], ch.4)

$$\langle T_\nu f, T_\nu g\rangle_\nu = c\langle f, g\rangle_{\mathscr{C}_{\nu/n}},$$

for some real constant c. Putting, f and g equal to the constant function 1 and applying, we see that $c=1$. Therefore, T_ν extends to a unitary operator

$$T_\nu : \mathscr{C}_{\nu/n} \to \overline{T_\nu(\bigoplus_m E_m(S^{n-1}))}$$

and we have proved the theorem. ∎

7. Realisation of the minimal representation $\pi_{(n-2)/2}$

In this section we show that the representation $\pi_{(n-2)/2}$ of H is irreducible by realising it as a complementary series representation.

We recall the space $\mathscr{C}_{\nu/n}$ from the previous section and the corresponding operator T_ν.

Theorem 19. *$T_{(n-2)/2}$ is a unitary H-intertwining operator from $\mathscr{C}_{(n-2)/2n}$ onto $\mathscr{H}_{(n-2)/2}$.*

Proof. Recall that

$$\mathscr{C}_{(n-2)/n} = \overline{\bigoplus_m E_m(S^{n-1})} \tag{97}$$

and that the sum is a decomposition into $SO(n)$-irreducible subspaces. If we let $\mathcal{P}_{(n-2)/n}$ denote the set of all finite sums in (97), $\sigma_{(n-2)/n}$ defines a representation of $\mathfrak{l}$ on $\mathcal{P}_{(n-2)/n}$. The polynomial $(\zeta_1 + i\zeta_2)^m$ is a highest weight vector in E_m for this representation. Moreover, the power series expansion of $K_{(n-2)/n}$ shows that $T_{(n-2)/2}$ is a polynomial in E_m. Since $T_{(n-2)/2}$ intertwines the $\mathfrak{l}$-actions, $T_{(n-2)/2}((\zeta_1 + i\zeta_2)^m)$ is a highest weight vector space for $\pi_{(n-2)/2}(\mathfrak{l})$, i.e.,

$$(T_{(n-2)/2}(\zeta_1 + i\zeta_2)^m)(z) = C_m(z + iz)^m, \tag{98}$$

for some constant C_m. We now determine C_m. Choose $z = w\frac{1}{2}(1, -i, 0, \ldots, 0)$, where w is a complex number with $|w| < 1$. In this case $zz^t = 0, (z + iz)^m = w^m$. We now compute $(T_{(n-2)/2}((\zeta_1 + i\zeta_2)^m)(z)$.

$$\begin{aligned}&\int_{S^{n-1}} K_{(n-2)/n}(z, \zeta)(\zeta_1 + i\zeta_2)^m \\ &\quad = \int_{S^{n-1}} (1 - w(\zeta_1 - i\zeta_2))^{-(n-2)/n}(\zeta_1 + i\zeta_2)^m d\sigma(\zeta)\end{aligned}$$

This integral only depends on the first two coordinates and can hence be converted to an integral over the unit disk, U (cf [21] Prop 1.4.4).

$$\begin{aligned}&\int_{S^{n-1}} (1 - w(\zeta_1 - i\zeta_2))^{-(n-2)/n}(\zeta_1 + i\zeta_2)^m d\sigma(\zeta) \\ &\quad = \frac{\Gamma\left(\frac{n-2}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)} \int_U (1 - w\overline{\zeta})^{-(n-2)/n}\zeta^m(1 - |\zeta|^2)^{(n-4)/2} dm(\zeta).\end{aligned}$$

We have the power series expansion

$$(1 - w\overline{\zeta})^{-(n-2)/2} = \sum_{k=0}^{\infty} \left(\frac{n-2}{2}\right)_k (z\overline{\zeta})^k.$$

Recall that $(1 - w\overline{\zeta})^{-n/2}$ is the reproducing kernel for the weighted Bergman space $\mathscr{H}_{n/2}(U)$, defined as

$$\mathscr{H}_{n/2}(U) = \{f \in \mathcal{O}(U) \mid \frac{\Gamma\left(\frac{n}{2}\right)}{\pi\Gamma\left(\frac{n-2}{2}\right)} \int_U |f(\zeta)|^2(1 - |\zeta|^2)^{(n-4)/2} dm(\zeta) < \infty\},$$

Polynomials of different degree are orthogonal in $\mathscr{H}_{n/2}(U)$ and hence we have

$$\begin{aligned}
&\int_U (1 - w\overline{\zeta})^{-(n-2)/n} \zeta^m (1 - |\zeta|^2)^{(n-4)/2} dm(\zeta) \\
&= \int_U \sum_{k=0}^{\infty} \left(\frac{n-2}{2}\right)_k (z\overline{\zeta})^k \zeta^m (1 - |\zeta|^2)^{(n-4)/2} dm(\zeta) \\
&= \int_U \sum_{k=0}^{\infty} \left(\frac{n-2}{2}\right) \left(\frac{n}{2}\right)_m (z\overline{\zeta})^m \zeta^m (1 - |\zeta|^2)^{(n-4)/2} dm(\zeta) \\
&= \pi w^m,
\end{aligned}$$

where the last equality follows from the reproducing property in $\mathscr{H}_{n/2}(U)$. Summing up, we have

$$(T_{(n-2)/2}(\zeta_1 + i\zeta_2)^m)(z) = \frac{n-2}{2\pi^2}(z_1 + iz_2)^m \tag{99}$$

From this and the intertwining of the $\mathfrak{l}$-action, it follows that

$$T_{(n-2)/2}\left(\bigoplus_m E_m(S^{n-1})\right) \subseteq \bigoplus_m E_m \tag{100}$$

To compute the norm of $T_{(n-2)/2}(p)$ where $p \in E_k(S^{n-1})$, we first fix $r < 1$ and consider the polynomial $T_{(n-2)/2}(p(rz))$. By definition

$$\begin{aligned}
T_{(n-2)/2}(p)(rz) &= \int_{S^{n-1}} K_\nu(rz, \zeta) p(\zeta) d\sigma(\zeta) \\
&= \int_{S^{n-1}} K_\nu(z, r\zeta) p(\zeta) d\sigma(\zeta). \qquad (101)
\end{aligned}$$

The norm is given by

$$\|T_{(n-2)/2}(p)(r\,\cdot)\|_\nu^2 = \int_{S^{n-1}} \int_{S^{n-1}} p(\zeta)\overline{p(\eta)} K_\nu(r\zeta, r\eta) d\sigma(\zeta) d\sigma(\eta).$$

Finally, we let $r \to 1$ and obtain

$$\|T_{(n-2)/2}(p)\|_\nu^2 = \int_{S^{n-1}} \int_{S^{n-1}} p(\zeta)\overline{p(\eta)} K_\nu(\zeta, \eta) d\sigma(\zeta) d\sigma(\eta).$$

From this and the orthogonality of the spaces E_k, it follows that $T_{(n-2)/2}$ maps $(\bigoplus_m E_m(S^{n-1})$ isometrically onto $(\bigoplus_m E_m)$. Hence it extends to a unitary operator from $\mathscr{C}_{(n-2)/2n}$ onto $\mathscr{H}_{(n-2)/2}$. ■

References

[1] Cowling, M., *Harmonic analysis on some nilpotent Lie groups (with application to the representation theory of some semisimple Lie groups)*, "Topics in modern harmonic analysis, Vol. I, II" Turin/Milan, 1982, 81–123, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983, 81–123.

[2] Faraut, J. and A. Korányi, *Function spaces and reproducing kernels on bounded symmetric domains*, J. Funct. Anal. **88** (1990), 64–89.

[3] —, "Analysis on symmetric cones," Oxford Mathematical Monographs, Oxford Science Publications, The Clarendon Press Oxford University Press, New York, 1994.

[4] Folland, G. B., "A course in abstract harmonic analysis," Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[5] Gasper, G., and M. Rahman, "Basic hypergeometric series", Encyclopedia of Mathematics and its Applications **35**, Cambridge University Press, 1990.

[6] Helgason, S., "Groups and geometric analysis," Mathematical Surveys and Monographs **83**, American Mathematical Society, Providence, RI, 2000.

[7] Howe, R., *Transcending classical invariant theory*, J. Amer. Math. Soc. **2** (1989), 535–552.

[8] Hua, L., *On the theory of Fuchsian functions of several variables*, Ann. of Math. (2) **47** (1946), 167–191.

[9] Jakobsen, H. P., and M. Vergne, *Restrictions and expansions of holomorphic representations*, J. Funct. Anal. **34** (1979), 29–53.

[10] Kashiwara, M., and M. Vergne, *On the Segal-Shale-Weil representations and harmonic polynomials*, Invent. Math. **44** (1978), 1–47.

[11] Knapp, A. W., "Representation theory of semisimple groups, an overview based on examples", Princeton University Press, Princeton, NJ, 2001.

[12] —, "Lie Groups beyond an Introduction," Progress in Mathematics, **140**, Birkhäuser Boston Inc., Boston, MA, 2002.

[13] Knapp, A. W., and D. A. Vogan, Jr., "Cohomological induction and unitary representations," Princeton Mathematical Series, 45, Princeton University Press, Princeton, NJ, 1995.

[14] Kobayashi, T., and B. Ørsted, *Analysis on the minimal representation of $O(p,q)$, II.* Branching laws, Adv. Math. **180** (2003), 513–550.

[15] Loos, O., "Bounded symmetric domains and Jordan pairs," Lecture Notes, University of California, Irvine, 1977.

[16] Murphy, G. J., "C^*-algebras and operator theory," Academic Press Inc., Boston, MA, 1990.

[17] Naĭmark, M. A., "Normed rings," P. Noordhoff N. V., Groningen, 1964.

[18] Neretin, Yu. A., *Beta integrals and finite orthogonal systems of Wilson polynomials*, Mat. Sb. **193** (2002), 131–148.

[19] —, *Plancherel formula for Berezin deformation of L^2 on Riemannian symmetric space*, J. Funct. Anal. **189** (2002), 336–408.

[20] Rossi, H., and M. Vergne, *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group*, Acta Math. **136** (1976), 1–59.

[21] Rudin, Walter, "Function theory in the unit ball of $\mathbf{C}^n$," Grundlehren der Mathematischen Wissenschaften **241**, Springer-Verlag, New York, 1980.

[22] —, "Functional analysis," International Series in Pure and Applied Mathematics, McGraw-Hill Inc., New York, 1991.

[23] Stein, E. M., and G. Weiss, "Introduction to Fourier analysis on Euclidean spaces," Princeton Mathematical Series **32**, Princeton University Press, Princeton, N.J., 1971.

[24] Unterberger, A., and H. Upmeier, *The Berezin transform and invariant differential operators*, Comm. Math. Phys. **164** (1994), 563–597.

[25] van Dijk, G., and M. Pevzner, *Berezin kernels of tube domains*, J. Funct. Anal. **181** (2001), 189–208.

[26] Wallach, N. R., *The analytic continuation of the discrete series. I, II*, Trans. Amer. Math. Soc. **251** (1979), 1–17, 19–37.

[27] Wilson, J. A., *Some hypergeometric orthogonal polynomials*, SIAM J. Math. Anal. **11** (1980), 690–701.

[28] Zhang, G., *Berezin transform on real bounded symmetric domains*, Trans. Amer. Math. Soc. **353** (2001), 3769–3787.

[29] —, *Tensor products of minimal holomorphic representations*, Represent. Theory **5** (2001), 164–190.

[30] —, *Branching coefficients of holomorphic representations and Segal-Bargmann transform*, J. Funct. Anal. **195** (2002), 306–349.

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