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Decomposition and Multiplicities for Quasiregular Representations of Algebraic Solvable Lie Groups

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Abstract. We obtain an explicit irreducible decomposition for the quasiregular representation τ of a connected algebraic solvable Lie group induced from a co-normal Levi factor. In the case where the multiplicity function is unbounded, we show that τ is a finite direct sum of subrepresentations τ_ϵ where for each ϵ, τ_ϵ is either infinite or has finite but unbounded multiplicity. We obtain a criterion by which the cases of bounded multiplicity, finite unbounded multiplicity, and infinite multiplicity are distinguished.
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0. Introduction

Let N be a connected, simply connected nilpotent Lie group, and let H be a connected abelian group acting on $\mathfrak{n}$ by automorphisms in such a way that $\mathrm{ad}(\mathfrak{h})$ is completely reducible. The resulting semi-direct product $G = N \rtimes H$ is solvable, and if it is also exponential, then the irreducible decomposition of monomial unitary representations of G can be understood precisely in terms of co-adjoint orbit parameters [8, 10]. In the case where $\tau = \mathrm{ind}_H^G(1)$ and G is algebraic and exponential, then a number of precise results regarding the decomposition of τ have been obtained [11, 6]. In particular, the question of the existence of admissible vectors in the case where H has trivial stabilizers is settled in [4] by means of an explicit decomposition for τ. We are concerned in this paper with the following situation where G is not exponential. Let U be a torus in $\mathrm{Aut}(\mathfrak{n}_{\mathbb{C}})$ that is defined over $\mathbb{R}$; we assume that $H = U(\mathbb{R})_0$ is the connected component of the set of real points of U. The group G is not exponential here, but it is Type

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1 and acts regularly on $\hat{N}$. Again for $\tau = \mathrm{ind}_H^G(1)$, the decomposition of τ is obtained in [9] (where the context is more general) in terms of parameters for $\hat{G}$ that constitute a fiber space over the base $\hat{N}/H$. Motivated in part by the question of admissibility in this context, the aim of the present work is two-fold. First, to give a natural construction for this decomposition in terms of an explicit manifold that parametrizes (a.e.) $\hat{N}/H$, an explicit measure $\tilde{\mu}$ on this manifold, and an explicit intertwining operator Φ. Second, to describe the multiplicity function for τ in precise terms, and in particular to obtain a criterion for the case where it is finite but unbounded.

Since τ is naturally realized in $L^2(N)$ so that its restriction to N is the regular representation, a starting point for this analysis is a concrete Plancherel formula for $L^2(N)$. Originally this is obtained in [13], where $\hat{N}$ is explicitly parametrized by a cross-section for coadjoint orbits in $\mathfrak{n}^*$. Since we are ultimately interested in an explicit parametrization for $\hat{N}/H$, we then consider the natural action of H on $\mathfrak{n}^*/N \simeq \hat{N}$, with the hope of describing this action in terms of the cross-section. However, the cross-section used in [13] is not H-invariant in general. In order to construct an explicit cross-section for coadjoint orbits in $\mathfrak{n}^*$ that is H-invariant, we apply a method of stratification and parametrization of coadjoint orbits first developed in [7] for the case of exponential groups, and then slightly but significantly generalized in [1]. As a result of the work in [1], one obtains a cross-section for each stratum (or "layer") in $\mathfrak{n}^*$ that is simply described and well-behaved under certain projection maps. As usual, the construction depends only upon a certain choice of Jordan-Hölder basis for the complexification of the Lie algebra. In the present work we show that by making this choice of basis so as to consist of eigenvectors for $\mathrm{ad}(H)$, the resulting orbital cross-section in each layer is indeed H-invariant. In particular, specializing to the minimal Zariski-open layer, we obtain an H-invariant cross-section Λ that parametrizes almost all of $\hat{N}$, and thus the action of H on $\hat{N}$ is understood in explicit terms as the action of H on Λ. Moreover, there is a closed subgroup K of H that coincides exactly with the stabilizer H_λ in H for all $\lambda \in \Lambda$. The preceding constructions are carried out in Section 1.

In Section 2, we specialize to the class of G that are algebraic in the sense described above. Then the quotient space Λ/H is described by means of an explicit algebraic submanifold Σ of Λ, and a finite subgroup F of H acting on Σ, so that the map $H\lambda \mapsto H\lambda \cap \Sigma$ is a homeomorphism of Λ/H onto Σ/F. For each H-orbit $\mathcal{O}^H \subset \Lambda$, a natural semi-invariant measure ω is defined on $\mathcal{O}^H$ and an explicit measure $\tilde{\mu}$ on Σ is defined so that for any fundamental domain Σ_0 for Σ/F,

$$\int_\Lambda f(\lambda)\, d\mu(\lambda) = \int_{\Sigma_0} \int_{\mathcal{O}_\sigma^H} f(\lambda)\, d\omega_\sigma(\lambda)\, d\tilde{\mu}(\sigma)$$

Here $\tilde{\mu}$ is explicitly described in terms of the usual Pfaffian and a Lebesgue measure on Σ_0. The stage is then set for an explicit decomposition of the quasi-regular representation τ, which is taken up in Section 3, and as in [11] this depends upon an understanding of the action of K on each $\mathfrak{n}/\mathfrak{n}(\lambda), \lambda \in \Sigma_0$. We write Λ as a finite disjoint union $\Lambda = \Lambda^\epsilon$ where $\epsilon \in \{1, -1\}^m$ are "sign indices" measuring the positivity (or lack thereof) of the Vergne polarizations $\mathfrak{p}(\lambda)$ associated to $\lambda \in \Lambda_\epsilon$. Setting $\mathfrak{e}(\lambda) = (\mathfrak{p}(\lambda) + \overline{\mathfrak{p}(\lambda)}) \cap \mathfrak{n}$ and $\mathfrak{d}(\lambda) = \mathfrak{p}(\lambda) \cap \overline{\mathfrak{p}(\lambda)} \cap \mathfrak{n}$, we construct irreducible representations π_λ associated with λ by inducing from a Bargmann-Fock representation of $E(\lambda)$. For $\lambda \in \Lambda^\epsilon$, the actions of K in $\mathfrak{n}/\mathfrak{n}(\lambda)$ (or on $\mathfrak{n}/\mathfrak{d}(\lambda)$) are isomorphic, and hence the Weil representations γ_λ are isomorphic. Using methods borrowed from [9], an intertwining operator is defined that obtains a finite decomposition $\tau \simeq \oplus_\epsilon \tau_\epsilon$ where

$$\tau_\epsilon = \int_{\Sigma_0^\epsilon}^{\otimes} \int_{\hat{K}}^{\otimes} m_\epsilon(\eta) \cdot \rho_\lambda^{\overline{\eta}} \, d\eta \, d\tilde{\mu}(\lambda).$$

Here $m_\epsilon(\eta)$ is the multiplicity of $\eta \in \hat{K}$ in the decomposition of γ_λ, and $\rho_\lambda^{\overline{\eta}}$ is the irreducible representation of G induced from an extension $\tilde{\pi}_\lambda \otimes \overline{\eta}$ of π_λ to NK corresponding to $\overline{\eta}$. Since the K-actions on $\mathfrak{n}/\mathfrak{d}(\lambda)$ are constant on each Λ^ϵ, the multiplicity functions depend only upon the index ϵ.

In Section 5 we turn to the analysis of the multiplicity functions. The irreducible representation π_λ of N is realized in an L^2-space where γ_λ is simply described, and we show that the real issue is the multiplicities for the characters of the identity component $K''_\circ$ in the anisotropic subgroup K'' of K; note that $K''_\circ \simeq \mathbb{T}^s$ for some s. By evaluating a (convenient) basis for the Lie algebra $\mathfrak{k}''$ at the roots of $\mathfrak{k}''$ in $\mathfrak{n}/\mathfrak{d}(\lambda)$, we codify this action in an "action matrix" P. For $h \in \hat{K}''_\circ = \mathbb{Z}^s$, the value $m_\epsilon(h)$ is the number of integer solutions to the diophantine system $Pn = h$ that lie in a convex cone E^ϵ determined by ϵ. This number is finite if and only if the intersection of the real solution set $\mathcal{S}(P, h)$ for $Px = h$ with E^ϵ is bounded. In particular, if K acts with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$ (in other words, if the image of K in $\mathrm{Sp}(\mathfrak{n}_\mathbb{C}/\mathfrak{n}(\lambda)_\mathbb{C})$ is Cartan), then P is invertable and m_ϵ is bounded (with value 2^r a.e., given by the rank of the split subgroup K' of K, see also [11, Lemma 3.3]). In the case where K does not act with full rank, then m_ϵ is unbounded but not necessarily infinite: see for example [9, Section 8, example (vii)]. When P is not invertable but $S(P, h) \cap E^\epsilon$ is bounded for all h, then m_ϵ is finite everywhere, and this condition depends only upon P and the sign index ϵ. We prove a precise criterion for unbounded finite multiplicity in terms of the relationship between the action of $\mathfrak{k}$ on $\mathfrak{n}/\mathfrak{d}(\lambda)$ and the cone E^ϵ. We obtain the following result, which is stated more precisely in Section 5 as Theorem 5.4.

Theorem 0.1. *Let $G = N \rtimes H$ be a real algebraic solvable Lie group with N simply connected nilpotent and H a connected Levi factor, and let $\tau = ind_H^G$. Let K be the generic stabilizer in H. Then one of the following obtains.*

(1) If K acts with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$, then τ has uniform multiplicity 2^r, where

r is the split rank of K.

(2) If K does not act with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$, then τ is infinite.

(3) If K acts with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$, but not with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$, then τ is a finite direct sum of subrepresentations τ_ϵ, such that for each ϵ, either τ_ϵ has finite unbounded multiplicity, or τ_ϵ is infinite.

We conclude in Section 6 with four examples to illustrate both methods and notations.

1. An *H*-invariant Orbital Cross-section

Let N be a real, connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let $\mathfrak{l}$ be the complexification of $\mathfrak{n}$, and for $Z \in \mathfrak{l}$ let $\Re Z$ and $\Im Z$ denote the elements in $\mathfrak{n}$ for which $Z = \Re Z + i\Im Z$ (we apply the same notation to complex numbers also.) Choose an ordered basis $\{Z_1, \ldots, Z_n\}$ for $\mathfrak{l}$ with the properties that

(i) For each $1 \le j \le n$, $\mathfrak{l}_j = \mathbb{C}\text{-span}\{Z_1, Z_2, ..., Z_j\}$ is an ideal in $\mathfrak{l}$.

(ii) If $\mathfrak{l}_j \neq \overline{\mathfrak{l}_j}$ then $\mathfrak{l}_{j+1} = \overline{\mathfrak{l}_{j+1}}$ and $Z_{j+1} = \overline{Z_j}$.

(iii) if $\mathfrak{l}_j = \overline{\mathfrak{l}_j}$ and $\mathfrak{l}_{j-1} = \overline{\mathfrak{l}_{j-1}}$, then $Z_j \in \mathfrak{n}$.

We shall find the following notation useful. Define $I = \{1 \le j \le n \mid \mathfrak{l}_j = \bar{\mathfrak{l}}_j\}$, $I' = \{j \in I \mid j-1 \in I\}$, and $I'' = I - I'$. For each $1 \le j \le n$ set $j' = \max\{k \in I \mid k < j\}$ and $j'' = \min\{k \in I \mid k \ge j\}$.

An element $X \in \mathfrak{n}$ can be written as $X = z_1 Z_1 + z_2 Z_2 + \cdots + z_n Z_n$ and can be identified with the element $x = (x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ setting $x_j = z_j$ if $j \in I'$, and $x_j = \Re z_j, x_{j+1} = \Im z_j$, if $j \notin I$. Let $\mathfrak{n}$ have the Lebesgue measure obtained by this identification.

Let $\mathfrak{n}^*$ be the linear dual of $\mathfrak{n}$; elements of $\mathfrak{n}^*$ are extended to $\mathfrak{l}$ in the natural way. For $\ell \in \mathfrak{n}^*$, write $\ell_j = \ell(Z_j)$, and $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$. Note that if $j \notin I$, then $\ell_{j+1} = \overline{\ell_j}$. Thus ℓ is identified with an element of $\mathbb{C}^n$ and is in turn identified with an element ξ of $\mathbb{R}^n$ by setting $\xi_j = \ell_j$ if $j \in I'$, and $\xi_j = \Re \ell_j, \xi_{j+1} = \Im \ell_j$ if $j \notin I$. Let $\mathfrak{n}^*$ have the corresponding Lebesgue measure via this identification.

Let H be a closed, abelian subgroup of $\mathrm{Aut}(N)$ with Lie algebra $\mathfrak{h}$; H acts linearly on $\mathfrak{n}$ and $\mathfrak{n}^*$ as usual, and we denote all actions multiplicatively. We assume that for each $a \in H$, the basis elements Z_j are eigenvectors of a,. For each $a \in H$ we set

$$aZ_j = \delta_j(a) Z_j, \ 1 \le j \le n,$$

and we denote the differential $\mathbf{d}\delta_j$ by γ_j. Let $D(n, \mathbb{C})$ be the torus of all diagonal elements in $GL(n, \mathbb{C})$, and for $a \in H$ put

$$\delta(a) = \mathrm{diag}(\delta_1(a), \delta_2(a), \ldots, \delta_n(a)) \in D(n, \mathbb{C}).$$

We assume that the action of H on $\mathfrak{n}$ is effective, and so we can identify H with its image $\delta(H) \subset D(n,\mathbb{C})$. Let G be the semi-direct product of N by H, and $\mathfrak{g} = \mathfrak{n} + \mathfrak{h}$ its Lie algebra. The inverse of the modular function of G is $|\delta| := |\delta_1\delta_2\cdots\delta_n|$. Note that $\mathfrak{l}_j$ is an ideal in $\mathfrak{g}_{\mathbb{C}}$, $1 \leq j \leq n$. We denote the actions of G on $\mathfrak{n}$ and $\mathfrak{n}^*$ multiplicatively as well.

For any subset $\mathfrak{t}$ of $\mathfrak{l}$, if f is a linear functional defined on $[\mathfrak{l}, \mathfrak{t}]$, then set

$$\mathfrak{t}^f = \{Z \in \mathfrak{g} \mid f[Z,T] = 0 \text{ holds for every } T \in \mathfrak{t}\}.$$

If $\mathfrak{t}$ is an ideal in $\mathfrak{l}$, then $\mathfrak{t}^f$ is a subalgebra of $\mathfrak{l}$. Recall that for any $\ell \in \mathfrak{n}^*$, the Lie algebra $\mathfrak{g}(\ell)$ of its stabilizer $G(\ell)$ in G is $\mathfrak{n}^\ell$, and the Lie algebra $\mathfrak{n}(\ell)$ of its stabilizer $N(\ell)$ in N is $\mathfrak{n}^\ell \cap \mathfrak{n}$. We apply the stratification procedure as described in [7] to the Lie algebra $\mathfrak{n}$; in [1], it is observed that this procedure does not require that the chosen basis of $\mathfrak{n}_{\mathbb{C}}$ be real (as is the assumption in [7]). Thus we have the following.

(1) To each $\ell \in \mathfrak{n}^*$ there is associated a set $\mathbf{e}(\ell) \subset \{1, 2, \ldots, n\}$ defined by

$$\mathbf{e}(\ell) = \{1 \leq j \leq n \mid \mathfrak{l}_j \not\subset \mathfrak{l}_{j-1} + \mathfrak{l}^\ell\}.$$

Note that since $\overline{\mathfrak{l}^\ell} = \mathfrak{l}^\ell$, then for each index j, $j'' \in \mathbf{e}(\ell)$ implies $j \in \mathbf{e}(\ell)$. Note also that the number of elements in the index set $\mathbf{e}(\ell)$ is even since it is the dimension of the coadjoint orbit of N through ℓ. For a subset $\mathbf{e}$ of $\{1, 2, \ldots, n\}$, the set $\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{n}^* \mid \mathbf{e}(\ell) = \mathbf{e}\}$ is N-invariant. The non-empty $\Omega_{\mathbf{e}}$ are determined by polynomials as follows: to each index set $\mathbf{e}$ one associates the skew-symmetric matrix

$$M_{\mathbf{e}}(\ell) = \big[\ell[Z_i, Z_j]\big]_{i,j\in\mathbf{e}}.$$

Setting

$$Q_{\mathbf{e}}(\ell) = \det\, M_{\mathbf{e}}(\ell),$$

one has a total ordering $\prec$ on the set $\mathcal{E} = \{\mathbf{e} \mid \Omega_{\mathbf{e}} \neq \emptyset\}$ such that

$$\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^* \mid Q_{\mathbf{e}'}(\ell) = 0 \text{ for all } \mathbf{e}' \prec \mathbf{e}, \text{ and } Q_{\mathbf{e}}(\ell) \neq 0\}.$$

(2) Set $d = |\mathbf{e}|/2$. To each ℓ there is associated a "polarizing sequence" of subalgebras

$$\mathfrak{l} = \mathfrak{p}_0(\ell) \supset \mathfrak{p}_1(\ell) \supset \cdots \supset \mathfrak{p}_d(\ell) = \mathfrak{p}(\ell),$$

and an index *sequence pair* $\mathbf{i}(\ell) = \{i_1 < i_2 < \cdots < i_d\}$ and $\mathbf{j}(\ell) = \{j_1, j_2, \ldots, j_d\}$, having values in $\mathbf{e}(\ell)$, defined recursively for $1 \leq k \leq d$ by

$$i_k = \min\{1 \leq j \leq n \mid \mathfrak{l}_j \cap \mathfrak{p}_{k-1}(\ell) \not\subset \mathfrak{p}_{k-1}(\ell)^\ell\},$$

$$\mathfrak{p}_k(\ell) = \left(\mathfrak{p}_{k-1}(\ell) \cap \mathfrak{l}_{i_k}\right)^\ell \cap \mathfrak{p}_{k-1}(\ell),$$

and

$$j_k = \min\{1 \le j \le n \mid \mathfrak{l}_j \cap \mathfrak{p}_{k-1}(\ell) \not\subset \mathfrak{p}_k(\ell)\}.$$

For each k, $i_k < j_k$, and $\mathbf{e}(\ell)$ is the disjoint union of the values of $\mathbf{i}(\ell)$ and $\mathbf{j}(\ell)$. The subalgebra $\mathfrak{p}(\ell)$ is the complex Vergne polarization associated to ℓ and to the given Jordan- Hölder sequence for $\mathfrak{l}$. Note that $\overline{\mathfrak{p}(\ell)}$ does not necessarily coincide with $\mathfrak{p}(\ell)$.

Since $\mathbf{i}(\ell)$ must be increasing, it is determined by $\mathbf{e}(\ell)$ and $\mathbf{j}(\ell)$. For any such splitting of $\mathbf{e}$ into such a sequence pair $(\mathbf{i}, \mathbf{j})$ we have the N-invariant set $\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \Omega_{\mathbf{e}} \mid \mathbf{j}(\ell) = \mathbf{j}\}$. We refer to these sets as "fine layers", and to the collection of non-empty $\Omega_{\mathbf{e},\mathbf{j}}$ as the fine stratification of $\mathfrak{n}^*$. For $1 \le k \le d$, if we set

$$M_{\mathbf{e},k}(\ell) = \left[\ell[Z_i, Z_j]\right]_{i,j \in \{i_1, j_1, i_2, j_2, \ldots, i_k, j_k\}}$$

let $\mathbf{Pf}_{\mathbf{e},k}(\ell)$ denote the Pfaffian of $M_{\mathbf{e},k}(\ell)$, and let

$$\mathbf{P}_{\mathbf{e},\mathbf{j}}(\ell) = \mathbf{Pf}_{\mathbf{e},1}(\ell)\mathbf{Pf}_{\mathbf{e},2}(\ell) \cdots \mathbf{Pf}_{\mathbf{e},d}(\ell).$$

Then there is a total ordering $\prec\!\prec$ on the pairs $\mathbf{e}, \mathbf{j}$ such that

$$\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \mathfrak{g}^* \mid \mathbf{P}_{\mathbf{e}',\mathbf{j}'}(\ell) = 0 \text{ for all } (\mathbf{e}', \mathbf{j}') \prec\!\prec (\mathbf{e}, \mathbf{j}) \text{ and } \mathbf{P}_{\mathbf{e},\mathbf{j}}(\ell) \neq 0\}.$$

Lemma 1.1. *For $a \in H$ and $1 \le k \le d$, one has*

$$\mathbf{Pf}_{\mathbf{e},k}(a \cdot \ell) = \left(\prod_{l=1}^{k} \delta_{i_l}(a)^{-1} \delta_{j_l}(a)^{-1}\right) \mathbf{Pf}_{\mathbf{e},k}(\ell).$$

In particular, the fine layers are H-invariant.

Proof. Let $a \in H$ and set $\mathfrak{s}_k = \operatorname{span}\{Z_{i_1}, Z_{j_1}, \ldots, Z_{i_k}, Z_{j_k}\}$. Let $\sigma_k(W, \ell)$ denote the projection of W into the subspace $\mathfrak{s}_k^\ell$ parallel to $\mathfrak{s}_k$. It is easily seen that $a \cdot \mathfrak{s}_k^\ell = (a \cdot \mathfrak{s}_k)^{a \cdot \ell}$ and since our basis consists of eigenvectors for a, then $a \cdot \mathfrak{s}_k = \mathfrak{s}_k$ and we have $a \cdot \mathfrak{s}_k^\ell = \mathfrak{s}_k^{a \cdot \ell}$. Now it follows that $a \circ \sigma_k(\cdot\ , a \cdot \ell) \circ a^{-1} = \sigma_k(\cdot, \ell)$ and hence for any $W \in \mathfrak{l}$, $a^{-1} \cdot \sigma_k(W, a \cdot \ell) = \sigma_k(a^{-1} \cdot W, \ell)$, $1 \le k \le d$. In particular, we have

$$\begin{aligned} a \cdot \ell[\sigma_{k-1}(Z_{i_k}, a \cdot \ell), \sigma_{k-1}(Z_{j_k}, a \cdot \ell)] &= \ell[\sigma_{k-1}(a^{-1} \cdot Z_{i_k}, \ell), \sigma_{k-1}(a^{-1} \cdot Z_{j_k}, \ell)] \\ &= \delta_{i_k}(a)^{-1} \delta_{j_k}(a)^{-1}\ \ell[\sigma_{k-1}(Z_{i_k}, \ell), \sigma_{k-1}(Z_{j_k}, \ell)] \end{aligned}$$

But $\mathbf{Pf}_{\mathbf{e},1}(\ell) = \ell[Z_{i_1}, Z_{j_1}]$ and

$$\mathbf{Pf}_{\mathbf{e},k}(\ell) = \mathbf{Pf}_{\mathbf{e},k-1}(\ell)\ \ell[\sigma_{k-1}(Z_{i_k}, \ell), \sigma_{k-1}(Z_{j_k}, \ell)],\ k = 2, 3, \ldots d.$$

The desired formula follows. ∎

Now suppose that $Z_j \in \mathfrak{n}$ holds for $1 \le j \le n$, and fix a fine layer Ω. Then it is well-known that a cross-section for the coadjoint orbits in Ω is $\Omega \cap \{\ell \mid \ell_j = 0, \forall j \in \mathbf{e}\}$, but it is clear that such a cross-section is not necessarily H-invariant if H has non-real roots. However, if each Z_j is an eigenvector for the elements $a \in H$, then we shall see that the methods of [1, 7] obtain an H-invariant cross-section.

We begin by describing the construction of [7, Lemma 1.3] (see also [5, Lemma 1.2.1]), which proceeds by means of a case-by-case analysis. To this end, and following the notation of [7, page 248], we define subsets of $K = \{1, 2, \ldots d\}$ as follows. We set $K_0 = \{1 \le k \le d \mid i_k - 1 \in I \text{ and } i_k \in I\}$, $K_1 = \{1 \le k \le d \mid i_k \notin I \text{ and } i_k + 1 \notin \mathbf{e}\}$, $K_2 = \{1 \le k \le d \mid i_k - 1 \in \mathbf{j} \setminus I\}$, $K_3 = \{1 \le k \le d \mid i_k \notin I \text{ and } i_k + 1 \in \mathbf{j}\}$, $K_4 = \{1 \le k \le d \mid i_k \notin I \text{ and } i_k + 1 \in \mathbf{i}\}$, and $K_5 = \{1 \le k \le d \mid i_k - 1 \in \mathbf{i} \setminus I\}$. One observes that if $k \in K_2$, then $i_k - 1 = j_h$ where $1 \le h < k$. Second, it is shown in [7, page 252] that if $k \in K_3$ then $i_k + 1 = j_k$. Third, note that the fact that $\mathbf{i}$ is an increasing sequence implies that if $k \in K_4$, then $i_k + 1 = i_{k+1}$, and $K_5 = K_4 + 1$. It follows from these observations that $K = \cup_{N=0}^{5} K_N$ as a disjoint union. We have the following.

Lemma 1.2. ([1, Lemma 3.1], [7, Lemma 1.3]) *Let $\mathfrak{n}$ be a nilpotent Lie algebra over $\mathbb{R}$, and choose an adaptable basis for $\mathfrak{l} = \mathfrak{n}_c$. Let $\Omega = \Omega_{\mathbf{e},\mathbf{j}}$ be a fine layer with $2d$ the dimension of the G-orbits in Ω. Assume $d > 0$. Then one has a construction for rational functions $V_k : \Omega \to \mathfrak{l}$ and $U_k : \Omega \to \mathfrak{l}$, $1 \le k \le d$, that satisfy the following conditions.*

(i) For each $\ell \in \Omega$, $U_k(\ell) \in \mathfrak{l}_{j_k''} - \mathfrak{l}_{j_k'}$ and $V_k(\ell) \in \mathfrak{l}_{i_k''} - \mathfrak{l}_{i_k'}$

(ii) $\ell[U_h(\ell), U_k(\ell)] = \ell[V_h(\ell), V_k(\ell)] = 0, 1 \le h, k \le d$.

(iii) $\ell[U_h(\ell), V_k(\ell)] = 0$ if and only if $h \ne k$, $1 \le h, k \le d$.

(iv) There is a covering $\mathcal{C}$ of Ω by finitely many Zariski-open subsets and for each $O \in \mathcal{C}$ and $1 \le k \le d$, a continuous function $\phi_k^O : O \to \mathbb{T}$, such that for each $\ell \in O$, the elements $\{\phi_k^O(\ell)^{-1} U_k(\ell)$ and $\phi_k^O(\ell)^{-1} V_k(\ell)$ are real (i.e., they belong to $\mathfrak{n}$.)

(v) For $1 \le k \le d$, if $k \in K_0 \cup K_1 \cup K_2$, then $\mathfrak{h}_k(\ell) = \mathfrak{h}_{k-1}(\ell) \cap \{V_k(\ell)\}^\ell$ holds for each $\ell \in \Omega$. If $k \in K_4$, then $\mathfrak{h}_{k+1}(\ell) = \mathfrak{h}_{k-1}(\ell) \cap \{V_k(\ell), V_{k+1}(\ell)\}^\ell$ holds for each $\ell \in \Omega$.

Set $\mathfrak{m}_0(\ell) = (0)$, and for each $1 \le k \le d$, set

$$\mathfrak{m}_k(\ell) = \mathbb{C}\text{-span}\{V_1(\ell), V_2(\ell), \ldots, V_k(\ell), U_1(\ell), U_2(\ell), \ldots, U_k(\ell)\}.$$

so that for each $\ell \in \Omega$, $\mathfrak{l} = \mathfrak{m}_k(\ell) \oplus \mathfrak{m}_k(\ell)^\ell$. For $Z \in \mathfrak{l}, \ell \in \Omega$, let $\rho_k(\cdot, \ell)$ be the projection of $\mathfrak{l}$ onto $\mathfrak{m}_k(\ell)^\ell$ parallel to $\mathfrak{m}_k(\ell)$, with $\rho_0(\cdot, \ell)$ the identity mapping.

It follows easily from the preceding that $\rho_k(\cdot,\ell)$ has the following properties (for each $1 \le k \le d, \ell \in \Omega$).

(a) For each $Z \in \mathfrak{l}$, $\rho_k(\overline{Z},\ell) = \overline{\rho_k(Z,\ell)}$.

(b) ρ_k satisfies the recursion formula

$$\rho_k(Z,\ell) = \rho_{k-1}(Z,\ell) - \frac{\ell[\rho_{k-1}(Z,\ell),U_k(\ell)]}{\ell[V_k(\ell),U_k(\ell)]}\,V_k(\ell) - \frac{\ell[\rho_{k-1}(Z,\ell),V_k(\ell)]}{\ell([U_k(\ell),V_k(\ell)]}\,U_k(\ell).$$

(c) $\rho_k(\mathfrak{l},\ell) \subset \mathfrak{l}^{\ell}_{i'_{k+1}}$, holds for $1 \le k \le d-1$ and $\rho_d(\mathfrak{l},\ell) \subset \mathfrak{l}(\ell)$. Also, $\rho_k(\mathfrak{l}_j,\ell) \subset \mathfrak{l}_{j''}, 1 \le j \le n$.

(d) For any $W,Z \in \mathfrak{l}$, $\ell[\rho_k(W,\ell),\rho_k(Z,\ell)] = \ell[W,\rho_k(Z,\ell)] = \ell[\rho_k(W,\ell),Z]$

There are two more properties of the function ρ_k that emerge from the above and that we shall need later.

Lemma 1.3. [1, Lemma 3.2] *One has each of the following.*

(a) If $k \notin K_4$, then $\mathfrak{m}_k(\ell)^{\ell} \subset \mathfrak{p}_k(\ell)$, and hence (by definition) $\rho_k(\cdot,\ell)$ maps $\mathfrak{l}$ into $\mathfrak{p}_k(\ell)$.

(b) For each $1 \le k \le d$, $\rho_{k-1}(\cdot,\ell)$ maps $\mathfrak{l}_{i'_k}$ into $\mathfrak{l}^{\ell}$.

An implicit part of the proof of [7, Lemma 1.3] is the construction of rational functions $Z_{i_k} : \Omega \to \mathbf{C}\text{-span}\{Y_1,Y_2\}$ and $Z_{j_k} : \Omega \to \mathbf{C}\text{-span}\{X_1,X_2\}$ such that $V_k(\ell) = \rho_{k-1}(Z_{i_k}(\ell),\ell))$ and $U_k(\ell) = \rho_{k-1}(Z_{j_k}(\ell),\ell))$. An important insight of [1] is the utility of these functions in describing coadjoint orbit cross-sections. They are defined case by case, as follows.

$k \in K_0$. We have $Z_{i_k}(\ell) = Z_{i_k}$. (Note that Z_{i_k} is real in this case.)

$k \in K_1$. We have

$$Z_{i_k}(\ell) = \frac{1}{2}\Big(\ell[\rho_{k-1}(Z_{j_k},\ell),\overline{Z}_{i_k}]Z_{i_k} + \ell[\rho_{k-1}(\overline{Z}_{j_k},\ell),Z_{i_k}]\overline{Z}_{i_k}\Big)$$

$k \in K_2$. Here we have $i_k - 1 = j_r$ for some $1 \le r < k$ and we have

$$Z_{i_k}(\ell) = \frac{1}{2i}\Big(\ell[\overline{Z}_{j_r},V_r(\ell)]Z_{j_r} - \ell[Z_{j_r},V_r(\ell)]\overline{Z}_{j_r}\Big).$$

$k \in K_3$. Here we can take $Z_{i_k}(\ell) = \Im Z_{i_k}$.

$k \in K_4$. It is not necessarily true here that $Z_{j_{k+1}} = \overline{Z}_{j_k}$, but it is true that $j_{k+1} > {j_k}'$. Accordingly this case splits into two subcases.

Subcase (a). $Z_{j_{k+1}} = \overline{Z}_{j_k}$. Here $Z_{i_k}(\ell) = \Re Z_{i_k}$ and $Z_{i_{k+1}}(\ell) = \Im Z_{i_k}$.

Subcase (b): $Z_{j_{k+1}} \neq \overline{Z}_{j_k}$. In this case one has $j_{k+1} > j''_k$ ([7, page 250]). For the index i_k, this case is the same as $k \in K_1$: one has

$$Z_{i_k}(\ell) = \frac{1}{2}\Big(\ell[\rho_{k-1}(Z_{j_k},\ell), \overline{Z}_{i_k}]Z_{i_k} + \ell[\rho_{k-1}(\overline{Z}_{j_k},\ell), Z_{i_k}]\overline{Z}_{i_k}\Big)$$

As for the index i_{k+1}, we define

$$Z_{i_{k+1}}(\ell) = \frac{1}{2i}\Big(\ell[\rho_{k-1}(Z_{j_k},\ell), \overline{Z}_{i_k}]Z_{i_k} - \ell[\rho_{k-1}(\overline{Z}_{j_k},\ell), Z_{i_k}]\overline{Z}_{i_k}\Big)$$

Note that in this subcase because $j_{k+1} > j''_k$, it follows that $\rho_k(Z_{i_{k+1}}(\ell),\ell) = \rho_{k-1}(Z_{i_{k+1}}(\ell),\ell)$, that is, that $V_{k+1}(\ell) = \rho_{k-1}(Z_{i_{k+1}}(\ell),\ell)$.

For future reference we write $K_4 = K_{4a} \cup K_{4b}$ and $K_5 = K_{5a} \cup K_{5b}$ according to the subcases (a) and (b) above. The covering sets referenced in Proposition 1.2 are formed by writing

$$Z_{i_k}(\ell) = \beta_1(\ell)\Re Z_{i_k} + \beta_2(\ell)\Im Z_{i_k}$$

for each $k \in K_1 \cup K_{4b}$. For each such k, select $t_k = 1$ or $t_k = 2$. Then a covering set $O = O_t$ is a set $O_t = \{\ell \in \Omega \mid \beta_{t_k}(\ell) \neq 0, k \in K_1 \cup K_{4b}\}$.

Now that we have defined $Z_{i_k}(\ell)$, and hence $V_k(\ell)$, for all possible cases, it is shown in [5] that one definition for $Z_{j_k}(\ell)$ will suffice. Thus in each case above we can take

$$Z_{j_k}(\ell) = \frac{1}{2}\Big(\ell[\overline{Z}_{j_k}, V_k(\ell)]Z_{j_k} + \ell[Z_{j_k}, V_k(\ell)]\overline{Z}_{j_k}\Big).$$

The following three results are proved in [1].

Lemma 1.4. *Let $\mathfrak{p} = \mathfrak{p}_d(\ell)$ be the complex Vergne polarization associated with the chosen adaptable basis. Then*

$$\mathfrak{p} = \mathfrak{p} \cap \overline{\mathfrak{p}} + \text{ span } \{\rho_{k-1}(Z_{i_k},\ell) \mid k \in K_3\}.$$

Lemma 1.5. [1, Lemma 3.3] *Let Ω be a fine layer whose orbits have dimension $2d > 0$. Let $k, 1 \le k \le d$ be a subindex such that $k \notin K_5$, let $X \in \mathfrak{l}_{j''_k} - \mathfrak{l}_{j'_k}, Y \in \mathfrak{l}_{i''_k} - \mathfrak{l}_{i'_k}$, and set $\beta(\ell) = \ell[X, \rho_{k-1}(Y,\ell)],\ \ell \in \Omega$. Then β is N-invariant on Ω. In particular, the functions $Z_j(\ell), j \in \mathbf{e}$ defined above are N-invariant, and the functions $\ell \mapsto \ell[Z_j, V_k(\ell)]$ are N-invariant. Moreover, each covering set O is N-invariant, and the continuous functions ϕ_k^O are N-invariant.*

Theorem 1.1. [1, Theorem 4.5 (specialized to the nilpotent case)] *The subset*

$$\Lambda = \{\ell \in \Omega \mid \ell(Z_j(\ell)) = 0, \textit{ for all } j \in \mathbf{e}\}$$

is a cross-section for the coadjoint orbits in Ω.

Note that even in the generic layer, the above cross-section need not be flat; see Section 6, Example 6.2. The following consequence of our cross-section description shall be useful later.

Corollary 1.2. *For each $Z \in \mathfrak{l}$, $\ell \in \Lambda$, we have $\ell\big(\rho_k(Z,\ell)\big) = \ell(Z), 0 \le k \le d$.*

Proof. The result is true for $k=0$ by definition of ρ_0. Assume that the result is true for $k-1$. Then $\ell\big(U_k(\ell)\big) = \ell\big(\rho_{k-1}(Z_{j_k}(\ell),\ell)\big) = \ell\big(Z_{j_k}(\ell)\big) = 0$ and similarly $\ell\big(V_k(\ell)\big) = 0$. Hence

$$\ell\big(\rho_k(Z,\ell)\big) = \ell\Big(\rho_{k-1}(Z,\ell) - c(\ell)U_k(\ell) - d(\ell)V_k(\ell)\Big) = \ell\big(\rho_{k-1}(Z,\ell)\big) = \ell(Z).$$

■

We have seen in Lemma 1.1 that the fine layers Ω are invariant under that action of H. We claim that the cross-sections Λ are H-invariant also. This claim will follow from the next result.

Lemma 1.6. *Let Ω be a fine layer with $d > 0$. For $\ell \in \Omega$, we have the following.*

(1) If $k \ge 1$ and $k \notin K_3 \cup K_{4a} \cup K_{5a}$, then we have homomorphisms $\nu_{i_k} : H \to \mathbb{C}^*$ and $\nu_{j_k} : H \to \mathbb{C}^*$ such that for any $a \in H$, $a^{-1}Z_{i_k}(a\ell) = \nu_{i_k}(a)Z_{i_k}(\ell)$ and $a^{-1}Z_{j_k}(a\ell) = \nu_{j_k}(a)Z_{j_k}(\ell)$. Moreover, the functions ν_{i_k} and ν_{j_k} are defined as follows. One has $\nu_{j_k}(a) = |\delta_{j_k}(a)|^{-2}\nu_{i_k}(a)$ in all cases, while ν_{i_k} is defined casewise by

(i) $\nu_{i_k}(a) = \delta_{i_k}(a)^{-1}$, if $k \in K_0$,

(ii) $\nu_{i_k}(a) = |\delta_{i_k}(a)|^{-2}\delta_{j_k}(a)^{-1}$, if $k \in K_1 \cup K_{4b}$,

(iii) $\nu_{i_k}(a) = \nu_{i_{k-1}}(a)$, if $k \in K_{5b}$ (whence $k-1 \in K_{4b}$), and

(iv) $\nu_{i_k}(a) = |\delta_{j_r}(a)|^{-2}\delta_{i_r}(a)$, if $k \in K_2$ (where $r < k$ is defined by $i_k - 1 = j_r \notin I$.)

(2) If $k \notin K_{4a}$, then

(a) $\mathfrak{m}_k(a\ell) = a\mathfrak{m}_k(\ell)$,

(b) $\mathfrak{m}_k(a\ell)^{a\ell} = a\big(\mathfrak{m}_k(\ell)^{\ell}\big)$, and

(c) $\rho_k(a^{-1}W,\ell) = a^{-1}\rho_k(W,a\ell)$ holds for each $W \in \mathfrak{l}$.

Proof. We begin by establishing that for each k, the statements (2b) and (2c) follow from (2a). Suppose that for some $0 \le k \le d$, $a \in H$, we have $\mathfrak{m}_k(a\ell) = a\mathfrak{m}_k(\ell)$. Then $W \in \mathfrak{m}(a\ell)^{a\ell}$ iff $a\ell[W,aZ] = 0$ holds for all $Z \in \mathfrak{m}_k(\ell)$, iff $\ell[a^{-1}W,Z] = 0$ holds for all $Z \in \mathfrak{m}_k(\ell)$, iff $a^{-1}W \in \mathfrak{m}_k(\ell)^{\ell}$. Now set $P = a^{-1} \circ \rho_k(\cdot,a\ell) \circ a$; then P is a projection, and the preceding shows that the image of P is $\mathfrak{m}_k(\ell)^\ell$. If $W \in \mathfrak{m}_k(\ell)$, then $aW \in \mathfrak{m}_k(a\ell)$ and so by definition of $\rho_k(\cdot, a\ell)$ we have $\rho_k(aW, a\ell) = 0$. Hence $P(W) = a^{-1}\rho_k(aW, a\ell) = 0$ and it follows that $P = \rho_k(\cdot, \ell)$. The identity (2c) follows.

Secondly, we show that in (1), if one assumes that (2c) holds for $k-1$ and that $a^{-1}Z_{i_k}(a\ell) = \nu_{i_k}(a)Z_{i_k}(\ell)$ holds, then the identities $a^{-1}V_k(a\ell) = \nu_{i_k}(a)V_k(\ell)$, $a^{-1}Z_{j_k}(a\ell) = \nu_{j_k}(a)Z_{j_k}(\ell)$, and $a^{-1}U_k(a\ell) = \nu_{j_k}(a)U_k(\ell)$ follow.

Suppose that for some $1 \le k \le d, k \notin K_3 \cup K_{4a} \cup K_{5a}$, $a \in H$, we have $a^{-1}Z_{i_k}(a\ell) = \nu_{i_k}(a)Z_{i_k}(\ell)$ and that $\rho_{k-1}(a^{-1}W, \ell) = a^{-1}\rho_{k-1}(W, a\ell)$ holds for each $W \in \mathfrak{l}$. We then have $a^{-1}\rho_{k-1}(Z_{j_k}, a\ell) = \delta_{j_k}(a)\rho_{k-1}(Z_{j_k}, \ell)$, and

$$\begin{aligned} a^{-1}V_{i_k}(a\ell) &= a^{-1}\rho_{k-1}(Z_{i_k}(a\ell), a\ell) = \rho_{k-1}(a^{-1}Z_{i_k}(a\ell), \ell) \\ &= \rho_{k-1}(\nu_{i_k}(a)Z_{i_k}(\ell), \ell) \\ &= \nu_{i_k}(a)V_k(\ell). \end{aligned}$$

Using the formula for $Z_{j_k}(\ell)$ given above, we have

$$\begin{aligned} a^{-1}Z_{j_k}(a\ell) &= a^{-1}\Big\{\frac{1}{2}\Big(a\ell[\overline{Z}_{j_k}, V_k(a\ell)]Z_{j_k} + a\ell[Z_{j_k}, V_k(a\ell)]\overline{Z}_{j_k}\Big)\Big\} \\ &= \frac{1}{2}\Big(\ell[a^{-1}\overline{Z}_{j_k}, a^{-1}V_k(a\ell)]a^{-1}Z_{j_k} + \ell[a^{-1}Z_{j_k}, a^{-1}V_k(a\ell)]a^{-1}\overline{Z}_{j_k}\Big) \\ &= \frac{1}{2}\Big(\ell[\overline{\delta}_{j_k}(a)^{-1}\overline{Z}_{j_k}, \nu_{i_k}(a)V_k(\ell)]\delta_{j_k}(a)^{-1}Z_{j_k} \\ &\qquad\qquad + \ell[\delta_{j_k}(a)^{-1}Z_{j_k}, \nu_{i_k}(a)V_k(\ell)]\overline{\delta}_{j_k}(a)^{-1}\overline{Z}_{j_k}\Big) \\ &= |\delta_{j_k}(a)|^{-2}\nu_{i_k}(a)Z_{j_k}(\ell). \end{aligned}$$

Now just as the identity for $V_k(\ell)$, the identity $a^{-1}U_k(a\ell) = \nu_{j_k}(a)U_k(\ell)$ follows.

Having established these preliminary relations between the above identities, we proceed by induction on k, $0 \le k \le d$. The statements (1) and (2) are trivially true when $k = 0$. Suppose then that $k \ge 1$ and that the lemma holds for smaller k. Observe that if $k \notin K_3 \cup K_{4a} \cup K_{5a}$, then $k-1 \notin K_{4a}$, and hence we have the identity (2c) for $k-1$.

Therefore, in light of the relations established above, it remains to prove the following statements for k:

(a) if $k \notin K_3 \cup K_{4a} \cup K_{5a}$, then for $a \in H$, $a^{-1}Z_{i_k}(a\ell) = \nu_{i_k}(a)Z_{i_k}(\ell)$ where ν_{i_k} is as claimed, and

(b) if $k \notin K_{4a}$, then $\mathfrak{m}_k(a\ell) = a\mathfrak{m}_k(\ell)$ holds for $a \in H$.

We consider several cases.

Case 0. Suppose that $k \in K_0$. In this case $Z_{i_k}(\ell) = Z_{i_k}$, so (a) is clear. As for (b), in this case we have $\mathfrak{m}_k(\ell) = \mathfrak{m}_{k-1}(\ell) + (V_k(\ell), U_k(\ell))$. By induction and the above observations we have $\mathfrak{m}_k(a\ell) = \mathfrak{m}_{k-1}(a\ell) + (V_k(a\ell), U_k(a\ell)) = a\mathfrak{m}_{k-1}(\ell) + a(\nu_{i_k}(a)V_k(\ell), \nu_{j_k}(a)U_k(\ell)) = a\mathfrak{m}_k(\ell)$, so (b) is proved.

Case 1. Suppose that $k \in K_1$. Here again $k-1 \notin K_{4a}$, so we have the identity (2c) for $k-1$.

$$\begin{aligned}
a^{-1}Z_{i_k}(a\ell) &= \frac{1}{2}\Big(a\ell[Z_{j_k}, \rho_{k-1}(\overline{Z}_{i_k}, a\ell)]a^{-1}Z_{i_k} + a\ell[Z_{j_k}, \rho_{k-1}(Z_{i_k}, a\ell)]a^{-1}\overline{Z}_{i_k}\Big) \\
&= \frac{1}{2}\Big(\ell[a^{-1}Z_{j_k}, a^{-1}\rho_{k-1}(\overline{Z}_{i_k}, a\ell)]a^{-1}Z_{i_k} + \ell[a^{-1}Z_{j_k}, a^{-1}\rho_{k-1}(Z_{i_k}, a\ell)]a^{-1}\overline{Z}_{i_k}\Big) \\
&= \frac{1}{2}\Big(\ell[a^{-1}Z_{j_k}, \rho_{k-1}(a^{-1}\overline{Z}_{i_k}, \ell)]a^{-1}Z_{i_k} + \ell[a^{-1}Z_{j_k}, \rho_{k-1}(a^{-1}Z_{i_k}, \ell)]a^{-1}\overline{Z}_{i_k}\Big) \\
&= |\delta_{i_k}(a)|^{-2}\delta_{j_k}(a)^{-1}\frac{1}{2}\Big(\ell[Z_{j_k}, \rho_{k-1}(\overline{Z}_{i_k}, \ell)]Z_{i_k} + \ell[Z_{j_k}, \rho_{k-1}(Z_{i_k}, \ell)]\overline{Z}_{i_k}\Big) \\
&= \nu_{i_k}(a)Z_{i_k}(\ell)
\end{aligned}$$

where $\nu_{i_k}(a) = |\delta_{i_k}(a)|^{-2}\delta_{j_k}(a)^{-1}$. Thus (a) is proved. As for (b), we have $\mathfrak{m}_k(\ell) = \mathfrak{m}_{k-1}(\ell) + (V_k(\ell), U_k(\ell))$ just as in Case 0, and the proof of (b) is the same as that case.

Case 2. Suppose that $k \in K_2$. Let $i_k - 1 = j_r$ where $r < k$. Observe that in this case $r \notin K_3 \cup K_{4a} \cup K_{5a}$, and hence we have the identity $a^{-1}V_r(a\ell) = \nu_{i_r}(a)V_r(\ell)$. In a similar way as Case 1 we find

$$\begin{aligned}
a^{-1}Z_{i_k}(a\ell) &= \frac{1}{2i}\Big(\ell[a^{-1}\overline{Z}_{j_r}, a^{-1}V_r(a\ell)]a^{-1}Z_{j_r} - \ell[a^{-1}Z_{j_r}, a^{-1}V_r(a\ell)]a^{-1}\overline{Z}_{j_r}\Big) \\
&= \nu_{i_k}(a)Z_{i_k}(\ell)
\end{aligned}$$

where in this case $\nu_{i_k}(\ell) = |\delta_{j_r}(a)|^{-2}\delta_{i_r}(a)^{-1}$. The proof of the identity (b) is the same as the preceding cases.

Case 3. Suppose that $k \in K_3$, so that $Z_{j_k} = \overline{Z}_{i_k}$. Here we need only prove that (b) holds, and the point here (as in the cases where $k \in K_{4a}$ and $k \in K_{5a}$ also) is that $\mathfrak{m}_k(\ell)$ can be rewritten in a more convenient form. Indeed, since

$$V_k(\ell) = \frac{1}{2i}\Big(\rho_{k-1}(Z_{i_k}, \ell) - \rho_{k-1}(Z_{j_k}, \ell)\Big)$$

and

$$U_k(\ell) = \frac{1}{2}\Big(\rho_{k-1}(Z_{i_k}, \ell) + \rho_{k-1}(Z_{j_k}, \ell)\Big),$$

then we have

$$\mathfrak{m}_k(\ell) = \mathfrak{m}_{k-1}(\ell) + (\rho_{k-1}(Z_{i_k}, \ell), \rho_{k-1}(Z_{j_k}, \ell)).$$

Now as in prior cases, $k-1 \notin K_{4a}$ so we have the identity (2c) for $k-1$. Hence $a^{-1}\rho_{k-1}(Z_{i_k}, a\ell) = \delta_{i_k}(a)^{-1}\rho_{k-1}(Z_{i_k}, \ell)$ and $a^{-1}\rho_{k-1}(Z_{j_k}, a\ell) = \delta_{j_k}(a)^{-1}\rho_{k-1}(Z_{j_k}, \ell)$ and

$$\begin{aligned}
a^{-1}\mathfrak{m}_k(a\ell) &= a^{-1}\mathfrak{m}_{k-1}(a\ell) + (a^{-1}\rho_{k-1}(Z_{i_k}, a\ell), s^{-1}\rho_{k-1}(Z_{j_k}, a\ell)) \\
&= \mathfrak{m}_{k-1}(\ell) + (\delta_{i_k}(a)^{-1}(Z_{i_k}, \ell), \delta_{j_k}(a)^{-1}\rho_{k-1}(Z_{j_k}, \ell)) \\
&= \mathfrak{m}_k(\ell).
\end{aligned}$$

Case 4. Suppose that $k \in K_{4b}$. We have $k-1 \notin K_{4a}$ and since the formulae for $Z_{i_k}(\ell)$ and $\mathfrak{m}_k(\ell)$ are the same as Case 1, the proof in this case is identical to that of Case 1 as well.

Case 5. Suppose that $k \in K_5$. Note that in this case we have $k-2 \notin K_{4a}$. We consider two subcases.

Subcase 5(a). Suppose that $k \in K_{5a}$. By construction, the complex span of the elements $V_{k-1}(\ell), V_k(\ell), U_{k-1}(\ell), U_k(\ell)\}$ coincides with the complex span of $\{\rho_{k-2}(Z_{i_{k-1}}, \ell), \rho_{k-2}(Z_{j_{k-1}}, \ell), \rho_{k-2}(Z_{i_k}, \ell), \rho_{k-2}(Z_{j_k}, \ell))\}$, and hence

$$\mathfrak{m}_k(\ell) = \mathfrak{m}_{k-2}(\ell) + (\rho_{k-2}(Z_{i_{k-1}}, \ell), \rho_{k-2}(Z_{j_{k-1}}, \ell), \rho_{k-2}(Z_{i_k}, \ell), \rho_{k-2}(Z_{j_k}, \ell)).$$

Now an argument similar to that of Case 3 shows that $\mathfrak{m}_k(a\ell) = s\mathfrak{m}_k(\ell)$.

Subcase 5(b). Suppose that $k \in K_{5b}$. Here we have

$$Z_{i_k}(\ell) = \frac{1}{2i}\Big(\ell[Z_{j_{k-1}}, \rho_{k-2}(\overline{Z}_{i_{k-1}}, \ell)]Z_{i_{k-1}} - \ell[Z_{j_{k-1}}, \rho_{k-2}(Z_{i_{k-1}}, \ell)]\overline{Z}_{i_{k-1}}\Big)$$

and an argument similar to that of Case 2 shows that $a^{-1}Z_{i_k}(a\ell) = \nu_{i_k}(a)Z_{i_k}(\ell)$ and $\mathfrak{m}_k(a\ell) = a\mathfrak{m}_k(\ell)$. ∎

The following is almost immediate.

Proposition 1.3. *The cross-sections $\Lambda_{\mathbf{e},\mathbf{j}}$ are H-invariant.*

Proof. An examination of the definitions of the functions $Z_j(\ell), j \in \mathbf{e}$, shows that if $k \in K_3$, then the statement

$$\ell(Z_{i_k}(\ell)) = 0 \text{ and } \ell(Z_{j_k}(\ell)) = 0$$

is equivalent to

$$\ell_{i_k} = \ell_{j_k} = 0.$$

while if $k \in K_{4a}$, then

$$\ell(Z_{i_k}(\ell)) = \ell(Z_{j_k}(\ell)) = \ell(Z_{i_{k+1}}(\ell)) = \ell(Z_{j_{k+1}}(\ell)) = 0$$

is equivalent to the vanishing of each of $\ell_{i_k}, \ell_{j_k}, \ell_{i_{k+1}}$, and $\ell_{j_{k+1}}$. It follows from this and from Lemma 1.6 that for each $j \in \mathbf{e}$, we have a non-zero, semi-invariant function p_j on Ω such that $\Lambda = \{\ell \in \Omega \mid p_j(\ell) = 0, j \in \mathbf{e}\}$, and the proposition follows. ∎

Next we examine the restrictions of the preceding characters to stabilizer subgroups.

Lemma 1.7. *Suppose that a belongs to the stabilizer H_ℓ in H for some $\ell \in \Omega$. Then we have the following.*

(a) For each $1 \le k \le d$, $\delta_{j_k}(a) = \delta_{i_k}(a)^{-1}$.

(b) If $k \in K_3$ then $|\delta_{j_k}(a)| = 1$.

(c) If $k \in K_0 \cup K_1 \cup K_2 \cup K_{4b} \cup K_{5b}$, then $\nu_{i_k}(a)$ and $\nu_{j_k}(a)$ are both real.

(d) If $k \in K_0 \cup K_1 \cup K_2 \cup K_{4b} \cup K_{5b}$, then $\delta_{i_k}(a) = \nu_{i_k}(a)^{-1}$ and $\delta_{j_k}(a) = \nu_{j_k}(a)^{-1}$.

Proof. First of all, we observe that by the preceding lemma, for any $1 \le j \le n$

$$a\rho_k(Z_j, \ell) = \delta_j(a)\rho_k(Z_j, \ell).$$

Suppose that $k \notin K_5$. Using the definition of i_k and j_k and the properties of the functions ρ_k, we have

$$\ell[\rho_{k-1}(Z_{j_k}, \ell), \rho_{k-1}(Z_{i_k}, \ell)] \neq 0,$$

and hence

$$\begin{aligned} 0 \neq \ell[\rho_{k-1}(Z_{i_k}, \ell), \rho_{k-1}(Z_{j_k}, \ell)] &= a\ell[\rho_{k-1}(Z_{i_k}, a\ell), \rho_{k-1}(Z_{j_k}, a\ell)] \\ &= \delta_{i_k}(a)\delta_{j_k}(a)\ell[\rho_{k-1}(Z_{i_k}, \ell), \rho_{k-1}(Z_{j_k}, \ell)]. \end{aligned}$$

If $k \in K_5$, then replace $k-1$ by $k-2$ and repeat the preceding. Part (a) follows.

Now $k \in K_3$ means that $Z_{j_k} = \overline{Z}_{i_k}$, so $\delta_{j_k} = \overline{\delta_{i_k}}$ and part (b) follows. As for (c), suppose that $k \in K_0 \cup K_1 \cup K_2 \cup K_{4b} \cup K_{5b}$; the point here is that in this case $Z_{i_k}(\ell)$ and $Z_{j_k}(\ell)$ are "almost real": they belong to $\mathbb{C}\mathfrak{n}$. It follows immediately from the definitions of ν_{i_k} and ν_{j_k} and the fact that $a\ell = \ell$ that $\nu_{i_k}(a)$ and $\nu_{j_k}(a)$ belong to $\mathbb{R}$. Thus part (c) holds, and now the proof is completed by an examination of the formulae for ν_{i_k} and ν_{j_k} in each case, and using parts (a) and (c). The cases where $k \in K_0 \cup K_1 \cup K_{4b} \cup K_{5b}$ are straightforward. If $k \in K_2$, then let $r < k$ such that $i_k - 1 = j_r$. We have $r \in K_0 \cup K_1 \cup K_2 \cup K_{4b} \cup K_{5b}$, so by induction we may assume that the result holds for r (Note that $1 \notin K_2$ by definition of K_2.) Hence $\delta_{j_r}(a)$ is real and

$$\nu_{i_k}(a) = |\delta_{j_r}(a)|^{-2}\delta_{i_r}(a)^{-1} = \delta_{j_r}(a)^{-1} = \delta_{i_k}(a)^{1}.$$

Then using part (a) (the following calculation works for all cases),

$$\nu_{j_k}(a) = |\delta_{j_k}(a)|^{-2}\nu_{i_k}(a) = \delta_{j_k}(a)^{-2}\delta_{i_k}(a)^{-1} = \delta_{j_k}(a)^{-1}$$

∎

From now on we let $\Omega = \Omega_{\mathbf{e},\mathbf{j}}$ be the minimal (and hence Zariski-open) fine layer in $\mathfrak{n}^*$, with Λ its orbital cross-section. From Theorem 1.1 we have rational functions $Z_j : \Omega \to \mathfrak{l}, j \in \mathbf{e}$ such that Λ is a Zariski open subset of the algebraic set $V = \{\ell \in \mathfrak{n}^* \mid \ell(Z_j(\ell)) = 0, j \in \mathbf{e}\}$. We shall now define real coordinates for Λ and equip Λ with a Lebesgue measure. Recalling the index operations $j \mapsto j'$ and $j \mapsto j''$ defined at the beginning of this section, we have already observed that (see the definition of $\mathbf{e}$ above) that if $j'' \in \mathbf{e}$, then $j \in \mathbf{e}$ also. If the basis of $\mathfrak{l} = \mathfrak{n}_c$ consists entirely of elements in $\mathfrak{n}$ – or more generally, if $j \in \mathbf{e}$ implies $j'' \in \mathbf{e}$ – then V is just a subspace of $\mathfrak{n}^*$, that is, the cross-section is flat. However, it may happen that $j \in \mathbf{e}$ while $j'' \notin \mathbf{e}$. It is the presence of this case which results in a cross-section which is not so simple.

First we identify the indices j for which the coordinate ℓ_j does not vanish on Λ. Define the index sequence $\mathbf{u}$ by

$$\mathbf{u} = \{u_1 < u_2 < \cdots < u_c\} = \{1 \le j \le n \mid j-1 \in I \text{ and } j'' \notin \mathbf{e}\}.$$

The indices $\mathbf{u}$ identify the directions where there is a "non-jump index"; in fact, in terms of the index operation $j \mapsto j'$, we have

$$\mathbf{u} = \big(\{1, 2, \ldots n\} \setminus \mathbf{e}\big)' + 1.$$

Note also $\mathbf{u} \cap \mathbf{e} = \{j \in \mathbf{e} \mid j \notin I, j'' \notin \mathbf{e}\}$ consists of the indices referred to in the preceding paragraph.

For each $1 \le a \le c$, set $\mathbb{K}_a = \mathbb{R}$ if $u_a \in I$ and $\mathbb{K}_a = \mathbb{C}$ if $u_a \notin I$. Set $\lambda_a = \ell(Z_{u_a}), 1 \le a \le c$. We shall find it convenient to identify elements of Λ by their mixed real and complex coordinates, writing $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_c) \in \Lambda$ where $\lambda_a \in \mathbb{K}_a, 1 \le a \le c$. We point out that in the simpler case where none of the indices u_a belong to $\mathbf{e}$, this notation identifies Λ with an open subset of $\prod_{a=1}^c \mathbb{K}_a$ (this is the case in [4]). We shall also find it convenient in what follows to adopt a notation for the characters of the action of H on Λ: set $\chi_a = \delta_{u_a}^{-1}$.

For each $1 \le a \le c$, write $\lambda^a = (\lambda_1, \lambda_2, \ldots, \lambda_a)$, and set

$$\Lambda^a = \{\lambda^a \mid \lambda \in \Lambda\}.$$

Now if $u_a \notin \mathbf{e}$, then for each $\lambda \in \Lambda$ set $L_a(\lambda) = \mathbb{K}_a$. Suppose that $u_a \in \mathbf{e}$. For $j = u_a$, recall that we have defined the element $Z_j(\lambda) = \beta_1(\lambda)\Re Z_j + \beta_2(\lambda)\Im Z_j$. Since $j \in \mathbf{e}$ but $j'' \notin \mathbf{e}$, it follows (see [7]) that $\Im\big(\beta_1(\lambda)\overline{\beta_2}(\lambda)\big) = 0$. For each $\lambda \in \Lambda$ let $L_a(\lambda)$ be the real subspace of $\mathbb{C}$ defined by

$$L_a(\lambda) = \{z \in \mathbb{C} \mid \beta_1(\lambda)\Re z + \beta_2(\lambda)\Im z = 0\}.$$

It is shown in [5] that for each $\ell \in \Omega$, $\beta_1(\ell)$ and $\beta_2(\ell)$ depend only upon $\ell_1, \ldots, \ell_{j-1}$. Taking $\ell = \lambda \in \Lambda$ we see that $\beta_1(\lambda)$ and $\beta_2(\lambda)$, and hence $L_a(\lambda)$, depend only upon λ^{a-1}. Combining Theorem 1.1 with [5, Proposition 2.2.1], we have

Proposition 1.4. [5, Proposition 2.2.1] *For each* $1 \le a \le c$*, there is a dense open subset* $U_a(\lambda) = U_a(\lambda^{a-1})$ *of* $L_a(\lambda)$ *depending only upon* λ^{a-1} *such that*

$$\Lambda^a = \{\lambda^a = (\lambda_1, \lambda_2, \ldots \lambda_a) \mid \lambda^{a-1} \in \Lambda^{a-1} \textit{ and } \lambda_a \in U_a(\lambda)\}.$$

Set

$$\mathbf{u}^1 = \{u \in \mathbf{u} \mid u \in I \text{ or } u \in \mathbf{e}\} = \{u_a \in \mathbf{u} \mid \dim L_a(\lambda) = 1\}$$

and

$$\mathbf{u}^2 = \{u \in \mathbf{u} \mid u \notin I \text{ and } u \notin \mathbf{e}\} = \{u_a \in \mathbf{u} \mid \dim L_a(\lambda) = 2\}.$$

We define a Lebesgue measure $d\lambda^a$ on $\Lambda^a, 1 \le a \le c$ iteratively. Since $\mathfrak{n}$ is nilpotent, $u_1 = 1 \notin \mathbf{e}$ and we take $d\lambda^1$ to be Lebesgue measure on $L^1 = \mathbb{K}_1$. Assume that $1 < a \le c$ and that $d\lambda^{a-1}$ is defined. If $u_a \in \mathbf{u}^1$, denote by $d\lambda_a$ the one-dimensional Lebesgue measure on $L_a(\lambda^{a-1})$, while if $u_a \in \mathbf{u}^2$, denote also by $d\lambda_a$ the two dimensional Lebesgue measure on $L_a(\lambda^{a-1}) = \mathbb{C}$. For non-negative measurable functions f on Λ^a define

$$\int_{\Lambda^a} f(\lambda^a) d\lambda^a = \int_{\Lambda^{a-1}} \int_{U_a(\lambda^{a-1})} f(\lambda^{a-1}, \lambda_a)\, d\lambda_a\, d\mu_{a-1}(\lambda^{a-1}).$$

We denote the measure on Λ so obtained by $d\lambda$. Now let $\mathbf{Pf} = \mathbf{Pf}_{\mathbf{e},d}$; we have the following [5].

Proposition 1.5. [5, Corollary 2.2.6] *The Plancherel measure on* N *is given (up to a constant) by* $|\mathbf{Pf}(\lambda)|d\lambda$.

In the final portion of this section, we observe that the almost all elements of Λ have a common stabilizer in H. Set

$$K = \bigcap_{u \in \mathbf{u}} \ker(\delta_u);$$

since $\delta_{j''} = \overline{\delta_j}$, we have $K = \bigcap_{j \notin \mathbf{e}} \ker(\delta_j)$. Observe also that the Lie algebra $\mathfrak{k}$ of K is

$$\mathfrak{k} = \bigcap_{u \in \mathbf{u}} \ker \gamma_u$$

and is contained in $\mathfrak{n}^\ell$ for every $\ell \in \Lambda$.

Lemma 1.8. *Let* $\lambda \in \Lambda$ *such that* $\lambda_a \ne 0, 1 \le a \le c$*. Then* $K = H_\lambda$.

Proof. It is clear that $K \subset \operatorname{stab}_H(\lambda)$ holds for all $\lambda \in \Lambda$. On the other hand, if $h \in H$ but $h \notin K$, then for some $1 \le a \le c$, we have $\chi_a(h) \ne 1$ and hence $(h\lambda)_a \ne \lambda_a$. ∎

From now we denote by Λ those elements λ of our cross-section for which $\lambda_a \neq 0, 1 \leq a \leq c$. The natural inclusion of K in $\mathrm{Sp}(\mathfrak{n}/\mathfrak{n}(\lambda), \lambda \in \Lambda$ is associated with the characters $\delta_j, j \in \mathbf{e}$, and hence the following is expected.

Lemma 1.9. *One has* $K \subset \ker|\delta|$.

Proof. Let $a \in K$; by Lemma 1.7, we have $\delta_{i_k}(a) = \delta_{j_k}(a)^{-1}$. Now suppose that $j \notin \mathbf{e}$; then $\rho_d(Z_j, \lambda)$ belongs to $\mathfrak{n}(\lambda)$. By Corollary 1.2 we have $r_j(\lambda) = \lambda\big(\rho_d(Z_j, \lambda)\big) = \lambda_j$ and it is clear from the description of Λ that r_j is non-vanishing on Λ when $j \notin \mathbf{e}$. From part (c) of Lemma 1.6, we find that $r(s\lambda) = \delta_j(s)r(\lambda)$, and hence $\delta_j(s) = 1$. ■

2. The Connected Algebraic Case

For the remainder of this paper we assume that G is connected and algebraic, that is, that H satisfies the following. We suppose that $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, with

(i) $H = H'H''$ where $H' = \exp(\mathfrak{h}')$ and $H'' = \exp(\mathfrak{h}'')$

(ii) for each $A \in \mathfrak{h}'$ we have $\gamma_j(A) \in \mathbb{R}, 1 \leq j \leq n$,

(iii) for each $B \in \mathfrak{h}''$ we have $\gamma_j(B) \in i\mathbb{R}, 1 \leq j \leq n$,

(iv) for each $B \in \mathfrak{h}''$, $\gamma_j(B)/\gamma_k(B)$ is rational, $1 \leq j < k \leq n$.

Of course G is not exponential; we have the following.

Lemma 2.1. *One has*

$$\ker(\exp) = \{B \in \mathfrak{h}'' \mid \gamma_j(B) \in 2\pi i\mathbb{Z}, 1 \leq j \leq n\}.$$

In particular, H' is exponential.

Proof. It follows from the fact that N is exponential that $\ker(\exp) \subset \mathfrak{h}$. If $A \in \mathfrak{h}'$, then $e = \exp A$ implies $1 = \delta_j(\exp A) = e^{\gamma_j(A)}$ so $\gamma_j(A) = 0$. Hence for all $1 \leq j \leq n$ and any $t \in \mathbb{R}$, $\delta_j(\exp tA) = 1$. But recall that we have assumed that H acts effectively on $\mathfrak{n}$ so we have $\cap_{1\leq j\leq n} \ker(\delta_j) = (1)$. Hence $\exp(\mathbb{R}A) = \{e\}$ and $A = 0$.

Let $B \in \mathfrak{h}''$. If $e = \exp B$, then as above $1 = \delta_j(\exp B) = e^{\gamma_j(B)}$ so $\gamma_j(B) \in 2\pi i\mathbb{Z}$, while if $\gamma_j(B) \in 2\pi i\mathbb{Z}, 1 \leq j \leq n$, then $\delta(\exp B) = 1$ so $\exp B = e$. ■

For each subindex a, $1 \leq a \leq c$, put $\chi_a = \delta_{u_a}^{-1}$, and let α_a be its differential. Set $H_a = \cap\{\ker \chi_b \mid 1 \leq b \leq a\}$; the Lie algebra of H_a is $\mathfrak{h}_a = \cap_{1\leq b\leq a} \ker \alpha_b$.

Define $d_a = (d'_a, d''_a), 1 \leq a \leq c$ by

$$d'_a = \operatorname{rank}\Big(\Re(\alpha_a)|_{\mathfrak{h}_{a-1}}\Big)$$

and

$$d''_a = \operatorname{rank}\Big(\Im(\alpha_a)|_{\mathfrak{h}_{a-1}}\Big)$$

Let $\mathbf{a} = \{a_1 < a_2 < \cdots < a_p\} = \{1 \leq a \leq c \mid d_a \neq (0,0)\}$, $\mathbf{a}' = \{a'_1 < a'_2 < \cdots < a'_p\} = \{1 \leq a \leq c \mid d'_a = 1\}$ and $\mathbf{a}'' = \{a''_1 < a''_2 < \cdots < a''_q\} = \{1 \leq a \leq c \mid d''_a = 1\}$. Let $\{A_1, A_2, \ldots, A_p\} \subset \mathfrak{h}$ be a subset of $\mathfrak{h}'$ that is dual to the roots $\alpha_{a'_1}, \ldots, \alpha_{a'_p}$ in the sense that $\alpha_{a'_j}(A_k) = 1$ if $j = k$ and 0 if $j \neq k$. Let $S_j = \exp(\mathbb{R}A_j), 1 \leq j \leq p$ and set $S = S_1 S_2 \cdots S_p \subset H'$.

We shall say that an element $B \in \mathfrak{h}''$ is integral if $\gamma_j(B) \in i\mathbb{Z}$ holds for $1 \leq j \leq n$. We select integral elements $\{B_1, B_2, \ldots, B_q\} \subset \mathfrak{h}''$ as follows. Let $\{\tilde{B}_1, \tilde{B}_2, \ldots, \tilde{B}_q\}$ be a set of elements of $\mathfrak{h}''$ dual to the independent roots $\alpha_{a''_1}, \ldots, \alpha_{a''_q}$ in the sense that $\alpha_{a''_j}(\tilde{B}_k) = i$ if $j = k$ and 0 if $j \neq k$. Choose $B_k \in \mathbb{R}\tilde{B}_k$ such that the kernel of the map $t \mapsto \exp(tB_k)$ is $2\pi\mathbb{Z}$. Our choice of B_k means that $2\pi\mathbb{Z}B_k \subset \ker(\exp)$, so by Lemma 2.1, $\gamma_j(2\pi B_k) \in 2\pi i\mathbb{Z}$ and $\gamma_j(B_k) \in i\mathbb{Z}$ for $1 \leq j \leq n$. Thus B_k is integral. Set $T_k = \exp(\mathbb{R}B_k), 1 \leq k \leq q$, and put $T = T_1 T_2 \cdots T_q \subset H''$. We shall write elements of S and T as $s = s_1 s_2 \cdots s_p$ and $t = t_1 t_2 \cdots t_q$ where $s_j \in S_j$ and $t_k \in T_k$.

We have

$$\mathfrak{h} = \mathbb{R}\text{-span}\{A_1, A_2, \ldots, A_p, B_1, B_2, \ldots B_q\} \oplus \mathfrak{k},$$

and exponentiating,

$$H = S \cdot T \cdot K_\circ$$

as a direct product, where $K_\circ = \exp(\mathfrak{k})$ is the connected component of the identity in K. Put $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{h}'$, $\mathfrak{k}'' = \mathfrak{k} \cap \mathfrak{h}''$; by definition of $\mathfrak{h}'$ and $\mathfrak{h}''$ we have $\mathfrak{k} = \mathfrak{k}' \oplus \mathfrak{k}''$. We also have $\mathfrak{h}' = \mathbb{R}\text{-span}\{A_1, A_2, \ldots, A_p\} + \mathfrak{k}'$, and since H' is exponential, then $K' := K \cap H' = \exp(\mathfrak{k}')$ and $H' = S \cdot K'$.

Put $K'' = K \cap H''$; note that K'' is not necessarily connected. Put $K''_\circ = \exp(\mathfrak{k}'')$, $F_k = \ker \chi_{a''_k} \cap T_k, 1 \leq k \leq q$ and let F the finite subgroup of T defined by

$$F = F_1 F_2 \cdots F_q.$$

Lemma 2.2. *One has $K'' \cap F = K \cap F = K \cap T$ and $K'' = (K \cap F) \cdot K''_\circ$.*

Proof. Since $F \subset T \subset H''$, it is clear that $K'' \cap F = K \cap F$, and we have $K \cap F \subset K \cap T$; on the other hand if $t = t_1 t_2 \ldots t_q \in K \cap T$, then for each $1 \leq k \leq q$, by the definition of K and $T_1, T_2, \ldots T_q$, we have that

$$1 = \delta_{u_{a''_k}}(t)^{-1} = \chi_{a''_k}(t) = \chi_{a''_k}(t_k)$$

so $t_k \in F_k$ and $t \in F$. Thus $K \cap T = K \cap F$.

Now let $b \in K''$, then $b \in H''$ so $b = \exp(B)$ with $B \in \mathfrak{h}''$. Write

$$B = r_1 B_1 + \cdots + r_q B_q + B_0$$

where $B_0 \in \mathfrak{k}''$. Then $b = t_1 t_2 \ldots t_q b_0$ where $t_k = \exp(r_k B_k) \in T_k$ and $b_0 \in K_0''$. Now for each $1 \leq k \leq q$,

$$1 = \delta_{u_{a_k''}}(b)^{-1} = \chi_{a_k''}(b) = \chi_{a_k''}(t_k)$$

so $t_k \in F_k$. Thus $t_1 t_2 \ldots t_q \in K \cap F$. ■

Let $\mathbb{S}$ denote the multiplicative group of positive real numbers, and $\mathbb{T}$ the multiplicative group of complex numbers of modulus one. For each $1 \leq j \leq p$, we have the canonical isomorphism $\iota_j' : S_j \to \mathbb{S}$ defined by $\iota_j'(\exp(yA_j)) = e^y, y \in \mathbb{R}$, and from now on we identify S_j with $\mathbb{S}$ in this way. Similarly, for each $1 \leq k \leq q$ identify T_k with $\mathbb{T}$ by $\iota_k''(\exp(\theta B_k)) = e^{i\theta}, \theta \in \mathbb{R}$. Thus the subgroup S is identified with the direct product $\mathbb{S}^p$ and T with the q-torus $\mathbb{T}^q$. Note that for $s = s_1 s_2, \cdots s_p \in S$, we have $\chi_{a_j'}(s) = s_j, 1 \leq j \leq p$. For each $1 \leq k \leq q$, we have $\alpha_{a_k''}(B_k) = i m_k$ where $m_k \in \mathbb{Z}$, so that

$$\chi_{a_k''}(t) = t_k^{m_k}$$

holds for all $t = t_1 t_2 \cdots t_q \in T$. Thus F_k is identified with the subgroup $\mathbb{F}(m_k)$ of m_k-th roots of unity in $\mathbb{T}$.

The Haar measure on S will be given by

$$d\nu_S(s) = \frac{ds_1 ds_2 \cdots ds_p}{s_1 s_2 \cdots s_p}.$$

The Haar measure ν_T on T will be the product of the usual Lebesgue probability measure on T_k when identified with $\mathbb{T}$ as above; thus

$$\int_T f(t) d\nu_T(t) = \frac{1}{(2\pi)^q} \int_0^{2\pi} \int_0^{2\pi} \cdots \int_0^{2\pi} f(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_q}) d\theta_1 d\theta_2 \cdots d\theta_q.$$

For simplicity we use the notation $d\nu(s)$ for $d\nu_S(s)$ and dt for $d\nu_T(t)$.

The action of H on Λ is given by the actions of S and T; with this in mind we define a cross-section in Λ for this action. Set

$$\begin{aligned} \Sigma \;=\; & \{\lambda \in \Lambda \mid |\lambda_a| = 1 \text{ if } d_a = (1,0), \lambda_a > 0 \text{ if } d_a = (0,1), \\ & \text{and } \lambda_a = 1, \text{ if } d_a = (1,1)\}. \end{aligned}$$

Using the iterative method by which Λ is described above, we describe Σ explicitly as follows.

Proposition 2.1. *For $1 \leq a \leq c$ let*

$$\Sigma^a = \{(\lambda_1, \lambda_2, \ldots, \lambda_a) \mid \lambda \in \Sigma\},$$

and define a subset $V_a(\lambda) = V_a(\lambda^{a-1})$ of $U_a(\lambda)$ by

$$V_a(\lambda) = \begin{cases} U_a(\lambda), & \text{if } d_a = (0,0), \\ \{\lambda_a \in U_a(\lambda) \mid |\lambda_a| = 1\}, & \text{if } d_a = (1,0), \\ \{\lambda_a \in U_a(\lambda) \mid \lambda_a > 0\}, & \text{if } d_a = (0,1), \\ \{1\}, & \text{if } d_a = (1,1). \end{cases}$$

Then for each a,

$$\Sigma^a = \{(\lambda_1, \lambda_2, \ldots, \lambda_a) \mid \lambda^{a-1} \in \Sigma^{a-1}, \lambda_a \in V_a(\lambda)\}. \tag{2.1}$$

In the case where $d_a = (1,0)$ and $\dim(L_a(\lambda)) = 1$, then $V_a(\lambda)$ is the two-point set $\mathbb{T} \cap L_a(\lambda)$. If $d_a = (1,0)$ and $\dim(L_a(\lambda)) = 2$ then $V_a(\lambda)$ is a full-measure subset of $\mathbb{T}$, while if $d_a = (0,1)$ and $\dim(L_a(\lambda)) = 2$ then $V_a(\lambda)$ is a full-measure subset of $\mathbb{S}$.

Proof. The equality 2.1 follows easily by induction on $a, 1 \leq a \leq c$, using the definition of Σ and Proposition 1.4.

Suppose that $d_a = (1,0)$; observe that $U_a(\lambda)$ is invariant under the real dilations D_a since Λ is invariant under H. Hence if $L_a(\lambda)$ is one-dimensional (this occurs if $u_a \in I$ or if $u_a \notin I$ but $u_a \in \mathbf{e}$), then $U_a(\lambda) = L_a(\lambda) \setminus \{0\}$ and $V_a(\lambda)$ consists of the two points in $U_a(\lambda)$ that have unit modulus. If instead $L_a(\lambda) = \mathbb{C}$, then since $U_a(\lambda)$ is dilation-invariant and has full measure in $\mathbb{C}$ it follows that $V_a(\lambda)$ has full measure in $\mathbb{T}$. Suppose next that $d_a = (0,1)$. Again $U_a(\lambda)$ is an open, full-measure subset of $\mathbb{C}$ which is now invariant under rotations. Hence $V_a(\lambda)$ is an open full-measure subset of the positive reals. ∎

Now it is easily seen that Σ is F-invariant. Indeed, let $t \in F$, $t = t_1 t_2 \cdots t_q$, and let $\lambda \in \Sigma$. If $a \in \mathbf{a}''$ then $(t \cdot \lambda)_a = \chi_a(t)\lambda_a = \lambda_a$ while if $d_a = (1,0)$, then $|(t \cdot \lambda)_a| = |\chi_a(t)\lambda_a| = 1$. The set Σ/F of F-orbits in Σ will be our parameter set for H-orbits in Λ. For each $\lambda \in \Lambda$, define $P(\lambda) \subset \Lambda$ as follows. Fix $\lambda \in \Lambda$. For each $1 \leq j \leq p$, define $s_j(\lambda) \in S_j$ by $s_j(\lambda) = 1/|\lambda_{a'_j}|$ and set $s(\lambda) = s_1(\lambda)s_2(\lambda) \cdots s_p(\lambda)$. For each $1 \leq k \leq q$, let $F_k(\lambda)$ be the finite subset of T_k defined by

$$F_k(\lambda) = \left(1/\text{sign}(\lambda_{a''_k})\right)^{1/m_k},$$

and set $F(\lambda) = F_1(\lambda) \times F_2(\lambda) \times \cdots \times F_q(\lambda) \subset T$. (Here $\text{sign}(z) = z/|z|$ for $z \neq 0$ and for $z \in \mathbb{T}$, $z^{1/m}$ denotes the set of m^{-th} roots of z in $\mathbb{T}$.) Define

$$P(\lambda) = \{s(\lambda)t(\lambda) \cdot \lambda \mid t(\lambda) \in F(\lambda)\}.$$

Lemma 2.3. *For each $\lambda \in \Lambda$, $P(\lambda)$ is an element of Σ/F, and $P(\lambda) = H\lambda \cap \Sigma$.*

Proof. Fix $\lambda \in \Lambda$. We begin by showing that $P(\lambda) \subset \Sigma$. Let $\lambda' = s(\lambda)t(\lambda)\cdot\lambda \in P(\lambda)$; we check the coordinates λ'_a for which $d_a \neq (0,0)$. Suppose that $d_a = (1,0)$, say $a = a'_j$. Then $\chi_a(s_j(\lambda)) = s_j(\lambda) = 1/|\lambda_{a'_j}|$, so

$$\lambda'_a = \chi_a(s(\lambda)t(\lambda))\lambda_a = \chi_a(s_j(\lambda))\chi_a(t(\lambda))\lambda_a = \chi_a(t(\lambda))\mathrm{sign}(\lambda_a).$$

If $d_a = (0,1)$, say $a = a''_k$, then $t_k(\lambda) \in (1/\mathrm{sgn}(\lambda_a))^{1/m_k}$, and so $\chi_a(t_k(\lambda)) = t_k(\lambda)^{m_k} = 1/\mathrm{sgn}(\lambda_a)$. Hence

$$\begin{aligned} P_a(\lambda) &= \chi_a(s(\lambda)t(\lambda))\lambda_a = \chi_a(s(\lambda))\chi_a(t_k(\lambda))\lambda_a = \chi_a(s(\lambda))(1/\mathrm{sgn}(\lambda_a))\lambda_a \\ &= \chi_a(s(\lambda))|\lambda_a|. \end{aligned}$$

Finally if $d_a = (1,1)$, say $a = a'_j = a''_k$, then

$$\chi_a(s(\lambda)t(\lambda)) = \chi_a(s_j(\lambda)t_k(\lambda)) = (1/|\lambda_a|)\,(1/\mathrm{sgn}(\lambda_a)) = 1/\lambda_a$$

so $\lambda'_a = \chi_a(s(\lambda)t(\lambda))\lambda_a = 1$. Thus $\lambda' \in \Sigma$.

Next, we show that in fact $P(\lambda)$ is an F-orbit in Σ. let λ' and λ'' be elements of $P(\lambda)$: $\lambda' = s(\lambda)t'(\lambda)\cdot\lambda$ and $\lambda'' = s(\lambda)t''(\lambda)\cdot\lambda$. For each $1 \le k \le q$, $t'_k(\lambda)$ and $t''_k(\lambda)$ both belong to $\left(1/\mathrm{sgn}(\lambda_{a''_k})\right)^{1/m_k}$ and hence $t_k = t'_k(\lambda)/t''_k(\lambda) \in F_k(m_k)$. Thus

$$\lambda'' = s(\lambda)t''(\lambda)\lambda = s(\lambda)t''_1(\lambda)\cdots t''_q(\lambda)\lambda = t_1\cdots t_q s(\lambda)t'(\lambda)\lambda = t_1\cdots t_q\lambda'.$$

On the other hand if $\lambda' \in P(\lambda)$ and $\lambda'' \in F\lambda'$, then we have $t = t_1\cdots t_q \in F$ such that $\lambda'' = t\lambda'$. Writing $\lambda' = s(\lambda)t(\lambda)\cdot\lambda$, we have $t_k t_k(\lambda) \in \left(1/\mathrm{sgn}(\lambda_{a''_k})\right)^{1/m_k}, 1 \le k \le q$, so

$$\lambda'' = t\lambda' = s(\lambda)t_1t_1(\lambda)t_2t_2(\lambda)\cdots t_qt_q(\lambda)\cdot\lambda \in P(\lambda).$$

Thus the set $P(\lambda)$ belongs to Σ/F.

Since by definition $P(\lambda) \subset H\lambda$, we have $P(\lambda) \subset H\lambda \cap \Sigma$. To finish the proof, it is enough to show that $P(\lambda)$ is an H-invariant function. Let $\lambda \in \Lambda$ and set $\lambda' = b\lambda$ where $b \in H$. We may assume that $b = st$, where $s \in S$ and $t \in T$. Observe that for each $1 \le j \le p$, since $\chi_{a'_j}(s) = s_j$, then

$$s_j(\lambda') = 1/|\lambda'_{a'_j}| = 1/s_j|\lambda_{a'_j}| = s_j^{-1}s_j(\lambda).$$

Hence $s(\lambda') = s^{-1}s(\lambda)$. Similarly, for each $1 \le k \le q$, we have the equality of the finite subsets of $\mathbb{T}$:

$$\left(1/\mathrm{sgn}(\lambda'_{a''_k})\right)^{1/m_k} = \left(1/t_k^{m_k}\mathrm{sgn}(\lambda_{a''_k})\right)^{1/m_k} = t_k^{-1}\left(1/\mathrm{sgn}(\lambda_{a''_k})\right)^{1/m_k}.$$

Hence for each $t(\lambda') \in F(\lambda')$, we have $t(\lambda) \in F(\lambda)$ such that $t(\lambda') = t^{-1}t(\lambda)$. It follows that

$$\begin{aligned} P(\lambda') &= \{s(\lambda')t(\lambda') \cdot \lambda' \mid t(\lambda') \in F(\lambda')\} \\ &= \{s^{-1}s(\lambda)t^{-1}t(\lambda) \cdot \lambda' \mid t(\lambda) \in F(\lambda)\} \\ &= \{s(\lambda)t(\lambda) \cdot \lambda \mid t(\lambda) \in F(\lambda)\} = P(\lambda). \end{aligned}$$

This completes the proof. ∎

The following is almost immediate from the preceding and the definition of $P(\lambda)$.

Proposition 2.2. *The map $\eta : \Lambda/H \to \Sigma/F$ defined by $\eta(H\lambda) = P(\lambda)$ is a bijection; indeed, η is a homeomorphism of quotient topologies.*

Proof. That η is injective follows from Lemma 2.3. To see that η is surjective, let $\lambda \in \Sigma$. Then the definition of Σ shows that $s_j(\lambda) = 1, 1 \le j \le p$, and $F_k(\lambda) = F_k$. Hence $P(\lambda) = F\lambda$ by definition of P. It is clear that η is bicontinuous. ∎

For $m \in \mathbb{N}$ set $\mathbb{T}(m) = \{e^{i\theta} \mid 0 \le \theta < 2\pi/m\}$. For each $1 \le k \le q$ define $I_k \subset T_k$ to be the set of elements in T_k that are identified with $\mathbb{T}(m_k)$, and set $I = I_1 I_2 \cdots I_q \subset T$. Note that I is a fundamental domain for the action of F on T, and that the map $S \times I \times \Sigma \to \Lambda$ given by $(s, t, \sigma) \mapsto st \cdot \sigma$ is a Borel isomorphism.

We define a Lebesgue measure $d\sigma^a$ on $\Sigma^a, 1 \le a \le c$ by the iterative method used in the definition of $d\lambda$:

$$\int_{\Sigma^a} f(\sigma^a)\, d\sigma^a = \int_{\Sigma^{a-1}} \int_{V_a(\sigma)} f(\sigma^{a-1}, \sigma_a) d\sigma_a d\sigma^{a-1}$$

where $d\sigma_a$ is the natural measure on $V_a(\sigma)$: if $d_a = (0,0)$ then $d\sigma_a = d\lambda_a$. If $d_a = (1,0)$ and $L_a(\sigma)$ is one-dimensional, then $d\sigma_a$ is point mass measure on the two-point set $V_a(\sigma)$, while if $d_a = (1,0)$ and $L_a(\lambda)$ is two-dimensional, then $d\sigma_a$ is the counterclockwise line integral over $V_a(\sigma)$. If $d_a = (0,1)$ then $d\sigma_a$ is just Lebesgue measure on the positive reals, while if $d_a = (1,1)$ then $d\sigma_a$ is just point mass measure on $\{1\}$. Thus we have the Lebesgue measure $d\sigma$ on Σ.

We shall write the integral on Λ as an iterated integral over Σ, S, and I. For $s \in S$ define $J_a(s) = \chi_a(s)$ if $u_a \in \mathbf{u}^1$ and $J_a(s) = |\chi_a(s)|^2$ if $u_a \in \mathbf{u}^2$, and set $J(s) = J_1(s)J_2(s) \cdots J_c(s)$. We use the notation $\sigma'' = \sigma_{a_1''}\sigma_{a_2''} \cdots \sigma_{a_q''}$ and $m = m_1 m_2 \cdots m_q$.

Lemma 2.4. *For any non-negative Borel-measurable function f on Λ, one has*

$$\int_\Lambda f(\lambda)\, d\lambda = m \int_\Sigma \int_S \int_I f(st \cdot \sigma)\, dt\ J(s) d\nu(s)\ \sigma'' d\sigma.$$

Proof. Using the notation $\Theta(s,t,\sigma) = st\cdot\sigma$, we examine the coordinate functions $\Theta_a, 1 \le a \le c$. Fix $1 \le a \le c$ and let $j(a) = \max\{1 \le j \le p \mid a'_j \le a\}$, $k(a) = \max\{1 \le k \le q \mid a''_k \le a\}$. We have

$$\Theta_a(s,t,\sigma) = \begin{cases} \chi_a(s_1 s_2 \cdots s_{j(a)} t_1 t_2 \cdots t_{k(a)})\sigma_a, & \text{if } d_a = (0,0) \\ s_{j(a)}\sigma_a, & \text{if } d_a = (1,0) \\ t_{k(a)}^{m_{k(a)}}\sigma_a, & \text{if } d_a = (0,1) \\ t_{k(a)}^{m_{k(a)}} s_{j(a)}, & \text{if } d_a = (1,1) \end{cases}$$

Set $S^a = \{s^a = (s_1, s_2, \dots s_{j(a)}, 1, 1, \dots, 1) \mid s_j \in S_j\}$ and similarly define T^a. Denote the natural Haar measures on S^a and T^a by $d\nu(s^a)$ and dt^a, respectively. Set $I^a = I \cap T^a$. Set $J^a(s) = J_1(s)\cdots J_a(s)$. Let $m^a = m_1 m_2 \cdots m_{k(a)}$, and $(\sigma'')^a = \sigma_{a''_1}\sigma_{a''_2}\cdots\sigma_{a''_{k(a)}}$. Set $\Theta^a = (\Theta_1, \Theta_2, \dots, \Theta_a)$; note that $\Theta^a = \Theta^a(s,t,\sigma)$ depends only upon s^a, t^a, and σ^a. Also for simplicity, we denote $U_a(\lambda) = U_a, V_a(\lambda) = V_a$. We now proceed iteratively as in the definitions of $d\lambda$ and $d\sigma$. Assume that

$$\int_{\Lambda^{a-1}} f(\lambda^{a-1})\, d\lambda^{a-1} = m^{a-1}\int_{\Sigma^{a-1}}\int_{S^{a-1}}\int_{I^{a-1}} f\left(\Theta^{a-1}(s,t,\sigma)\right)\, dt^{a-1}J^{a-1}(s)d\nu(s^{a-1})(\sigma'')^{a-1}d\sigma^{a-1}.$$

To show that the same formula holds for a, we consider several cases.

Case 0. Suppose that $d_a = (0,0)$. Then $j(a-1) = j(a), k(a-1) = k(a)$, $S^a = S^{a-1}$, $I^a = I^{a-1}$, $V_a = U_a$, and $d\sigma_a = d\lambda_a$. Moreover, we have

$$\int_{V_a} f(\sigma_a)\, d\sigma_a = \int_{V_a} f(\Theta_a(s,t,\sigma))J_a(s)\, d\sigma_a$$

Hence

$$\begin{aligned} \int_{\Lambda^a} f(\lambda^a)\, d\lambda^a &= \int_{\Lambda^{a-1}}\left(\int_{V_a} f(\lambda^{a-1},\sigma_a)\, d\sigma_a\right)\, d\lambda^{a-1} \\ &= m^{a-1}\int_{\Sigma^{a-1}}\int_{S^{a-1}}\int_{I^{a-1}}\left(\int_{V_a} f(\Theta^{a-1}(s,t,\sigma),\Theta_a(s,t,\sigma))\; J_a(s)d\sigma_a\right) \\ &\qquad dt^{a-1}J^{a-1}(s)d\nu(s^{a-1})(\sigma'')^{a-1}d\sigma^{a-1} \\ &= m^a\int_{\Sigma^a}\int_{S^a}\int_{I^a} f(\Theta^a(s,t,\sigma))\; dt^a\; J^a(s)d\nu(s^a)\;(\sigma'')^a d\sigma^a \end{aligned}$$

Case 1. Suppose next that $d_a = (1,0)$, so that $a = a'_j$ with $j = j(a)$. Then $T^a = T^{a-1}$ and $(\sigma'')^{a-1} = (\sigma'')^a$, but $S^a \simeq S^{a-1}\times S_j$ and $J^a(s) = J^{a-1}(s)J_a(s)$. We have

$$\int_{U_a} f(\lambda_a)\, d\lambda_a = \int_{V_a}\int_{S_j} f(\Theta_a(s,t,\sigma))\; J_a(s)\; d\nu(s_j)\; d\sigma_a,$$

and hence

$$\int_{\Lambda^a} f(\lambda^a)\ d\lambda^a = \int_{\Lambda^{a-1}} \left(\int_{U_a} f(\lambda^{a-1}, \lambda_a)\ d\lambda_a \right)\ d\lambda^{a-1}$$

$$= m^{a-1} \int_{\Sigma^{a-1}} \int_{S^{a-1}} \int_{I^{a-1}} \left(\int_{V_a} \int_{S_j} f(\Theta^{a-1}(s,t,\sigma), \Theta_a(s,t,\sigma))\ J_a(s)\ d\nu(s_j)\ d\sigma_a \right)$$
$$dt^{a-1} J^{a-1}(s) d\nu(s^{a-1}) (\sigma'')^{a-1} d\sigma^{a-1}$$

$$= m^a \int_{\Sigma^{a-1}} \int_{V_a} \left(\int_{S^{a-1}} \int_{I^a} \int_{S_j} f(\Theta^a(s,t,\sigma))\ dt^a\ J_a(s) d\nu(s_j)\ J^{a-1}(s) d\nu(s^{a-1}) \right)$$
$$(\sigma'')^a d\sigma_a d\sigma^{a-1}$$

$$= m^a \int_{\Sigma^a} \int_{S^a} \int_{I^a} f(\Theta^a(s,t,\sigma))\ dt^a\ J^a(s) d\nu(s^a)\ (\sigma'')^a d\sigma^a$$

Case 2. Suppose next that $d_a = (0,1)$ so that $a = a''_k$ with $k = k(a)$. Then $S^a = S^{a-1}$ and $J^a(s) = J^{a-1}(s)$, but $T^a = T^{a-1} \cdot T_k$, $(\sigma'')^a = (\sigma'')^{a-1}\sigma_a$, and $m^a = m^{a-1}m_k$. We have

$$\int_{U_a} f(\lambda_a)\ d\lambda_a = m_k \int_{V_a} \int_{I_k} f(\Theta_a(s,t,\sigma))\ \ dt_k\ \sigma_a d\sigma_a,$$

hence

$$\int_{\Lambda^a} f(\lambda^a)\ d\lambda^a = \int_{\Lambda^{a-1}} \left(\int_{U_a(\lambda^{a-1})} f(\lambda^{a-1}, \lambda_a)\ d\lambda_a \right)\ d\lambda^{a-1}$$

$$= \int_{\Sigma^{a-1}} \int_{S^{a-1}} \int_{I^{a-1}} \left(\int_{V_a} \int_{I_k} f(\Theta^{a-1}(s,t,\sigma), \Theta_a(s,t_k,\sigma))\ m_k dt_k \sigma_a d\sigma_a \right)$$
$$m^{a-1} dt^{a-1} J^{a-1}(s) d\nu(s^{a-1}) (\sigma'')^{a-1} d\sigma^{a-1}$$

$$= m^{a-1} \int_{\Sigma^{a-1}} \int_{V_a} \left(m_k \int_{S^a} \int_{I^{a-1}} \int_{I_k} f(\Theta^a(s,t,\sigma))\ dt_k\ dt^{a-1}\ J^a(s) d\nu(s^a) \right)$$
$$\sigma_a d\sigma_a\ (\sigma'')^{a-1} d\sigma^{a-1}$$

$$= m^a \int_{\Sigma^a} \int_{S^a} \int_{I^a} f(\Theta^a(s,t,\sigma))\ J^a(s) d\nu(s^a)\ dt^a\ (\sigma'')^a d\sigma^a$$

Case 3. Finally, if $d_a = (1,1)$, then $a = a'_j = a''_k$ with $j = j(a)$ and $k = k(a)$. Here $S^a \simeq S^{a-1} \times S_j$, $T^a = T^{a-1} \simeq T_k$, $m^a = m^{a-1}m_k$, and since $\sigma_a = 1$ in this case, $(\sigma'')^a = (\sigma'')^{a-1}\sigma_a = (\sigma'')^{a-1}$. The calculation is a combination of Cases 1 and 2. ∎

Let $\Sigma_0 \subset \Sigma$ be a fundamental domain for the action of F on Σ so that $F/F \cap K \times \Sigma_0 \to \Sigma$ defined by $(\dot{\epsilon}, \gamma) \mapsto \epsilon\gamma$ is a Borel isomorphism. A natural choice for Σ_0 is the following. For a positive integer m set $\mathbb{C}(m) = \{z \in \mathbb{C} \setminus \{0\} \mid \operatorname{sign}(z) \in \mathbb{T}(m)\}$. For each $1 \le a \le c$ set

$$F^a = F \cap \bigcap_{b=1}^{a} \ker \chi_b.$$

Assume that $\Sigma_0^{a-1} \subset \Sigma^{a-1}$ is defined. If $F^a = F^{a-1}$, then set $\Sigma_0^a = \{(\sigma_1, \ldots \sigma_a) \in \Sigma^a \mid (\sigma_1, \ldots, \sigma_{a-1}) \in \Sigma_0^{a-1}\}$. If $F^a \neq F^{a-1}$, then $\chi_a(F^{a-1}) = \mathbb{F}(m)$ for some m, and set

$$\Sigma_0^a = \{(\sigma_1, \sigma_2, \ldots, \sigma_a) \mid (\sigma_1, \sigma_2, \ldots, \sigma_{a-1}) \in \Sigma_0^{a-1} \text{ and } \sigma_a \in V_a(\sigma) \cap \mathbb{C}(m)\}.$$

Given $\sigma \in \Sigma$, suppose that $\epsilon^{a-1} \in F^{a-1}$ and $\sigma^{a-1} \in \Sigma_0^{a-1}$ such that $\epsilon^{a-1}\sigma^{a-1} = \sigma^{a-1}$. Choose $\epsilon_a \in F_a$ and $\sigma_a \in V_a(\sigma) \cap \mathbb{C}(m)$ such that $\chi_a(\epsilon_a)\chi_a(\epsilon^{a-1})\sigma_a = \sigma_a$. This iterative argument shows that $F\Sigma_0 = \Sigma$, and if $\sigma \in \Sigma_0$ and $\epsilon \neq 1 \in F$, then $\epsilon \in F^{a-1} \setminus F^a$ for some a, and then by construction $\chi_a(\epsilon)\sigma_a \notin \mathbb{C}(m)$. Hence $\epsilon\Sigma_0 \cap \Sigma_0 = \emptyset$ if $\epsilon \neq 1$.

We have

$$\int_\Sigma \phi(\sigma) d\sigma = \sum_{\dot\epsilon \in F/F\cap K} \int_{\Sigma_0} \phi(\epsilon\gamma) d\gamma. \tag{2.2}$$

Now recall that we have $H = S \cdot T \cdot K_\circ$ where $K_\circ$ is the connected component of the identity in K. Note that $S \cap K = (1)$ by definition of S. It follows that the map $S \times (T/K \cap T) \to H/K$ defined by $(s, \dot t) \mapsto \dot{st}$ is a continuous isomorphism of groups. Now $K \cap T = K \cap F$ and I is a fundamental domain in T for the action of F. Hence the image of I in $T/K \cap T$ is a fundamental domain for the action of $F/K \cap F$ and the map $I \times F/K \cap F \to T/K \cap T$ defined by $(t, \dot\epsilon) \mapsto t\dot\epsilon$ is a Borel isomorphism. Moreover, the prescription

$$\int_{T/K\cap T} \phi(\dot t)\ d\dot t := \sum_{\dot\epsilon \in F/K\cap F} \int_I \phi(t\dot\epsilon)\ dt$$

defines a Haar measure on $T/K\cap T$. Hence we have the natural Borel isomorphism

$$H/K \simeq S \times I \times F/K \cap F$$

and a Haar measure on H/K is given by

$$\int_{H/K} \phi(\dot a)\ d\dot a = \sum_{\dot\epsilon \in F/F\cap K} \int_S \int_I \phi(\epsilon stK)\ dt\ d\nu(s)$$

Now for the H-orbit $\mathcal{O}_\sigma$ of $\sigma \in \Sigma_0$ define the measure ω_σ on $\mathcal{O}_\sigma$ by

$$\int_{\mathcal{O}_\sigma} \phi(\lambda)\ d\omega_\sigma(\lambda) = \int_{H/K} \phi(a\sigma)\ |\delta(a)|^{-1}\ d\dot a.$$

(Note that $|\delta(a)|$ is constant on K-cosets.) Finally, set $|\delta_{\mathbf{e}}| = \prod_{j \in \mathbf{e}} |\delta_j|$ and $d\tilde\mu(\sigma) = m\sigma''|\mathbf{Pf}(\sigma)|d\sigma$. Combining these observations with Lemma 2.4 yields the following.

Proposition 2.3. *For any non-negative measurable function f on Λ one has*

$$\int_\Lambda f(\lambda)\ |\mathbf{Pf}(\lambda)| d\lambda = \int_{\Sigma_0}\int_{\mathcal{O}_\sigma} f(\lambda)\ d\omega_\sigma(\lambda)\ d\tilde{\mu}(\sigma)$$

Proof. By Lemma 2.4 and the preceding decomposition (2.2) of $d\sigma$, we have

$$\begin{aligned}\int_\Lambda f(\lambda)\ d\lambda &= m\int_\Sigma\int_S\int_I f(st\cdot\sigma)\ dt\ J(s)d\nu(s)\ \sigma'' d\sigma\\ &= m\sum_{F/K\cap F}\int_{\Sigma_0}\int_S\int_I f(st\epsilon\cdot\sigma)\ dt\ J(s)d\nu(s)\ \sigma'' d\sigma.\end{aligned}$$

Now with Lemma 1.1, we have

$$\begin{aligned}\int_\Lambda f(\lambda)|\mathbf{Pf}(\lambda)|d\lambda &= m\sum_{F/K\cap F}\int_{\Sigma_0}\int_S\int_I f(st\epsilon\cdot\sigma)\ |\mathbf{Pf}(stf\cdot\sigma)|\ dt\ J(s)d\nu(s)\ \sigma'' d\sigma\\ &= m\sum_{F/K\cap F}\int_{\Sigma_0}\int_S\int_I f(st\epsilon\cdot\sigma)\ |\delta_{\mathbf{e}}(s)|^{-1}|\mathbf{Pf}(\sigma)|\ dt\ J(s)d\nu(s)\ \sigma'' d\sigma\\ &= m\int_{\Sigma_0}\left(\sum_{F/K\cap F}\int_S\int_I f(st\epsilon\cdot\sigma)\ dt\ |\delta_{\mathbf{e}}(s)|^{-1}J(s)\ d\nu(s)\right)|\mathbf{Pf}(\sigma)|\sigma'' d\sigma\end{aligned}$$

and the proof is finished upon observing that $J(s) = \prod_{j\notin\mathbf{e}} |\delta_j(s)|^{-1}$, and hence $|\delta(s)|^{-1} = |\delta_{\mathbf{e}}(s)|^{-1}J(s)$. ■

3. Explicit realizations of irreducible representations

Denote by $\hat{N}$ the Borel space of unitary equivalence classes of irreducibe unitary representations of N, and let $\kappa : \mathfrak{n}^*/N \to \hat{N}$ be the canonical Kirillov correspondence. With the preceding constructions in place, we associate to each $\lambda\in\Lambda$ an irreducible representation π_λ whose equivalence class is $\kappa(N\lambda)$, as follows.

Recall that we have fixed an adaptable basis $\mathcal{B} = \{Z_1, Z_2, \ldots, Z_n\}$ for $\mathfrak{l} = \mathfrak{n}_c$, and we have Ω the minimal (Zariski open) fine layer in $\mathfrak{n}^*$. Recall also the subindex set K_3 for which

$$\mathfrak{p}(\ell) = \mathfrak{p}(\ell)\cap\overline{\mathfrak{p}(\ell)} + \text{ span }\{\rho_{k-1}(Z_{i_k}, \ell)\mid k\in K_3\}$$

where $\mathfrak{p}(\ell)$ is the complex Vergne polarization associated with $\ell\in\Omega$ and $\mathcal{B}$. Write $K_3 = \{h_1 < h_2 < \cdots < h_m\}$. For $\ell\in\Omega$ and $l = 1, 2, \ldots m$, define

$$W_l(\ell) = \rho_{h_l - 1}(Z_{i_{h_l}}, \ell),$$

$$\xi_l(\ell) = \ell[U_{h_l}(\ell), V_{h_l}(\ell)] = \frac{i}{2}\ \ell[W_l(\ell), \overline{W_l(\ell)}],$$

and

$$\epsilon_l(\ell) = \operatorname{sign}\big(\xi_l(\ell)\big), 1 \le l \le m.$$

For each $\ell \in \Omega$ set $\epsilon(\ell) = (\epsilon_1(\ell), \epsilon_2(\ell), \dots, \epsilon_m(\ell))$. We write the layer Ω as a disjoint union of open sets: for each $\epsilon = (\epsilon_1, \dots, \epsilon_m) \in \{\pm 1\}^m$ set

$$\Omega^\epsilon = \{\ell \in \Omega \mid \epsilon(\ell) = \epsilon\}.$$

Note that in many situations (for example, when N is a Heisenberg group and $Z_3 = \overline{Z}_2$) some of the sets Ω^ϵ are empty.

Lemma 3.1. *For each sign index ϵ, the set Ω^ϵ is G-invariant.*

Proof. It follows from Lemma 1.5 that Ω^ϵ is N-invariant, and from Lemma 1.6 that Ω^ϵ is H-invariant: let $a \in H$; then

$$W_l(a\ell) = \rho_{h_l-1}(Z_{i_{h_l}}, a\ell) = a\rho_{h_l-1}(a^{-1}Z_{i_{h_l}}, \ell) = \delta_{i_{h_l}}(a)^{-1}\ aW_l(\ell)$$

and $\overline{W_l(a\ell)} = \overline{\delta_{i_{h_l}}}(a)^{-1}\ aW_l(\ell)$. ■

Let $\epsilon \in \{\pm 1\}^m$. If $j \notin \{i_k, j_k : k \in K_3\}$, then set $Z_j^\epsilon = Z_j$. If $j = i_{h_l}$ (with $h_l \in K_3$), then define Z_j^ϵ and Z_{j+1}^ϵ as follows. If $\epsilon_l = 1$ set $Z_j^\epsilon = Z_j, Z_{j+1}^\epsilon = Z_{j+1}$, while if $\epsilon_l = -1$, then $Z_j^\epsilon = \overline{Z}_j = Z_{j+1}$ and $Z_{j+1}^\epsilon = \overline{Z}_{j+1} = Z_j$. It is clear that $\mathcal{B}^\epsilon = \{Z_1^\epsilon, Z_2^\epsilon, \dots, Z_n^\epsilon\}$ is also an adaptable basis for $\mathfrak{l}$. Put

$$\mathfrak{l}_j^\epsilon = \operatorname{span}\{Z_1^\epsilon, Z_2^\epsilon, \dots, Z_j^\epsilon\},\ 1 \le j \le n,$$

and let $\mathfrak{p}^\epsilon(\ell) = \sum_{j=1}^n (\mathfrak{l}_j^\epsilon)^\ell \cap \mathfrak{l}_j^\epsilon$ be the corresponding complex Vergne polarization at ℓ.

Lemma 3.2. *For each $\ell \in \Omega^\epsilon$, $\mathfrak{p}^\epsilon(\ell)$ is a positive polarization at ℓ.*

Proof. Let $\ell \in \Omega^\epsilon$ and let $Y \in \mathfrak{p}^\epsilon(\ell)$. By Lemma 1.4 we have $Y = W + \sum_{k \in K_3} a_k \rho_{k-1}(Z_{i_k}^\epsilon, \ell)$ where $W \in \mathfrak{p}^\epsilon(\ell) \cap \overline{\mathfrak{p}^\epsilon(\ell)}$, $a_k \in \mathbb{C}$. Now $\overline{\rho_{k-1}(Z_{i_k}^\epsilon, \ell)} = \rho_{k-1}(\overline{Z}_{i_k}^\epsilon, \ell)$ and

$$\begin{aligned} i\ \ell[\rho_{k-1}(Z_{i_k}^\epsilon, \ell), \rho_{k-1}(\overline{Z}_{i_k}^\epsilon, \ell)] &= \epsilon_k\ i\ \ell[\rho_{k-1}(Z_{i_k}, \ell), \rho_{k-1}(\overline{Z}_{i_k}, \ell)] \\ &= \big|\ \ell[\rho_{k-1}(Z_{i_k}, \ell), \rho_{k-1}(\overline{Z}_{i_k}, \ell)]\big|. \end{aligned}$$

Since $\mathfrak{p}^\epsilon(\ell) \cap \overline{\mathfrak{p}^\epsilon(\ell)} \subset \big(\mathfrak{p}^\epsilon(\ell) + \overline{\mathfrak{p}^\epsilon(\ell)}\big)^\ell$ and for $k \ne k' \in K_3$,

$$\ell[\rho_{k-1}(Z_{i_k}^\epsilon, \ell), \rho_{k'-1}(\overline{Z}_{i_{k'}}^\epsilon, \ell)] = 0,$$

then we have

$$\begin{aligned} i\ \ell[Y,\overline{Y}] &= \ell\Big[W + \sum_{k\in K_3} a_k \rho_{k-1}(Z^{\epsilon}_{i_k},\ell),\ \overline{W} + \sum_{k\in K_3} \overline{a_k}\rho_{k-1}(\overline{Z}^{\epsilon}_{i_k},\ell)\Big] \\ &= \sum_{k\in K_3} |a_k|^2\ i\ \ell[\rho_{k-1}(Z^{\epsilon}_{i_k},\ell),\rho_{k-1}(\overline{Z}^{\epsilon}_{i_k},\ell)] \\ &= \sum_{k\in K_3} |a_k|^2\ \big|\ell[\rho_{k-1}(Z_{i_k},\ell),\rho_{k-1}(\overline{Z}_{i_k},\ell)]\big| > 0 \end{aligned}$$

■

We set $\Lambda^\epsilon = \Lambda \cap \Omega^\epsilon$ and $\Sigma^\epsilon = \Sigma \cap \Omega^\epsilon$. For each $\lambda \in \Lambda^\epsilon$, $H\lambda \cap \Sigma \subset \Sigma^\epsilon$, so F leaves Σ^ϵ invariant, and so if Σ_0 is a fundamental domain for Σ/F, then $\Sigma^\epsilon_0 = \Sigma_0 \cap \Sigma^\epsilon$ is a fundamental domain for Σ^ϵ/F.

Now fix $\lambda \in \Lambda^\epsilon$. Set $\mathfrak{d}(\lambda)_{\mathbb{C}} = \mathfrak{p}^\epsilon(\lambda) \cap \overline{\mathfrak{p}^\epsilon(\lambda)}$, $\mathfrak{d}(\lambda) = \mathfrak{d}(\lambda)_{\mathbb{C}} \cap \mathfrak{n}$ and $\mathfrak{e}(\lambda) = \big(\mathfrak{p}^\epsilon(\lambda) + \overline{\mathfrak{p}^\epsilon(\lambda)}\big) \cap \mathfrak{n}$. Note that $\mathfrak{d}(\lambda)$ and $\mathfrak{e}(\lambda)$ are independent of $\epsilon(\lambda)$ and as is well-known, $[\mathfrak{e}(\lambda),\mathfrak{e}(\lambda)] \subset \mathfrak{d}(\lambda)$. Let $D(\lambda)$ and $E(\lambda)$ the corresponding analytic subgroups of N. We realize the irreducible representation corresponding to the N-orbit of λ by an explicit version of holomorphic induction as follows.

First we define complex coordinates on $E(\lambda)$. Let $\alpha^\circ_\lambda : \mathbb{C}^m \times D(\lambda) \to E(\lambda)$ be defined by

$$\alpha^\circ_\lambda(w,d) = \exp\Big(\Re\big(w_1\overline{W_1(\lambda)}\big) + \cdots + \Re\big(w_m\overline{W_m(\lambda)}\big)\Big)\ d.$$

For each $\epsilon \in \{\pm 1\}^m$ and $1 \le l \le m$, set $W^\epsilon_l(\lambda) = \rho_{h_l-1}(Z^\epsilon_{i_{h_l}},\lambda)$ and

$$\xi^\epsilon_l(\lambda) = \epsilon_l \xi_l(\lambda) = \frac{i}{2}\ \lambda[W^\epsilon_l(\lambda),\overline{W}^\epsilon_l(\lambda)].$$

Note that $\mathfrak{p}^\epsilon(\lambda) = \mathfrak{d}(\lambda)_{\mathbb{C}} + \mathbb{C}\text{-span}\{W^\epsilon_l(\lambda) : 1 \le l \le m\}$. Writing $w_l = x_l + iy_l$, define the usual complex derivative by

$$\partial_l = \frac{1}{2}\left(\frac{\partial}{\partial x_l} - i\frac{\partial}{\partial y_l}\right)$$

and put $\partial^{\epsilon_l}_l = \partial_l$ or $\overline{\partial}_l$, if $\epsilon_l = 1$ or -1, respectively. Define the algebra $\mathcal{A}^\epsilon(\mathbb{C}^m)$ of "ϵ-holomorphic" functions on $\mathbb{C}^m$ by

$$\mathcal{A}^\epsilon(\mathbb{C}^m) = \{p \in C^\infty(\mathbb{C}^m) \mid \partial^{-\epsilon_l}_l p = 0, 1 \le l \le m\}.$$

Now set $\epsilon = \epsilon(\lambda)$ so that $\xi^\epsilon_l(\lambda) > 0$, $1 \le l \le m$. Define $\mathcal{H}^\circ_\lambda = (\mathcal{A}^\epsilon(\mathbb{C}^m), \|\cdot\|_\lambda)$ where

$$\|p\|^2_\lambda = \int_{\mathbb{C}^m} |p(w)|^2\ \exp\left(-\frac{1}{2}\sum_l \xi^\epsilon_l(\lambda)|w_l|^2\right)\ dwd\overline{w}.$$

Write $w^{\epsilon_l}_l = w_l$ or $w^{\epsilon_l}_l = \overline{w}_l$ according as $\epsilon_l = +1$ or $\epsilon_l = -1$ respectively. Let $k = (k_1,k_2,\ldots,k_m)$ be a multi-index of non-negative integers and put

$$\psi^k_\lambda(w) = c^k_\lambda\ (w^{\epsilon_1}_1)^{k_1}(w^{\epsilon_2}_2)^{k_2}\cdots(w^{\epsilon_m}_m)^{k_m}$$

where c^k_λ is a normalizing constant. Then $\{\psi^k_\lambda \mid k_l \geq 0, 1 \leq l \leq m\}$ is a complete orthonormal set in $\mathcal{H}^\circ_\lambda$. Define the unitary representation π°_λ of $E(\lambda)$ in $\mathcal{H}^\circ_\lambda$ by

$$\big(\pi^\circ_\lambda(w', d')p\big)(w) = \\ p(w - w')\chi_\lambda(d') \exp\left(\frac{1}{2}\sum_l \xi^\epsilon_l(\lambda)\overline{w}'_l w_l\right) \exp\left(-\frac{1}{4}\sum_l \ \xi^\epsilon_l(\lambda))|w'_l|^2\right).$$

We show that for $\lambda \in \Lambda^\epsilon$, the representation π°_λ is isomorphic with the representation obtained from $\mathfrak{p}^\epsilon(\lambda)$ via holomorphic induction. For $X \in \mathfrak{e}(\lambda)$ define the differential operator $R(X)$ on $E(\lambda)$ by

$$R(X)\phi = \frac{d}{dt}\Big|_{t=0} \phi\big(\cdot \exp(tX)\big).$$

We can then define $R(W)$ for $W \in \mathfrak{e}(\lambda)_\mathbb{C}$ by extending in the obvious way.

Proposition 3.1. *The unitary representation π°_λ is irreducible and its equivalence class corresponds to the $E(\lambda)$-coadjoint orbit of $\lambda|_{E(\lambda)}$.*

Proof. In terms of the preceding coordinates and notations, we find that

$$R(W^\epsilon_l(\lambda)) = 2\partial^{-\epsilon_l}_l + \frac{i}{2}\epsilon_l w^{\epsilon_l}_l R(Z^\epsilon_l(\lambda)),$$

where $Z^\epsilon_l(\lambda) = \frac{i}{2}\,[W^\epsilon_l(\lambda), \overline{W}^\epsilon_l(\lambda)]$. Define

$$\psi_0(w, d) = \chi_\lambda(d)^{-1} \exp\left(-\frac{1}{4}\sum_{l=1}^m \ \xi^\epsilon_l(\lambda)|w_l|^2\right).$$

We compute easily that $R(W^{\epsilon_l}_l(\lambda))\psi_0(w, d) = 0$, $1 \leq l \leq m$. It follows that $\psi_0 \circ (a^\circ_\lambda)^{-1}$ belongs to the Hilbert space $\mathcal{H}(E(\lambda), D(\lambda), \chi_\lambda, \mathfrak{p}(\lambda))$ for holomorphic induction. Recall that $\mathcal{H}(E(\lambda), D(\lambda), \chi_\lambda, \mathfrak{p}^\epsilon(\lambda))$ is the completion of the subset $\mathcal{D}(E(\lambda), D(\lambda), \chi_\lambda, \mathfrak{p}^\epsilon(\lambda))$ consisting of all smooth functions ϕ on $E(\lambda)$ satisfying $R(W)\phi = -i\lambda(W)\phi$ for all $W \in \mathfrak{p}^\epsilon(\lambda)$, and

$$\int_{\mathbb{C}^m} \ |\phi(\alpha^\circ_\lambda(w, e)|^2 \ dwd\overline{w} < \infty.$$

Moreover (see for example [2, Theorem I.2.7]), one has

$$\mathcal{H}(E(\lambda), D(\lambda), \chi_\lambda, \mathfrak{p}(\lambda)) \\ = \{\phi \in \mathcal{H}(E, D, \chi_\lambda) \mid \phi\big(a^\circ_\lambda(w, d)\big) = p(w)\psi_0(w, d) \text{ for some } p \in \mathcal{A}^\epsilon(\mathbb{C}^m)\}.$$

Thus $\mathcal{H}^\circ_\lambda$ is naturally isomorphic with $\mathcal{H}(E(\lambda), D(\lambda), \chi_\lambda, \mathfrak{p}(\lambda))$ via the map

$$p \mapsto \big(p\psi_0\big) \circ (a^\circ_\lambda)^{-1}.$$

and it is a standard calculation to show that π°_λ is isomorphic with the holomorphically induced representation. ∎

The irreducible representation π_λ of N associated with λ will be induced from π_λ°. Just as with π_λ° we realize π_λ by a precise construction.

First we identify indices belonging to the sequence $\mathbf{j}$ which are "supplementary" to the subalgebras $\mathfrak{e}(\lambda)$. Let $\mathbf{j}'$ denote the subsequence of $\mathbf{j}$ consisting of the indices $\{j = j_k \in \mathbf{j} \cap I \mid k \notin K_3\} \cup \{j \in \mathbf{j} \mid j \notin I, j+1 \notin \mathbf{j}\}$ and write

$$\mathbf{j}' = \{j_{k_1}, j_{k_2}, \ldots, j_{k_p}\}. \tag{3.1}$$

We decompose $\mathbf{j}'$ into disjoint subsequences $\mathbf{j}^r$ and $\mathbf{j}^c$ where $\mathbf{j}^c$ consists of those indices $j \in \mathbf{j}'$ such that $j-1 \notin I$ (and hence $j-1 \in \mathbf{j}$).

Next, let $O \in \mathcal{C}$ be a covering set, as defined in Lemma 1.2. We use the continuous N-invariant functions ϕ_k^O of Lemma 1.2 to define an N-invariant, smoothly-varying supplementary basis for $\mathfrak{e}(\lambda)$ in $\mathfrak{n}$. Fix $1 \le l \le p$ and $j = j_{k_l}$. If $j \in I$, then set $X_l^O(\lambda) = Z_j$. If $j \notin I$ (and hence $j+1 \notin \mathbf{j}$), then, referring to notations of Lemma 1.2 and to the comments following it, set

$$X_l^O(\lambda) = \phi_k^O(\lambda)^{-1} \frac{Z_j(\lambda)}{|\ell[Z_j(\lambda), V_k(\lambda)]|^{1/2}}$$

where k is the subindex for j in $\mathbf{j}$. From Lemma 1.2, we have that $X_l^O(\lambda)$ is real, and from Lemma 1.5, we have that $X_l^O(\lambda)$ is N-invariant.

Now from the definition of the sequence $\mathbf{j}$, and the construction of the elements $X_l^O(\lambda)$, it is clear that the set

$$\{X_l^O(\lambda), \overline{X_l^O(\lambda)} \mid 1 \le l \le p\} \;\cup\; \{\rho_{k-1}(Z_{j_k}, \lambda) \mid k \in K_3\}$$

is a basis of $\mathfrak{n}_{\mathbb{C}}$ modulo $\mathfrak{p}(\lambda)$. By Lemma 1.4 we have

$$\{\rho_{k-1}(Z_{j_k}, \lambda) \mid k \in K_3\}$$

is a basis for $\mathfrak{e}(\lambda)_{\mathbb{C}} = \mathfrak{p}(\lambda) + \overline{\mathfrak{p}(\lambda)}$ modulo $\mathfrak{p}(\lambda)$. Hence $\{X_l^O(\lambda), \overline{X_l^O(\lambda)} \mid 1 \le l \le p\}$ is a basis for $\mathfrak{n}_{\mathbb{C}}$ modulo $\mathfrak{e}(\lambda)_{\mathbb{C}}$, and $\{\Re(X_l^O(\lambda)), \Im(X_l^O(\lambda)) \mid 1 \le l \le p\}$ is a basis for $\mathfrak{n}$ modulo $\mathfrak{e}(\lambda)$.

Now fix $1 \le l \le p$ and $j = j_{k_l}$. If $j \in \mathbf{j}^r$, put

$$\alpha_{\lambda,l}^O(x) = \exp\big(xX_l^O(\lambda)\big),\ x \in \mathbb{R},$$

while if $j \in \mathbf{j}^c$ then set

$$\alpha_{\lambda,l}^O(x) = \exp\Big(\Re\big(xZ_j\big)\Big),\ x \in \mathbb{C}.$$

Set

$$\mathcal{X} = \{(x_1, x_2, \ldots, x_p) \mid x_l \in \mathbb{C} \text{ if } j_{k_l} \in \mathbf{j}^c, \text{ and } x_l \in \mathbb{R} \text{ otherwise}\},$$

and define $\alpha_\lambda^O : \mathcal{X} \to N$ by

$$\alpha_\lambda^O(x_1, x_2, \ldots, x_p) = \alpha_{\lambda,1}^O(x_1)\alpha_{\lambda,2}^O(x_2)\cdots\alpha_{\lambda,p}^O(x_p)$$

Since N is nilpotent the following is immediate.

Lemma 3.3. *The map*

$$x \mapsto \alpha^O_\lambda(x)E(\lambda)$$

is a diffeomorphism of $\mathcal{X}$ *onto* $N/E(\lambda)$.

Write dx for the Lebesgue measure on $\mathcal{X}$. Define the measure $d\nu_\lambda(\dot{n})$ on $N/E(\lambda)$ by

$$\int_{N/E(\lambda)} f(\dot{n})d\nu_\lambda(\dot{n}) = \int_{\mathcal{X}} f(\alpha^O_\lambda(x))dx.$$

Suppose that O' is another covering set containing λ. Then it follows from the definition of the continuous functions $\phi^O_k(\lambda)$ (see [7]) that when $j = j_k \notin I$ and $j+1 \notin \mathbf{j}$, then $\phi^{O'}_k(\lambda)^{-1}Z_{j_k}(\lambda) = \pm\phi^O_k(\lambda)^{-1}Z_{j_k}(\lambda)$. Hence $\alpha^{O'}_{\lambda,l}(x) = \alpha^O_{\lambda.l}(\pm x)$ and the definition of $d\nu_\lambda(\dot{n})$ is independent of the covering set O.

Now for each $a \in H$ define $c_\lambda(a) : N/E(\lambda) \to N/E(a\lambda)$ by $c_\lambda(a)(nE(\lambda)) = ana^{-1}E(a\lambda)$. We now compute a positive, multiplicative character $|\delta^1|$ on H such that

$$\int_{N/E(\lambda)} f(c_\lambda(a)\dot{n})\ |\delta^1(a)|d\nu_\lambda(\dot{n}) = \int_{N/E(a\lambda)} f(\dot{n})d\nu_{a\lambda}(\dot{n}).$$

Fix $\lambda \in \Lambda, a \in H$ and choose covering sets O and O' such that $\lambda \in O$ and $a\lambda \in O'$. We must compute the determinant of the Jacobian matrix for the map $\varphi(a) : \mathcal{X} \to \mathcal{X}$ defined by

$$\varphi(a) = (\alpha^{O'}_{a\lambda})^{-1} \circ c_\lambda(a) \circ \alpha^O_\lambda.$$

Fix $1 \le l \le p$ and $j = j_{k_l}$; if $j \in I$, then $a\alpha^O_{\lambda,l}(x)a^{-1} = \alpha^O_{\lambda,l}(\delta_j(a)x)$. If $j \notin I$, then we use Lemma 1.6. With k the subindex for j in $\mathbf{j}$, we have complex numbers $\nu_{i_k}(a)$ and $\nu_j(a)$ such that $a^{-1}\cdot V_k(a\lambda) = \nu_{i_k}(a)V_k(\lambda)$ and $a^{-1}\cdot Z_j(a\lambda) = \nu_j(a)Z_j(\lambda)$ where $\nu_j(a) = \nu_{i_k}(a)|\delta_j(a)|^{-2}$. Hence $a^{-1}\cdot X_l(a\lambda)$

$$\begin{aligned}
&= \phi^{O'}_k(a\lambda)^{-1}\frac{a^{-1}\cdot Z_j(a\lambda)}{|a\lambda[Z_j(a\lambda), V_k(a\lambda)]|^{1/2}} \\
&= \phi^{O'}_k(a\lambda)^{-1}\frac{\nu_j(a)Z_j(\lambda)}{|\nu_{i_k}(a)\nu_j(a)\lambda[Z_j(\lambda), V_k(\lambda)]|^{1/2}} \\
&= \phi^{O'}_k(a\lambda)^{-1}\nu_j(a)|\nu_{i_k}(a)\nu_j(a)|^{-1/2}\frac{Z_j(\lambda)}{|\lambda[Z_j(\lambda), V_k(\lambda)]|^{1/2}} \\
&= \phi^{O'}_k(a\lambda)^{-1}\mathrm{sign}(\nu_{i_k}(a))|\delta_j(a)|^{-1}\frac{Z_j(\lambda)}{|\lambda[Z_j(\lambda), V_k(\lambda)]|^{1/2}} \\
&= \left(\phi^{O'}_k(a\lambda)^{-1}\phi^O_k(\lambda)\ \mathrm{sign}(\nu_{i_k}(a))\right)\ |\delta_j(a)|^{-1}\phi^O_k(\lambda)^{-1}\frac{Z_j(\lambda)}{|\lambda[Z_j(\lambda), V_k(\lambda)]|^{1/2}} \\
&= \pm|\delta_j(a)|^{-1}X_l(\lambda)
\end{aligned}$$

where we have also used the fact that $a^{-1}\cdot X_l(a\lambda)$ is real. Hence in this case

$$a\alpha^O_{\lambda,l}(x)a^{-1} = \alpha^{O'}_{a\lambda,l}(\pm|\delta_j(a)|x) \tag{3.2}$$

Hence $\varphi(a) = \operatorname{diag}\big(\varphi(a)_1, \varphi(a)_2, \dots, \varphi(a)_p\big)$ where $|\varphi(a)_l| = |\delta_{j_{k_l}}(a)|$ in each of the preceding cases.

Now set $\delta^1(a) = \prod_{k \notin K_3} \delta_{j_k}(a), a \in H$. The above shows that

$$|\delta^1(a)| = \prod_{j \in \mathbf{j}^r} |\delta_j(a)| \times \prod_{j \in \mathbf{j}^c} |\delta_j(a)|^2$$

is the determinant of the Jacobian matrix for $\varphi(a)$. Hence

$$\begin{aligned}\int_{N/E(\lambda)} f(c_\lambda(a)\dot{n})\ |\delta^1(a)|\ d\nu_\lambda(\dot{n}) &= \int_{\mathcal{X}} (f \circ c_\lambda(a) \circ \alpha_\lambda^O)(x)\ |\delta^1(a)|\ dx \\ &= \int_{\mathcal{X}} (f \circ \alpha_{a\lambda}^{O'} \circ \varphi(a))(x)\ |\delta^1(a)|\ dx \\ &= \int_{\mathcal{X}} (f \circ \alpha_{a\lambda}^{O'})(x) dx \\ &= \int_{N/E(h\lambda)} f(\dot{n})\ d\nu_{a\lambda}(\dot{n}).\end{aligned}$$

For each $\lambda \in \Lambda$, having fixed the relatively invariant measure $d\nu_\lambda$ on $N/E(\lambda)$, let π_λ be the representation of N induced from π_λ°, acting in the Hilbert space $\mathcal{H}_\lambda = L^2(N, E(\lambda), \mathcal{H}_\lambda^\circ, \pi_\lambda^\circ, d\nu_\lambda)$. We make two observations here about the explicit constructions above and the action of the stabilizer K.

Lemma 3.4. *For each $a \in K$ define the map $\varphi(a) = (\alpha_\lambda^O)^{-1} \circ c_\lambda(a) \circ \alpha_\lambda^O : \mathcal{X} \to \mathcal{X}$. Then $\varphi : K \to GL(\mathcal{X})$ is a representation of K that is isomorphic with the natural linear action of K on $\mathfrak{n}/\mathfrak{e}(\lambda)$. Moreover, $\varphi_l = \delta_{j_{k_l}}, 1 \le l \le p$; in particular, φ is independent of the choice of covering set and of λ.*

Proof. The map $\beta_\lambda^O : \mathcal{X} \to \mathfrak{n}/\mathfrak{e}(\lambda)$ defined by

$$\beta_\lambda^O(x_1, x_2, \dots, x_p) = \sum_{l=1}^p \log\big(\alpha_{\lambda,l}^O(x_l)\big) + \mathfrak{e}(\lambda)$$

is the indicated isomorphism. To show that $\varphi_l = \delta_{j_{k_l}}$, we need only consider the case where $j = j_{k_l} \notin I$. For this we apply preceding computation that resulted in equation (3.2):

$$\varphi(a)_l = \operatorname{sign}\ \big(\nu_{i_k}(a)\big)^{-1} |\delta_j(a)|,$$

where $j = j_k$. From Lemma 1.7 we know that $\nu_{i_k}(a)$ and $\delta_j(a) = \nu_{j_k}(a)^{-1}$ are both positive. The result follows. ∎

Thus φ defines an action of K on $\mathcal{X}$ which is independent of λ and O. We define the unitary representation $\gamma^{\mathcal{X}}$ of K on $L^2(\mathcal{X})$ by

$$\gamma^{\mathcal{X}}(a) f(x) = f\big(\varphi(a)^{-1} x\big) \cdot |\delta^1(a)|^{-1/2}$$

Lemma 3.5. *Given a choice of covering set O containing λ, we have a natural isomorphism of $\mathcal{H}_\lambda$ with $L^2(\mathcal{X}) \otimes \mathcal{H}_\lambda^\circ$.*

Proof. Given $f \in \mathcal{H}_\lambda$, we define $\tilde{A}_\lambda^O(f)$ as follows. For $v \in \mathcal{H}_\lambda^\circ$, and for a.e. $x \in \mathcal{X}$ put

$$\Big(\tilde{A}_\lambda^O(f)(v)\Big)(x) = \langle f(\alpha_\lambda^O(x)), v\rangle;$$

then the Cauchy-Schwartz inequality gives

$$\int_{\mathcal{X}} |\langle f(\alpha_\lambda^O(x)), v\rangle|^2 \; dx \leq \int_{\mathcal{X}} \|f(\alpha_\lambda^O(x))\|^2 \|v\|^2 \; dx = \|f\|^2\|v\|^2$$

so $\tilde{A}_\lambda^O(f)v$ defines an element of $L^2(\mathcal{X})$ and accordingly we have a linear map $\tilde{A}_\lambda^O(f) : \mathcal{H}_\lambda^\circ \to L^2(\mathcal{X})$. Let $\{v_j\}$ be an orthonormal basis for $\mathcal{H}_\lambda^\circ$: then

$$\begin{aligned}\sum_j \|\tilde{A}_\lambda^O(f)v_j\|^2 &= \sum_j \int_{\mathcal{X}} |\langle f(\alpha_\lambda^O(x)), v_j\rangle|^2 \; dx \\ &= \int_{\mathcal{X}} \sum_j |\langle f(\alpha_\lambda^O(x)), v_j\rangle|^2 \; dx \\ &= \int_{\mathcal{X}} \|f(\alpha_\lambda^O(x))\|^2 \; dx = \|f\|^2\end{aligned}$$

showing that $\tilde{A}_\lambda^O(f)$ is Hilbert-Schmidt and that $\tilde{A}_\lambda^O$ is an isometry.

Given an elementary tensor $g \otimes v \in L^2(\mathcal{X}) \otimes \mathcal{H}_\lambda^\circ$, define $f \in \mathcal{H}_\lambda$ as follows. For each $n \in N$, we have a unique point $x(n) \in \mathcal{X}$ and $e(n) \in E(\lambda)$ such that $n = \alpha_\lambda^O(x(n))e(n)$. Put

$$f(n) = g(x(n)) \; \pi_\lambda^\circ(e(n))^{-1}v.$$

Then $f \in \mathcal{H}_\lambda$ and $\tilde{A}_\lambda^O(f) = g \otimes v$. It follows that $\tilde{A}_\lambda^O$ is surjective. ■

Hence we may regard $\mathcal{H}_\lambda$ as a closed subspace of $L^2(\mathcal{X} \times \mathbb{C}^m)$ where the norm is given by

$$\|F\|^2 = \int_{\mathcal{X}} \int_{\mathbb{C}^m} |F(x,w)|^2 \; \exp\left(-\frac{1}{2}\sum_{l=1}^m \xi_l^\epsilon(\lambda)|w_l|^2\right) \; dw d\overline{w} \; dx, \;\; F \in L^2(\mathcal{X} \times \mathbb{C}^m).$$

We now describe the action of H on $\hat{N}$ in terms of the preceding explicit data. Let $a \in H$. Let $\mathbf{j}''$ be the subsequence of $\mathbf{j}$ defined by $\mathbf{j}'' = \{j_k \in \mathbf{j} : k \in K_3\}$; note that $\mathbf{j}''$ is disjoint from $\mathbf{j}'$, and recall the notation $K_3 = \{h_1 < h_2 < \cdots < h_m\}$. Put $\delta_l^\circ = \delta_{j_{h_l}}, 1 \leq l \leq m$ and set $\delta^\circ = (\delta_1^\circ, \delta_2^\circ, \ldots, \delta_m^\circ)$ and $|\delta^\circ| = \prod_{l=1}^m |\delta_l^\circ|$. Let $(\pi_\lambda^\circ)^a$ be the irreducible representation of $E(a\lambda)$ defined by $(\pi_\lambda^\circ)^a(n) = \pi_\lambda(a^{-1}na)$ and let $B(a,\lambda) : \mathcal{H}_\lambda^\circ \to \mathcal{H}_{a\lambda}^\circ$ be the map

$$\big(B(a,\lambda)p\big)(w) = p(\delta^\circ(a)^{-1}w) \; |\delta^\circ(a)|^{-1} = p(\delta_1^\circ(a)^{-1}w_1, \ldots, \delta_m^\circ(a)^{-1}w_m) \; |\delta^\circ(a)|^{-1}.$$

Lemma 3.6. *The operators $B(a,\lambda)$ are unitary and for each $a \in H$, $\lambda \in \Lambda$, $B(a,\lambda)$ intertwines the representations $(\pi^\circ_\lambda)^a$ and $\pi^\circ_{a\lambda}$. Moreover they satisfy the relation*

$$B(a,b\lambda)\circ B(b,\lambda) = B(ab,\lambda)$$

for each $a,b \in H, \lambda \in \Lambda$.

Proof. By Lemma 1.6 we have

$$(a\lambda)\big(W_l(a\lambda)\big) = \delta_{i_{h_l}}(a)^{-1}\ \lambda\big(W_l(\lambda)\big) = \overline{\delta^\circ_l}(a)^{-1}\ \lambda\big(W_l(\lambda)\big)$$

so

$$(a\lambda)[W_l(a\lambda), \overline{W_l(a\lambda)}] = |\delta^\circ_l(a)|^{-2}\lambda[W_l(\lambda), \overline{W_l(\lambda)}]$$

and hence $\xi_l(a\lambda) = |\delta^\circ_l(a)|^{-2}\ \xi_l(\lambda)$. It follows that for each $a \in H$, $B(a,\lambda)$ is unitary:

$$\begin{aligned}
&\|B(a,\lambda)p\|^{\circ\,2}_{a\lambda}\\
&= \int_{\mathbb{C}^m} |p(\delta^\circ_1(a)^{-1}w_1,\ldots,\delta^\circ_m(a)^{-1}w_m)|^2\ |\delta^\circ(a)|^{-2}\exp\left(-\frac{1}{2}\sum_l\ \xi_l(a\lambda)|w_l|^2\right) dw d\overline{w}\\
&= \int_{\mathbb{C}^m} |p(\delta^\circ_1(a)^{-1}w_1,\ldots,\delta^\circ_m(a)^{-1}w_m)|^2\ |\delta^\circ(a)|^{-2}\\
&\qquad\qquad \exp\left(-\frac{1}{2}\sum_l |\delta^\circ_l(a)|^{-2}\ \xi_l(\lambda)|w_l|^2\right) dw d\overline{w}\\
&= \int_{\mathbb{C}^m} |p(w_1,\ldots,w_m)|^2\ |\delta^\circ(a)|^{-2}\\
&\qquad\qquad \exp\left(-\frac{1}{2}\sum_l |\delta^\circ_l(a)|^{-2}\ \xi_l(\lambda)|\delta^\circ_l(a)w_l|^2\right)\ |\delta^\circ(a)|^2\ dw d\overline{w}\\
&= \int_{\mathbb{C}^m} |p(w_1,\ldots,w_m)|^2\exp\left(-\frac{1}{2}\sum_l\ \xi_l(\lambda)|w_l|^2\right) dw d\overline{w}\\
&= \|p\|^{\circ\,2}_\lambda.
\end{aligned}$$

It is easy to check that $B(a,\lambda)\pi^\circ_\lambda\big(a^{-1}(w,d)a\big) = \pi^\circ_{a\lambda}(w,d)B(a,\lambda)$ holds for all $(w,d) \in \mathbb{C}^m \times D(\lambda)$ and that $B(a,b\lambda)\circ B(b,\lambda) = B(ab,\lambda)$. ■

Denote the unitary representation $B(\cdot,\lambda)|_K$ of K acting in $\mathcal{H}^\circ_\lambda$ by γ°_λ. Recall that by part (b) of Lemma 1.7, each δ°_l, when restricted to K, is a unitary character, $1 \le l \le m$. The unitary representation $\delta^\circ : K \to D(m,\mathbb{C})$ is equivalent to the linear action of K on $\mathfrak{e}(\lambda)/\mathfrak{d}(\lambda)$ via the map $\mathbb{C}^m \to \log\alpha^\circ_\lambda + \mathfrak{d}(\lambda)$. For any $p \in \mathcal{H}^\circ_\lambda$,

$$\big(\gamma^\circ_\lambda(a)p\big)(w) = p(\delta^\circ(a)^{-1}w),\ a \in K.$$

Let μ_λ° denote a Borel measure on $\hat{K}$ and m_λ° the non-vanishing multiplicity function associated with γ_λ° so that

$$\gamma_\lambda^\circ \simeq \int_{\hat{K}}^{\oplus} m_\lambda^\circ(\eta)\ \eta\ d\mu_\lambda^\circ(\eta).$$

Then μ_λ° is supported on $\hat{K}''$ (where $\hat{K}'' \subset \hat{K}$ in the usual way.)

Lemma 3.7. *The class of the measure μ_λ° and the multiplicity function m_λ° associated with γ_λ° depend only upon the sign index $\epsilon(\lambda)$.*

Proof. The monomials

$$\{(w^\epsilon)^k = (w_1^{\epsilon_1})^{k_1}(w_2^{\epsilon_2})^{k_2}\cdots(w_m^{\epsilon_m})^{k_m} \mid k_1 \geq 0, k_2 \geq 0, \ldots, k_m \geq 0\}$$

are a complete set of eigenfunctions for $\gamma_\lambda^\circ(a), a \in K$:

$$\gamma_\lambda^\circ(a)\Big((w^\epsilon)^k\Big) = \delta_1^\circ(a)^{-\epsilon_1 k_1}\delta_2^\circ(a)^{-\epsilon_2 k_2}\cdots\delta_m^\circ(a)^{-\epsilon_m k_m}\ (w^\epsilon)^k, a \in K.$$

Hence, if η belongs to the support of μ_λ°, then the multiplicity $m_\lambda^\circ(\eta)$ of a character $\eta \in \hat{K}$ in the irreducible decomposition of γ_λ° is just

$$m_\lambda^\circ(\eta) = \Big|\{(k_1, k_2, \ldots, k_m) \mid (\delta_1^\circ)^{-\epsilon_1 k_1}(\delta_2^\circ)^{-\epsilon_2 k_2}\cdots(\delta_m^\circ)^{-\epsilon_m k_m} = \eta\}\Big|.$$

■

For each $a \in H$ define $\pi_\lambda^a = \pi_\lambda(a^{-1}\cdot a)$. For $f \in \mathcal{H}_\lambda$, define $C(a,\lambda)f$ by

$$\big(C(a,\lambda)f\big)(n) = B(a,\lambda)\big(f(a^{-1}na)\big)\delta^1(a)^{-1/2}.$$

Lemma 3.8. *The operator $C(a,\lambda)$ is a unitary operator from $\mathcal{H}_\lambda$ to $\mathcal{H}_{a\lambda}$ and intertwines π_λ^a and $\pi_{a\lambda}$. Moreover, the operators $C(a,\lambda)$ satisfy*

$$C(a, b\lambda)\circ C(b,\lambda) = C(ab,\lambda) \tag{3.3}$$

Proof. For $y \in E(a\lambda)$, we have $a^{-1}ya \in E(\lambda)$. For $f \in \mathcal{H}_\lambda$ we have

$$\begin{aligned}\big(C(a,\lambda)f\big)(ny) &= B(a,\lambda)\big(f(a^{-1}naa^{-1}ya)\big)\delta^1(a)^{-1/2}\\ &= B(a,\lambda)\big(\pi_\lambda^\circ(a^{-1}ya)^{-1}f(a^{-1}xh)\big)\delta^1(a)^{-1/2}\\ &= \pi_{a\lambda}^\circ(y)^{-1}B(a,\lambda)f(a^{-1}xa)\delta^1(a)^{-1/2}\\ &= \pi_{a\lambda}^\circ(y)^{-1}\big(C(a,\lambda)f\big)(x).\end{aligned}$$

It follows that $C(a,\lambda)$ maps $\mathcal{H}_\lambda$ into $\mathcal{H}_{a\lambda}$. To see that $C(a,\lambda)$ is unitary,

$$\begin{aligned}\int_{N/E(a\lambda)}\|C(a,\lambda)f(n)\|^2 d\nu_{a\lambda}(\dot{n}) &= \int_{N/E(a\lambda)}\|f(a^{-1}na)\|^2\ \delta^1(a)^{-1}d\nu_{a\lambda}(\dot{n})\\ &= \int_{N/E(\lambda)}\|f(n)\|^2 d\nu_\lambda(\dot{n})\end{aligned}$$

and it is easily seen that $C(a,\lambda)$ intertwines π_λ^a and $\pi_{a\lambda}$. ■

The following is immediate from the preceding.

Corollary 3.2. *Denote by ι the natural injection $\iota : \Lambda \to \hat{N}$ so that $\iota(\lambda) = [\pi_\lambda]$. Then ι is equivariant with respect to the actions of H on Λ and $\hat{N}$. Hence for each $\lambda \in \Lambda$, $H_{[\pi_\lambda]} = H_\lambda = K$.*

4. Decomposition of the quasiregular representation

In this section we show how the explicit orbital parameters and realizations are combined with results in [9] to obtain an explicit decomposition of the quasiregular representation of $G = N \rtimes H$ induced from H. We begin by recalling the group Fourier transform on N in terms of the parameter set Λ and the realizations π_λ. For each $\lambda \in \Lambda$ and $f \in L^1(N) \cap L^2(N)$, set

$$\mathcal{F}(f)(\lambda) = \int_N f(n)\ \pi_\lambda(n)\ dn.$$

Then $\mathcal{F}(f)(\lambda)$ belongs to the space $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda$ of Hilbert-Schmidt operators on $\mathcal{H}_\lambda$. Now let μ be the Plancherel measure on Λ as in Proposition 1.5. Then $\{\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda\}_{\lambda\in\Lambda}$ is a measurable field of Hilbert spaces and we set

$$\mathbb{H} = \int_\Lambda^\oplus \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda\ d\mu(\lambda).$$

Now $\lambda \to \pi_\lambda$ is a Borel function from Λ to $Irr(N)$, $\mathcal{F}(\psi)$ belongs to $\mathbb{H}$, and the map

$$\mathcal{F} : L^1(N) \cap L^2(N) \to \mathbb{H}$$

as defined above extends to all of $L^2(N)$ as a unitary isomorphism. For $f \in L^2(N)$ we use the notation $\hat{f}(\lambda) = \mathcal{F}(f)(\lambda), \lambda \in \Lambda$.

Next we recall the quasiregular representation τ of G in $L^2(N)$. Let G have the Haar measure $d\nu_G(na) = dn\ |\delta(a)|^{-1}da$. We realize τ on $L^2(N)$ as follows. For $f \in L^2(N)$, set

$$(\tau(a)f)(n_0) = f(a^{-1}n_0a)|\delta(a)|^{-1/2},\ a \in H$$
$$(\tau(n)f)(n_0) = f(n^{-1}n_0),\ n \in N.$$

The representation $\hat{\tau} := \mathcal{F} \circ \tau \circ \mathcal{F}^{-1}$ is described in terms of the usual action of H on $\hat{N}$.

For $a \in H$ and $\lambda \in \Lambda^1$, let $D(a,\lambda) : \mathcal{B}(\mathcal{H}_\lambda) \to \mathcal{B}(\mathcal{H}_{a\lambda})$ be defined by

$$D(a,\lambda)(T) = C(a,\lambda) \circ T \circ C(a,\lambda)^{-1}.$$

A simple computation shows the following.

Proposition 4.1. *Let $f \in L^1(N) \cap L^2(N)$, $a \in H$, $n \in N$. Then for each $\lambda \in \Lambda$, one has* *(i)* $(\hat{\tau}(a)\hat{f})(\lambda) = D(a, a^{-1}\lambda)\big(\hat{f}(a^{-1}\lambda)\big)\ |\delta(a)|^{1/2}$, *and*

(ii) $(\hat{\tau}(n)\hat{f})(\lambda) = \pi_\lambda(n) \circ \hat{f}(\lambda)$.

Denote the unitary representation $C(\cdot, \lambda)|_K$ of K by γ_λ. Recall that given a covering set O containing λ, we have a natural isomorphism $\tilde{A}^O_\lambda : \mathcal{H}_\lambda \to L^2(\mathcal{X}) \otimes \mathcal{H}^\circ_\lambda$. It is easy to check that for each $a \in K$,

$$\tilde{A}^O_\lambda \circ \gamma_\lambda(a) \circ (\tilde{A}^O_\lambda)^{-1} = \gamma^{\mathcal{X}}(a) \otimes \gamma^\circ_\lambda(a).$$

We propose to write γ_λ as an outer tensor product of representations γ'_λ of K' and γ''_λ of K''. Recall that we have decomposed $\mathbf{j}'$ into disjoint subsequences $\mathbf{j}^r$ and $\mathbf{j}^c$ where $\mathbf{j}^c$ consists of those indices $j \in \mathbf{j}'$ such that $j - 1 \notin I$ (and hence $j - 1 \in \mathbf{j}$). Write

$$\mathbf{j}^c = \{j_{k''_1}, j_{k''_2}, \cdots, j_{k''_q}\}$$

and let U be the open subset of $\mathbb{R}^p$ defined by

$$U = \{y \in \mathbb{R}^p \mid y_l > 0 \text{ if } x_l \text{ is complex }\}.$$

Use polar coordinates for the complex coordinates of $\mathcal{X}$ by setting $y_l(x) = x_l$ if x_l is real, and $y_l(x) = |x_l|$ if x_l is complex, $1 \le l \le p$, while $z_l(x) = \text{sign}(x_{k''_l}), 1 \le l \le q$. Thus for $x = (x_1, x_2, \ldots, x_p) \in \mathcal{X}$, define $\sigma(x) \in U \times \mathbb{T}^q$ by $\sigma(x) = \big(y(x), z(x)\big)$. We have the resulting obvious isomorphism $S : L^2(\mathcal{X}) \to L^2(U, y''dy) \otimes L^2(\mathbb{T}^q)$ defined by

$$Sf(y, z) = f(\sigma^{-1}(y, z))$$

where $y'' = y_{k''_1} y_{k''_2} \cdots y_{k''_q}$. Writing $a \in K$ as $a = bc, b \in K', c \in K''$, we have

$$\sigma\big(\varphi(bc)x\big) = \big(\varphi'(b)y(x), \varphi''(c)z(x)\big).$$

where $\varphi' : K' \to D(p, \mathbb{R})$ and $\varphi'' : K'' \to D(q, \mathbb{C})$ are defined by $\varphi' = \varphi|_{K'}$ and

$$\varphi''_l(c) = \varphi_{k''_l}(c),\ 1 \le l \le q.$$

Note that by Lemma 3.4, the characters φ''_l are just the characters $\delta_j, j \in \mathbf{j}^c$. Define the representation γ' of K' in $L^2(U, y''dy)$ by

$$\gamma'(b)F(y) = F(\varphi'(b)^{-1}y)\delta^1(b)^{-1/2},\ b \in K'.$$

Similarly we have the representation γ'' of K'' in $L^2(\mathbb{T}^q)$ defined by

$$\gamma''(c)G(z)) = G(\varphi''(c)^{-1}z).$$

and it is clear that

$$S \circ \gamma^{\mathcal{X}} \circ S^{-1} = \gamma' \otimes \gamma''.$$

Moreover, since $K' \subset \ker(\gamma^\circ_\lambda)$, we can regard γ°_λ as a representation of K''. Set $\mathcal{H}' = L^2(U, y''dy)$ and

$$\mathcal{H}''_\lambda = L^2(\mathbb{T}^q) \otimes \mathcal{H}^\circ_\lambda.$$

Let $g : L^2(U, y''dy) \otimes L^2(\mathbb{T}^q) \otimes \mathcal{H}^\circ_\lambda \to \mathcal{H}' \otimes \mathcal{H}''_\lambda$ be the operation of reassociation: $g\big((F \otimes G) \otimes \psi\big) = F \otimes (G \otimes \psi)$. Thus, for a fixed covering set O, we have $B^O_\lambda : \mathcal{H}_\lambda \to \mathcal{H}' \otimes \mathcal{H}''_\lambda$ defined by $B^O_\lambda = g \circ S \otimes I \circ \tilde{A}^O_\lambda$,

$$B^O_\lambda : \mathcal{H}_\lambda \stackrel{\tilde{A}^O_\lambda}{\to} L^2(\mathcal{X}) \otimes \mathcal{H}^\circ_\lambda \stackrel{S\otimes I}{\to} \Big(L^2(U, y''dy) \otimes L^2(\mathbb{T}^q)\Big) \otimes \mathcal{H}^\circ_\lambda \stackrel{g}{\to} \mathcal{H}'_\lambda \otimes \mathcal{H}''_\lambda$$

and it follows that

$$B^O_\lambda \circ \gamma_\lambda \circ (B^O_\lambda)^{-1} = \gamma' \otimes \big(\gamma'' \otimes \gamma^\circ_\lambda\big). \tag{4.1}$$

Let $\eta \in \hat{K}$ and write $\eta = \xi \otimes \zeta$ where $\xi \in \hat{K}'$ and $\zeta \in \hat{K}''$. Let m_λ be the multiplicity function for γ_λ on $\hat{K}$; by (4.1), we have

$$m_\lambda(\eta) = m_\lambda(\xi \otimes \zeta) = m'(\xi) m''_\lambda(\zeta). \tag{4.2}$$

where m' and m''_λ are the multiplicity functions for γ' and $\gamma'' \otimes \gamma^\circ_\lambda$, respectively. We have already seen that the multiplicity function for γ°_λ depends only upon $\epsilon(\lambda)$; since γ' and γ'' are independent of λ, the following is immediate.

Proposition 4.2. *The measure class μ_λ and the positive multiplicity function m_λ on $\hat{K}$ for the irreducible decomposition of γ_λ depend only upon $\epsilon(\lambda)$.*

When $\epsilon = \epsilon(\lambda)$ we shall also write $m_\lambda = m_\epsilon$ and $m''_\lambda = m''_\epsilon$. Let $T_\lambda : \mathcal{H}'_\lambda \otimes \mathcal{H}''_\lambda \to \int^\oplus_{\hat{K}} \mathbb{C}^{m_\lambda(\eta)}\, d\mu_\lambda(\eta)$ be an isomoorphism effecting the irreducible decomposition of γ_λ. Then

$$A^O_\lambda = T \circ B^O_\lambda : \mathcal{H}_\lambda \to \int^\oplus_{\hat{K}} \mathbb{C}^{m_\lambda(\eta)}\, d\mu_\lambda(\eta) \tag{4.3}$$

is a unitary isomorphism such that for $b \in K', c \in K''$,

$$A^O_\lambda \circ \gamma_\lambda(bc) \circ \big(A^O_\lambda\big)^{-1} = \int_{\hat{K}' \times \hat{K}''} m'(\xi) m''_\epsilon(\zeta) \cdot \xi(b) \otimes \zeta(c)\, d\mu(\xi \otimes \zeta)$$

We now digress to recall two facts. First, suppose that $\mathcal{H}$ is a Hilbert space and that $\{\mathcal{K}_s\}_{s\in S}$ is a measurable field of Hilbert spaces over a measure space (S, ν). Then there is a unique ν-measurable field structure on $\{\mathcal{H} \otimes \mathcal{K}_s\}_{s\in S}$ for which $\{v_s\}_{s\in S}$ measurable in $\{\mathcal{K}_s\}_{s\in S}$ implies $\{u \otimes v_s\}_{s\in S}$ is measurable in $\{\mathcal{H} \otimes \mathcal{K}_s\}_{s\in S}$. Setting $\mathcal{K} = \int^\oplus_S \mathcal{K}_s d\nu(s)$, one has a canonical isomorphism

$$\mathcal{H} \otimes \mathcal{K} \simeq \int^\oplus_S \mathcal{H} \otimes \mathcal{K}_s\, d\nu(s) \tag{4.4}$$

that takes the elementary tensor $u \otimes \{v_s\}_{s\in S}$ to the vector field $\{u \otimes v_s\}_{s\in S}$. In a similar way, tensor products distribute over direct sums on the right as well.

Second, let H be any separable, locally compact group and K a closed subgroup of H. Let $d\nu(\dot{a})$ be a Borel measure on H/K, $\mathcal{V}$ a Hilbert space, and γ a unitary representation of K acting in $\mathcal{V}$. Let $L^2(H, K, \mathcal{V}, \gamma, d\nu)$ be the Hilbert space of Borel functions $f : H \to \mathcal{V}$ which satisfy

$$f(ab) = \gamma_\lambda(b)^{-1} f(a), a \in H, b \in K,$$

and

$$\int_{H/K} \|f(a)\|^2 \, d\nu(\dot{a}) < \infty$$

Let $\mathcal{W}$ be a Hilbert space; then γ also acts in $\mathcal{V} \otimes \mathcal{W}$ in the obvious way. We have the following.

Lemma 4.1. *There is a canonical isomorphism*

$$L^2(H, K, \mathcal{V}, \gamma, d\nu) \otimes \mathcal{W} \simeq L^2(H, K, \mathcal{V} \otimes \mathcal{W}, \gamma, d\nu).$$

Proof. Elementary tensors in $L^2(H, K, \mathcal{V}, \gamma, d\nu) \otimes \mathcal{W}$ map naturally and isometrically into $L^2(H, K, \mathcal{V} \otimes \mathcal{W}, \gamma, d\nu)$: for each $u \in L^2(H, K, \mathcal{V}, \gamma, d\nu)$ and $v \in \mathcal{V}$, define $f\big(u \otimes v\big)(a) = u(a) \otimes v, a \in H$. The mapping f extends to an isometry on $L^2(H, K, \mathcal{V}, \gamma, d\nu) \otimes \mathcal{W}$. Now choose an orthonormal basis $\{e_j\}$ for $\mathcal{W}$ and for $U \in L^2(H, K, \mathcal{V} \otimes \mathcal{W}, \gamma, d\nu)$, define $U_j \in L^2(H, K, \mathcal{V}, \gamma, d\nu)$ by $U_j(a) = U(a)(e_j), a \in H$. Then $\|U(a)\|_{HS}^2 = \sum \|U_j(a)\|^2$ and it is easy to check that

$$U = f\left(\sum_j U_j \otimes e_j\right).$$

∎

As is well-known, π_λ extends to a representation $\tilde{\pi}_\lambda$ of NK defined by the prescription

$$\tilde{\pi}_\lambda(na) = \pi_\lambda(n)\gamma_\lambda(a), n \in N, a \in K,$$

and for each character $\eta \in \hat{K}$, the representation $\operatorname{ind}_{NK}^G\big(\tilde{\pi}_\lambda \otimes \eta\big)$ is irreducible and isomorphic with the representation ρ_λ^η defined as follows. We realize ρ_λ^η in the Hilbert space $\mathcal{H}_{\rho_\lambda^\eta} = L^2(H, K, \mathcal{H}_\lambda, \gamma_\lambda \otimes \eta, |\delta(a)|^{-1} d\dot{a})$. For $f \in \mathcal{H}_{\rho_\lambda^\eta}$ and $a \in H$,

$$\rho_\lambda^\eta(b) f = f(b^{-1}a)|\delta(b)|^{1/2}, b \in H,$$

and

$$\rho_\lambda^\eta(n) f(a) = \pi_\lambda^a(n) f(a), \; n \in N.$$

The following is an concrete form of [9, Theorem 7.1], specialized to the present context. (See also [11].)

Theorem 4.3. *Let $G = N \rtimes H$ be an algebraic solvable group with N connected, simply connected nilpotent and H is a connected, abelian Levi factor in G. Let Λ be parameters for coadjoint orbits in $\mathfrak{n}^*$ as constructed above with $\Sigma_0 \subset \Lambda$ a fundamental domain for $\Sigma/F \simeq \Lambda/H$. Let $\tilde{\mu}$ be the explicit measure on Σ_0 defined above, and let $\{\pi_\lambda\}_{\lambda \in \Sigma_0}$ be the explicit field of irreducible representations of N constructed above. Write $\Sigma_0 = \cup_\epsilon \Sigma_0^\epsilon$ where $\Sigma_0^\epsilon = \{\lambda \in \Sigma_0 \mid \epsilon(\lambda) = \epsilon\}$. For* *each sign index ϵ for which $\Sigma_0^\epsilon \neq \emptyset$, let m_ϵ be the positive multiplicity function (as in Proposition 4.2) and μ_ϵ a measure on $\hat{K}$ such that for each $\lambda \in \Sigma_0^\epsilon$,*

$$\gamma_\lambda \simeq \int_{\hat{K}} m_\epsilon(\eta) \cdot \eta \, d\mu_\epsilon(\eta).$$

Then we have the decomposition

$$\tau \simeq \bigoplus_\epsilon \int_{\Sigma_0^\epsilon}^{\otimes} \int_{\hat{K}}^{\otimes} m_\epsilon(\eta) \cdot \rho_\lambda^{\overline{\eta}} \; d\mu_\epsilon(\eta) \; d\tilde{\mu}(\lambda)$$

implemented by an explicit isomorphism Φ.

Proof. For each $\lambda \in \Sigma_0$ with $\mathcal{O}_\lambda$ the H-orbit of λ, put

$$\mathbb{H}_\lambda = \int_{\mathcal{O}_\lambda}^{\oplus} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda \; d\omega_\lambda(\lambda).$$

By Proposition 2.3 we have an obvious and explicit isomorphism

$$\mathbb{H} \simeq \int_{\Sigma_0}^{\oplus} \mathbb{H}_\lambda \; d\tilde{\mu}(\lambda).$$

The formula for $\hat{\tau}$ obtains a unitary representation $\hat{\tau}_\lambda$ on $\mathbb{H}_\lambda$ and thus we have the decomposition:

$$\tau \simeq \int_{\Sigma_0}^{\oplus} \hat{\tau}_\lambda \; d\tilde{\mu}(\lambda).$$

Put

$$\mathcal{K}_\epsilon = \int_{\hat{K}}^{\oplus} \mathbb{C}^{m_\epsilon(\eta)} \; d\mu_\epsilon(\eta)$$

and

$$\mathcal{L}_\lambda^\epsilon = \mathcal{H}_\lambda \otimes \overline{\mathcal{K}}_\epsilon$$

Fix a covering set O and for $\lambda \in \Sigma_0^\epsilon \cap O$, let

$$A_\lambda = A_\lambda^O : \mathcal{H}_\lambda \to \mathcal{K}_\epsilon$$

be the intertwining operator for γ_λ defined above. To construct Φ we must construct, for each $\lambda \in \Sigma_0^\epsilon \cap O$, an isomorphism

$$\Phi_\lambda : \mathbb{H}_\lambda \to \int_{\hat{K}}^{\oplus} \mathcal{H}_{\rho_\lambda^{\overline{\eta}}} \otimes \mathbb{C}^{m_\epsilon(\eta)} \; d\mu_\epsilon(\eta)$$

that intertwines $\hat{\tau}_\lambda$ and $\int_{\hat{K}}^{\oplus} \rho_\lambda^{\overline{\eta}} \otimes 1_{n_\epsilon(\eta)} \; d\mu_\epsilon(\eta)$.

Fix $\lambda \in \Sigma_0^\epsilon \cap O$ and let $T = \{T_{\lambda'}\}$ be a measurable field belonging to $\mathbb{H}_\lambda$. For each $a \in H$ define

$$f^T(a) = C(a,\lambda)^{-1} T_{a\cdot\lambda} C(a,\lambda) A_\lambda^{-1}.$$

Note that $f^T(a) \in \mathcal{L}^\epsilon_\lambda$, which we identify with

$$\int_{\hat{K}} \mathcal{H}_\lambda \otimes \overline{\mathbb{C}}^{m_\epsilon(\eta)}\ d\mu_\epsilon(\eta)$$

via (4.4). Thus for $a \in H$ we write $f^T(a) = \{f^T(a)_\eta\}_{\eta\in\hat{K}}$. Now put

$$\tilde{\gamma} = \int_{\hat{K}}^{\oplus} \gamma_\lambda \otimes \overline{\eta}\ d\mu_\epsilon(\eta)$$

acting in $\mathcal{L}^\epsilon_\lambda$. We claim that $f^T : H \to \mathcal{L}^\epsilon_\lambda$ belongs to

$$\mathcal{M}_\lambda{:=}L^2(H, K, \mathcal{L}^\epsilon_\lambda, \tilde{\gamma}, |\delta(a)|^{-1}d\dot{a})$$

and that $\|f^T\| = \|T\|$. It is clear that f^T is Borel. To check the appropriate covariance property we use (3.3); for $b \in K$,

$$f^T(ab) = \gamma_\lambda(b)^{-1} f^T(a) A_\lambda \gamma_\lambda(b) A_\lambda^{-1}$$

and hence

$$f^T_\eta(ab) = \gamma_\lambda(b)^{-1} f^T_\eta(a)\eta(b) = (\gamma_\lambda(b) \otimes \overline{\eta}(b))^{-1}\big(f^T_\eta(a)\big).$$

To check $\|f^T\|$, choose an orthonormal basis $\{z^{(j)}\}$ for $\mathcal{K}^\epsilon$, set $v^{(j)} = A_\lambda^{-1}z^{(j)}$, and calculate that

$$\begin{aligned}
\int_H \|f^T(a)\|^2|\delta(a)|^{-1}d\dot{a} &= \int_H \sum_j \|f^T(a)(z^{(j)}\|^2_{HS}\ |\delta(a)|^{-1}d\dot{a}\\
&= \int_H \sum_j \|C(a,\lambda)^{-1}T_{a\cdot\lambda}C(a,\lambda)v^{(j)}\|^2\ |\delta(a)|^{-1}d\dot{a}\\
&= \int_H \sum_j \|T_{a\cdot\lambda}C(a,\lambda)v^{(j)}\|^2\ |\delta(a)|^{-1}d\dot{a}\\
&= \int_H \|T_{a\cdot\lambda}\|^2_{HS}\ |\delta(a)|^{-1}d\dot{a} = \|T\|^2
\end{aligned}$$

and the claim is verified. Now by (4.4) and Lemma 4.1, we have the canonical isomorphism

$$\mathcal{M}_\lambda \simeq \mathcal{H}_{\rho^{\overline{\eta}}_\lambda} \otimes \overline{\mathcal{K}}^\epsilon \simeq \int_{\hat{K}}^{\oplus} \mathcal{H}_{\rho^{\overline{\eta}}_\lambda} \otimes \overline{\mathbb{C}}^{n_\epsilon(\eta)}\ d\mu_\epsilon(\eta).$$

It remains to verify that the map $\Phi_\lambda : T \mapsto f^T$ has the appropriate intertwining property. Let $b \in H$, then for any $a \in H$ we have (again using (3.3))

$$\begin{aligned}
f^{\hat{\tau}_\lambda(b)T}(a) &= C(a,\lambda)^{-1}(\hat{\tau}_\lambda(b)T)_{a\cdot\lambda}C(a,\lambda)A_\lambda^{-1}|\delta(b)|^{1/2}\\
&= C(a,\lambda)^{-1}C(b,b^{-1}a\lambda)T_{b^{-1}a\cdot\lambda}C(b,b^{-1}a\lambda)^{-1}C(a,\lambda)A_\lambda^{-1}|\delta(b)|^{1/2}\\
&= C(b^{-1}a,\lambda)^{1}T_{b^{-1}a\cdot\lambda}C(b^{-1}a,\lambda)A_\lambda^{-1}|\delta(b)|^{1/2}\\
&= f^T(b^{-1}a)|\delta(b)|^{1/2}.
\end{aligned}$$

For $n \in N$, we have for any $a \in H$,

$$\begin{aligned} f^{\hat{\tau}_\lambda(n)T}(a) &= C(a,\lambda)^{-1}(\hat{\tau}_\lambda(n)T)_{a\cdot\lambda}C(a,\lambda)A_\lambda^{-1} \\ &= C(a,\lambda)^{-1}\pi_{a\cdot\lambda}(n)T_{a\cdot\lambda}C(a,\lambda)A_\lambda^{1} \\ &= \pi_\lambda^a(n)C(a,\lambda)^{-1}T_{a\cdot\lambda}C(a,\lambda)A_\lambda^{-1} \\ &= \pi_\lambda^a(n)f^T(a). \end{aligned}$$

■

5. Multiplicities

In this section we study the multiplicity function m_ϵ for the decomposition of τ, given in Theorem 4.3. For each sign index ϵ we have the positive multiplicity function m_ϵ and a measure μ_ϵ on $\hat{K}$ that give a decomposition of $\gamma_\lambda, \lambda \in \Sigma_0^\epsilon$. Recall that by (4.1) we have $\gamma_\lambda \simeq \gamma' \otimes \big(\gamma'' \otimes \gamma_\lambda^\circ\big)$ as an outer tensor product, and by (4.2) and Theorem 4.3 we have $m_\epsilon(\rho_\lambda^{\overline{\eta}}) = m_\epsilon(\eta) = m'(\xi)m''_\epsilon(\zeta)$ where $\eta = \xi \otimes \zeta$ with $\xi \in \hat{K}'$ and $\zeta \in \hat{K}''$. Since $\hat{K}''$ is countable discrete, then we may choose the measure μ_ϵ so that for some Borel subset Z^ϵ of $\hat{K}''$ and measure μ' on $\hat{K}'$, μ_ϵ is supported on $\hat{K}' \times Z^\epsilon$ and given on each piece $\hat{K}' \times \{\zeta\}, \zeta \in Z^\epsilon$ by μ'. With this in mind we study the multiplicity functions m' and m''_ϵ separately.

Recall that the representation γ' of K' is given by

$$\big(\gamma'(a)f\big)(y) = f(\varphi'(a)^{-1}y)|\delta^1(a)|, \; a \in K', f \in \mathcal{H}'_\lambda.$$

On the other hand, the representation $\gamma'' \otimes \gamma_\lambda^\circ$ of the compact subgroup K'' acts in $\mathcal{H}''_\lambda = L^2(\mathbb{T}^q) \otimes \mathcal{A}^\epsilon(\mathbb{C}^m)$: for $h \in L^2(\mathbb{T}^q)$ and $p \in \mathcal{A}^\epsilon(\mathbb{C}^m)$,

$$\big(\gamma'' \otimes \gamma_\lambda^\circ\big)(b)\big(h(z) \otimes p(w)\big) = h(\varphi''(b)^{-1}z) \otimes p(\delta^\circ(b)^{-1}w), \; b \in K''.$$

We simplify notation here and just denote elements of $\mathcal{H}'' \otimes \mathcal{H}_\lambda^\circ$ as $F(z,w)$ and write $\varphi''_{q+l} = \delta_l^\circ$, $1 \le l \le m$. Thus we have the homomorphism $\varphi'' : K'' \to D(p,\mathbb{C})$ such that

$$\big(\gamma'' \otimes \gamma_\lambda^\circ\big)(b)\big(F(z,w)\big) = F(\varphi''(b)^{-1}(z,w).$$

The components of φ' are given by characters δ_j where $j \in \mathbf{j}' = \{j_{k_1}, j_{k_2}, \ldots, j_{k_p}\}$, the subsequence of $\mathbf{j}$ defined in Section 3. Recall that $\mathbf{j}'$ is decomposed into the disjoint subsequences $\mathbf{j}^r$ and $\mathbf{j}^c$ where $\mathbf{j}^c$ consists of those indices $j \in \mathbf{j}'$ such that $j-1 \notin I$, and that q is the number of indices belonging to $\mathbf{j}^c$. We also have $\mathbf{j}''$, the subsequence of $\mathbf{j}$ consisting of those indices $j = j_k$ where $k \in K_3$; recall that we have written $\mathbf{j}'' = \{j_{h_1}, j_{h_2}, \ldots, j_{h_m}\}$. With this notation and referring to Lemma 3.4, we have that φ' is isomorphic with the linear action of K' on $\mathfrak{n}/\mathfrak{e}(\lambda)$, $(\varphi''_1, \ldots, \varphi''_q)$ is isomorphic with the linear action of K'' on $\mathfrak{n}/\mathfrak{e}(\lambda)$, and $(\varphi''_{q+1}, \ldots, \varphi''_{q+m})$ is isomorphic with the linear action of K'' on $\mathfrak{e}(\lambda)/\mathfrak{d}(\lambda)$.

If $\dim\big(\varphi'(K')\big) = p$, then we shall say that "K' acts with full rank". We have the following.

Lemma 5.1. *If K' acts with full rank, then $m' = 2^{p-q}$ holds μ'-a.e.. Otherwise, $m' = \infty$ holds μ'-a.e.*

Proof. We proceed by induction on p: if $p = 1$, and $\dim(\varphi'(K')) = 0$, then $\gamma' = 1$ and the result is trivial (note that here ν' is point mass measure at 1). Suppose that $\dim(\varphi'(K')) = 1 = p$. Choose $A \in \mathfrak{k}'$ such that $\varphi'(A) = 1$, and let $K_1' = \ker(\varphi')$. Write $p : K' \to K'/\ker(\varphi') \simeq \mathbb{R}$ for the canonical map and put $\gamma' = \tilde{\gamma}' \circ p$. We consider two cases: Case 1: $p = 1$ and $q = 0$, and Case 2: $p = 1$ and $q = 1$.
Case 1. For each $t \in \mathbb{R}$, we have

$$\big(\tilde{\gamma}'(\exp(tA))f\big)(y) = f(e^{-t}y)e^{-t/2},\ y \in \mathbb{R}.$$

which is isomorphic to two copies of the regular representation of $\mathbb{R}$, and hence in this case $m'(\eta') = 2$ a.e..

Case 2. For each $t \in \mathbb{R}$, we have

$$\big(\tilde{\gamma}'(\exp(tA)t)f\big)(s) = f(e^{-t}s)e^{-t},\ s \in \mathbb{S}.$$

(recall that we are using the measure sds on $\mathbb{S}$ here.) It is clear that $\tilde{\gamma}'$ is equivalent to the regular representation of $\mathbb{R}$ and so $m'(\eta') = 1$ a.e..

Suppose then that $p > 1$. We first assume that $p > q$. Choose an index l such that y_l runs through $\mathbb{R}$, and let

$$V = \{v \in \mathbb{R}^{p-1} \mid v = (y_1, y_2, \ldots, y_{l-1}, y_{l+1}, \ldots, y_p), y \in U\}$$

so that $U \simeq \mathbb{R} \times V$ and $\mathcal{H}' \simeq L^2(\mathbb{R}) \otimes L^2(V, v''dv)$. Let $J = \ker \varphi_l'$, and let $\mu : J \to D(p-1, \mathbb{R})$ be defined by

$$\mu = (\varphi_1'|_J, \varphi_2'|_J, \ldots, \varphi_{l-1}'|_J, \varphi_{l+1}'|_J, \ldots, \varphi_p'|_J).$$

For $a \in J$ and $g \in L^2(V, v''dv)$, define

$$\gamma_\mu'(a)g(v) = g(\mu(a)^{-1}v)\,\det(\mu(a))^{-1/2},\ a \in J.$$

By induction the result holds for γ_μ'. If $J = K'$, then $\dim(\varphi'(K')) < p$, $\gamma' = 1 \otimes \gamma_\mu'$ and $m' = \infty m_\mu' = \infty$. If $J \neq K'$, then choose $A \in \mathfrak{k}$ such that $\varphi_l'(A) = 1$ and $\mu(A) = 0$. For $h \in L^2(\mathbb{R})$ put

$$\gamma_1'(\exp(tA)h(u) = h(e^{-t}u)e^{-t/2},\ t \in \mathbb{R};$$

so that $\gamma' = \gamma_1' \otimes \gamma_\mu'$ and $m' = m_1'm_\mu'$. Now if $\dim(\varphi'(K')) < p$ in this case, then $\dim(\mu(J)) < p - 1$, and so by induction $m_\mu' = \infty$ and hence $m' = \infty$ a.e.. If

$\dim\big(\varphi'(K')\big) = p$, then $\dim(\mu(J)) = p-1$ and so by induction $m'_\mu = 2^{p-q-1}$ a.e.; but $m'_1 = 2$ a.e., so we are done.

Finally, if $p = q$, then repeat the above argument except that in this case γ'_1 acts in $L^2(\mathbb{S}, sds)$, and

$$\gamma'_1(\exp(tA)h(s) = h(e^{-t}s)e^{-t},\ t \in \mathbb{R}.$$

has multiplicity 1. ∎

We turn next to the representation $\gamma'' \otimes \gamma^\circ_\lambda$ of the compact subgroup K''.

Lemma 5.2. *The unitary homomorphism φ'' is injective.*

Proof. Let $b \in K''$ such that $\varphi''_l(b) = 1$ for $1 \le l \le q+m$. Since we have assumed that δ is injective, then it is enough to show that $\delta_j(b) = 1$ holds for all $1 \le j \le n$. Now by definition of K, we have $\delta_j(b) = 1$ for all $\mathbf{j} \notin \mathbf{e}$. If j is a value in $\mathbf{j}''$, then by definition of φ'' and $\mathbf{j}''$ we have $\delta_j(b) = 1$. If $j \in \mathbf{j}$ but j is not a value in $\mathbf{j}''$, then either $j \in I$ or $j \notin I$ and $j+1 \notin \mathbf{e}$. But now parts (c) and (d) of Lemma 1.7 imply that $\delta_j(b) = 1$ in these cases also. Hence by part (a) of Lemma 1.7, we have $\delta_j(b) = 1$ for all $j \in \mathbf{e}$. ∎

Write $K'' = (F \cap K) \cdot K''_\circ$, and write $F \cap K = G_1 G_2 \cdots G_r$ as a direct product where G_j is finite cyclic of order m_l. For $b \in K \cap F$ write $b = b_1 b_2 \cdots b_r$ where $b_j \in G_j$. Choose a basis $\{C_1, \ldots, C_s\}$ for $\mathfrak{k}''$ consisting of integral elements and such that for each k, $\ker\big(\exp|_{\mathbb{R}C_k}\big) = 2\pi\mathbb{Z}$. Put $K''_k = \exp(\mathbb{R}C_k)$ so that $K''_\circ = K''_1 K''_2 \cdots K''_s$. Accordingly we write an element $c \in K''$ as $c = c_1 c_2 \cdots c_s$.

Let $\phi_{n_1,n_2,\ldots,n_q}, n \in \mathbb{Z}^q$ be the canonical complete orthogonal system for $L^2(\mathbb{T}^q)$. Using the monomials described in the proof of Lemma 3.7, we have the natural complete orthogonal system for $\mathcal{H}''_\lambda$:

$$\Psi_n = \phi_{n_1,n_2,\ldots,n_q} \otimes \psi_{n_{q+1},n_{q+2},\ldots,j_{q+m}},$$

where

$$\psi_{n_{q+1},n_{q+2},\ldots,j_{q+m}} = (w_1^{\epsilon_1})^{n_{q+1}}(w_2^{\epsilon_2})^{n_{q+2}} \cdots (w_m^{\epsilon_m})^{n_{q+m}}.$$

Here $n = (n_1, n_2, \ldots, n_{q+m})$ belongs to the set

$$J = \{(n_1, n_2, \ldots, n_{q+m}) \in \mathbb{Z}^{q+m} \mid n_{q+l} \ge 0, 1 \le l \le m\},$$

and

$$m''_\epsilon(\zeta) = \big|\{n \in J \mid (\gamma'' \otimes \gamma^\circ_\lambda)(b)\Psi_n = \zeta(b)\Psi_n, b \in K''\}\big|.$$

Now take $\zeta = \zeta_{g,h} \in \hat{K}''$ where $g_i \in \mathbb{Z}/m_i\mathbb{Z}, 1 \le i \le r$ and $h \in \mathbb{Z}^s$, so that

$$\zeta_{g,h}(b_1 b_2 \cdots b_r) = b_1^{g_1} b_2^{g_2} \cdots b_r^{g_r},\ b \in K \cap F,$$

and

$$\zeta_{g,h}(c_1c_2\cdots c_s)=c_1^{h_1}c_2^{h_2}\cdots c_s^{h_s},\ c\in K_0''.$$

Since the elements $C_k\in\mathfrak{k}''$ are integral we have integers $p_{k,l}, 1\le k\le s, 1\le l\le q+m$, such that

$$\varphi_l''(c_k)^{-1}=c_k^{p_{k,l}}.$$

Indeed, the integers $p_{k,l}$ are also defined by

$$p_{k,l}=-\Im\big(\mathbf{d}\varphi_l''(C_k)\big)=i\mathbf{d}\varphi_l''(C_k)$$

(here $\mathbf{d}$ denotes the differential.) We shall say that P is the action matrix for $K_\circ''$. Write $n^\epsilon=[n_1,n_2,\ldots,n_q,\epsilon_1 n_{q+1}\ldots,\epsilon_m n_{q+m}]$ and observe that

$$(\gamma''\otimes\gamma_\lambda^\circ)(c_k)\Psi_n=c_k^{p_k\cdot n^\epsilon}\Psi_n$$

where

$$p_k\cdot n^\epsilon=p_{k,1}n_1+p_{k,2}n_2+\cdots+p_{k,q}n_q+p_{kq+1}\epsilon_1 n_{q+1}+p_{k,q+2}\epsilon_2 n_{q+2}+\cdots+p_{k,q+m}\epsilon_m n_{q+m}.$$

Similarly, we have integers $q_{i,l},\ 1\le i\le r, 1\le l\le q+m$, such that

$$\varphi_l''(b_i)^{-1}=b_i^{q_{i,l}},$$

and we have

$$(\gamma''\otimes\gamma_\lambda^\circ)(b_i)\Psi_n=b_i^{q_i\cdot n^\epsilon}\Psi_n.$$

Put $P=[p_{k,l}]$, $Q=[q_{i,l}]$, and $J^\epsilon=\{n^\epsilon\mid n\in J\}$. Writing n as a column vector, we see that the mulitplicity of ζ is equal to the number of common solutions for the diophantine systems $Qn=g$ and $Pn=h$ that belong to J^ϵ. Now denote the solution set (in $\mathbb{R}^{q+m}$) for $Px=h$ by $\mathcal{S}(P,h)$, and the (integer point) solution set for the system $Qn=g$ by $\mathcal{Z}(Q,g)$. We have

$$m_\epsilon''(\zeta)=\Big|\mathcal{Z}(Q,g)\cap\mathcal{S}(P,h)\cap J^\epsilon\Big|. \tag{5.1}$$

We shall see that the more important role is played by the set $\mathcal{S}(P,h)$.

Lemma 5.3. *There are matrices $L\in SL_s(\mathbb{Z})$ and $R\in SL_{q+m}(\mathbb{Z})$ such that*

$$LPR=\begin{bmatrix}1&0&\cdots&0&0&0&\cdots&0\\0&1&\cdots&0&0&0\cdots&0\\ \vdots&&\vdots&\vdots&&&&\\0&0&\cdots&1&0&0&\cdots&0\end{bmatrix}$$

Proof. It is well known that there are matrices L and R as above such that

$$LPR = \begin{bmatrix} r_1 & 0 \cdots & 0 & 0 & 0 \cdots & 0 \\ 0 & r_2 \cdots & 0 & 0 & 0 \cdots & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 \cdots & r_s & 0 & 0 \cdots & 0 \end{bmatrix}$$

where $r_1, r_2, \ldots, r_s$ integers, and for some s', $0 < r_a | r_{a+1}, 1 \leq a < s'$, and $r_a = 0, s' < a \leq s$. Now suppose that the result is false. Then we have $t = (t_1, t_2, \ldots, t_s)$ where $tLPR \in \mathbb{Z}^{q+m}$ but not all t_j are integers. Set $u = tL$. Then not all coordinates u_k of u are integers (since $L \in SL_s(\mathbb{Z})$) but $uP = (tLPR)R^{-1}$ belongs to $\mathbb{Z}^{q+m}$ and so

$$u_1 p_{1,l} + u_2 p_{2,l} + \cdots + u_s p_{s,l} \in \mathbb{Z}$$

holds for $1 \leq l \leq q+m$. Let $c_k = \exp(2\pi u_k C_k) \in K''_k, 1 \leq k \leq s$. Then $c = c_1 c_2 \cdots c_s \neq 1$, but

$$\varphi''_l(c) = e^{-2\pi i (u_1 p_{1,l} + u_2 p_{2,l} + \cdots + u_s p_{s,l})} = 1, 1 \leq l \leq q+m.$$

This contradicts Lemma 5.2. ■

Let $\mathcal{N}$ be the nullspace for P; then $\mathcal{N} = R(\mathcal{T})$ where $\mathcal{T} \subset \mathbb{R}^{q+m}$ is the nullspace for LPR. Of course

$$\mathcal{T} = \{x \in \mathbb{R}^{q+m} \mid x_j = 0, 1 \leq j \leq s\}.$$

For any subset $\mathcal{S}$ of $\mathbb{R}^{q+m}$ put $\mathcal{S}_{\mathbb{Z}} = \mathcal{S} \cap \mathbb{Z}^{q+m}$.

Lemma 5.4. *One has* $\mathcal{N}_{\mathbb{Z}} = R(\mathcal{T}_{\mathbb{Z}})$.

Proof. This follows immediately from the fact that both R and R^{-1} have integer entries and $\mathcal{T} = R^{-1}(\mathcal{N})$. ■

We say that "K'' acts with full rank " (on $\mathfrak{n}/\mathfrak{d}(\lambda)$) if $\dim\big(\varphi''(K'')\big) = q+m$. We are now ready to dispense with this case. Define $\iota : \mathbb{R}^s \to \mathbb{R}^{q+m}$ by $\iota(x_1, \ldots, x_s) = (x_1, \ldots, x_s, 0, 0, \ldots, 0)$. Then ι is a right inverse for LPR, and it follows that

$$z^\circ = z^\circ(h) = R\big(\iota(Lh)\big)$$

belongs to $S(P, h)_{\mathbb{Z}}$. The following is proved in a different form in [11, Theorem 3.2].

Proposition 5.1. *One has* $\dim\big(\varphi''(K'')\big) = q+m$ *if and only if* $s = q+m$. *In this case,* $m''_\epsilon = 1$ *and the support* Z^ϵ *of* μ''_ϵ *is* $Z^\epsilon = \{\zeta_{g,h} \in \hat{K}'' \mid z^\circ(h) \in \mathcal{Z}(Q, g) \cap J^\epsilon\}$.

Proof. We have $\dim\big(\varphi''(K'')\big) = \dim\big(\varphi''(K''_\circ)\big) = \operatorname{rank}(P)$. By Lemma 5.3, $s = q + m$ if and only if $\operatorname{rank}(P) = q + m$. In this case P is invertable and hence $\mathcal{S}(P, h) = \{z^\circ(h)\}$ so that the result follows from equation (5.1). ■

Now define the cone E^ϵ in $\mathbb{R}^{q+m}$ by

$$E^\epsilon = \{[x_1, x_2, \ldots, x_{q+m}]^t \mid \epsilon_l x_{q+l} \geq 0 \text{ holds for all } 1 \leq l \leq m\}.$$

It is clear that for any subset $\mathcal{S}$ of $\mathbb{R}^{q+m}$, we have $\mathcal{S} \cap J^\epsilon = \mathcal{S}_\mathbb{Z} \cap E^\epsilon$. Hence if $\mathcal{S}(P, h) \cap E^\epsilon$ is bounded, then

$$m''(\zeta) = \Big|\mathcal{Z}(Q, g) \cap \mathcal{S}(P, h) \cap J^\epsilon\Big| \leq \Big|\mathcal{S}(P, h) \cap J^\epsilon\Big| = \Big|\mathcal{S}(P, h)_\mathbb{Z} \cap E^\epsilon\Big| < \infty.$$

We claim that the boundedness of $\mathcal{S}(P, h) \cap E^\epsilon$ is necessary for finite multiplicity as well.

Lemma 5.5. *Suppose that $\mathcal{S}(P, h) \cap E^\epsilon$ is unbounded. Then $\mathcal{S}(P, h) \cap J^\epsilon$ is infinite.*

Proof. Set $\|y\| = \sup_{1 \leq j \leq q+m} |y_j|, y \in \mathbb{R}^{q+m}$, and

$$\|R\| = \sup_{\|y\|=1} \|Ry\|.$$

Choose any $M \geq \|R\|$. Since $\mathcal{S}(P, h) \cap E^\epsilon$ is unbounded, the coordinates $\epsilon_j z_j$ are arbitrarily large as z runs through $\mathcal{S}(P, h) \cap E^\epsilon$, so we have $z \in \mathcal{S}(P, h) \cap E^\epsilon$ such that $\|z - z^\circ\| > M$ and $\epsilon_l(z_{q+l} - z^\circ_{q+l}) > M$ for $1 \leq l \leq m$. Then $x := z - z^\circ$ belongs to $\mathcal{N} \cap E^\epsilon$; put $y = R^{-1}x \in \mathcal{T}$. Then the cube C with edge length 1 centered at y must contain points of $\mathcal{T}_\mathbb{Z}$, and so by Lemma 5.4, the neighborhood $R(C)$ of x is contained in E^ϵ and must contain elements $u \in \mathcal{N}_\mathbb{Z}$. These elements satisfy $\|u\| \geq M - \|R\|$.

Since M was arbitrary we see that $\mathcal{N}_\mathbb{Z} \cap E^\epsilon$ is unbounded and hence infinite. Hence there are infinitely many $x \in \mathcal{N}_\mathbb{Z} \cap E^\epsilon$ such that $\epsilon_j(x_j + z^\circ_j) > 0$ holds for all j and for such x, $z^\circ + x \in \mathcal{S}(P, h) \cap E^\epsilon$. ■

The following shows that the question of finite multiplicity is not affected by the set $\mathcal{Z}(Q, g)$.

Lemma 5.6. *Let $g \in \mathbb{Z}^e$ and $h \in \mathbb{Z}^d$ such that $\mathcal{Z}(Q, g) \cap \mathcal{S}(P, h) \neq \emptyset$. If $\mathcal{S}(P, h) \cap J^\epsilon$ is infinite, then $\mathcal{Z}(Q, g) \cap \mathcal{S}(P, h) \cap J^\epsilon$ is infinite.*

Proof. Observe that if $\mathcal{Z}(Q, g) \neq \emptyset$, say $n = [n_1, \ldots, n_{q+m}]^t \in \mathcal{Z}(Q, g)$, then for any point $n^\circ \in \mathbb{R}^{q+m}$

$$\{n + m_1 m_2 \cdots m_s k n^\circ \mid k \in \mathbb{Z}\} \subset \mathcal{Z}(Q, g).$$

Let $n \in \mathcal{Z}(Q,g) \cap \mathcal{S}(P,h)$ and suppose that $\mathcal{S}(P,h) \cap J^\epsilon$ is infinite. By the proof of Lemma 5.5 we have $n^\circ \in \mathcal{N} \cap J^\epsilon = \mathcal{N}_{\mathbb{Z}} \cap E^\epsilon$, and there is $k_0 \in \mathbb{Z}$ such that $\{n + m_1 m_2 \cdots m_s k n^\circ \mid k \geq k_0\} \subset J^\epsilon$. Hence

$$\{n + m_1 m_2 \cdots m_s k n^\circ \mid k \geq k_0\} \subset \mathcal{S}(P,h) \cap \mathcal{Z}(Q,g) \cap J^\epsilon.$$

∎

We combine the preceding lemmas to obtain the following.

Proposition 5.2. *Let ϵ be a sign index and let $\zeta = \zeta_{g,h} \in \hat{K}''$. Then $m''_\epsilon(\zeta) < \infty$ if and only if $\mathcal{S}(P,h) \cap E^\epsilon$ is bounded.*

Proof. Suppose that $m''_\epsilon(\zeta) < \infty$, so that $\mathcal{S}(P,h) \cap \mathcal{Z}(Q,\overline{k}) \cap J^\epsilon$ is finite. By Lemma 5.6, we have $\mathcal{S}(P,h) \cap J^\epsilon$ is finite, and hence by Lemma 5.5, $\mathcal{S}(P,h) \cap E^\epsilon$ is bounded. On the other hand, suppose that $\mathcal{S}(P,h) \cap E^\epsilon$ is bounded. Again by Lemma 5.5 we have $\mathcal{S}(P,h) \cap J^\epsilon$ is finite, so that $\mathcal{S}(P,h) \cap \mathcal{Z}(Q,\overline{k}) \cap J^\epsilon$ is finite. ∎

We have seen that when P is invertable, then $m''_\epsilon = 1$ holds. Let P_0 be the submatrix consisting of the the first q columns of P:

$$P_0 = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,q} \\ p_{21} & p_{22} & \cdots & p_{2,q} \\ \vdots & & \vdots & \vdots \\ p_{s1} & p_{s2} & \cdots & p_{s,q} \end{bmatrix}.$$

Thus P_0 describes the action of $K''_\circ$ in the direction of the indices belonging to $\mathbf{j}^c$, that is, the action of $K''_\circ$ on $\mathfrak{n}/\mathfrak{e}(\lambda)$ (for each λ). If $\operatorname{rank}(P_0) = q$, then we shall say that K'' acts on $\mathfrak{n}/\mathfrak{e}(\lambda)$ with full rank.

Write $\mathbb{R}^{q+m} = \mathcal{Q} \oplus \mathcal{M}$ where $\mathcal{Q} = \{(x \in \mathbb{R}^{q+m} \mid x_j = 0, q+1 \leq j \leq m\} \simeq \mathbb{R}^q$ and $\mathcal{M} = \{(x \in \mathbb{R}^{q+m} \mid x_j = 0, 1 \leq j \leq q\} \simeq \mathbb{R}^m$.

Lemma 5.7. *Suppose that K'' does not act on $\mathfrak{n}/\mathfrak{e}(\lambda)$ with full rank. Then $m''_\epsilon(\zeta) = \infty$ holds for all $\zeta \in \hat{K}''$ and for all sign indices ϵ.*

Proof. Let $\zeta = \zeta_{g,h} \in \hat{K}''$; observe that for each sign index ϵ,

$$\mathcal{S}(P,h) \cap \mathcal{Q} \subset \mathcal{S}(P,h) \cap E^\epsilon$$

holds. Now $\operatorname{rank}(P_0) < q$ means that $\mathcal{S}(P,h) \cap \mathcal{Q}$ has positive dimension, and hence is unbounded. Proposition 5.2 now says that $m''_\epsilon(\zeta) = \infty$. ∎

We sum up our results so far as follows.

Proposition 5.3. *If K acts with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$, then $m_\epsilon = 1$. On the other hand, if K does not act on $\mathfrak{n}/\mathfrak{e}(\lambda)$ with full rank, then $m_\epsilon = +\infty$.*

We turn to the case where K acts with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$ but not with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$. Hence we must consider the case where K' acts with full rank and K'' acts on $\mathfrak{n}/\mathfrak{e}(\lambda)$ with full rank, but K'' does not act with full rank on $\mathfrak{e}(\lambda)/\mathfrak{d}(\lambda)$. We begin with an algebraic criterion in order that $\mathcal{S}(h,P)\cap E^{\epsilon}$ is bounded. Set

$$\mathcal{C}^{\epsilon} = E^{\epsilon}\cap\mathcal{M}$$

and observe that $E^{\epsilon} = \mathcal{Q}\oplus\mathcal{C}^{\epsilon}$. We can identify $\mathcal{C}^{\epsilon}$ with a "generalized quadrant" in $\mathbb{R}^m$: $\mathcal{C}^{\epsilon} = \{x\in\mathbb{R}^m \mid \epsilon_l x_l \geq 0, 1\leq l\leq m\}$. Set $\operatorname{int}\big(\mathcal{C}^{\epsilon}\big) = \{x\in\mathcal{C}^{\epsilon} \mid x_{q+l}\epsilon_l > 0, 1\leq l\leq m\}$; so that when the above identification is made, $\operatorname{int}\big(\mathcal{C}^{\epsilon}\big)$ is the interior of $\mathcal{C}^{\epsilon}$.

Lemma 5.8. *Let $\mathcal{W}$ be a subspace of $\mathbb{R}^m$ and let $\mathcal{C}$ be a generalized quadrant in $\mathbb{R}^m$. Then for any $y\in\mathbb{R}^m$, $y+\mathcal{W}$ meets $\mathcal{C}$ if and only if $y\in\mathcal{C}+\mathcal{W}$. Moreover, $\big(y+\mathcal{W}\big)\cap\mathcal{C}$ is bounded for all y if and only if*

$$\mathcal{W}^{\perp}\cap\operatorname{int}(\mathcal{C})\neq\emptyset.$$

Proof. The first statement is obvious. As for the second, note first that $v\cdot w>0$ for all $v,w\in\operatorname{int}(\mathcal{C})$ so $\mathcal{W}^{\perp}\cap\operatorname{int}(\mathcal{C})\neq\emptyset$ implies $\mathcal{W}\cap\operatorname{int}(\mathcal{C})=\emptyset$.

Suppose that $\mathcal{W}^{\perp}\cap\operatorname{int}(\mathcal{C})\neq\emptyset$, and let $x=(x_1,x_2,\ldots,x_m)\in\mathcal{W}^{\perp}\cap\operatorname{int}(\mathcal{C})$. Set $\alpha=\min\{|x_j| \mid 1\leq j\leq m\}>0$ and $c=x_1y_1+x_2y_2+\cdots+x_my_m$. For any $u=(u_1,u_2,\ldots,u_m)\in\big(y+\mathcal{W}\big)\cap\mathcal{C}$ we have

$$x\cdot u = x_1u_1+x_2u_2+\cdots+x_mu_m = c,$$

but also $x_ju_j\geq 0$ for all j so

$$|u_j|\leq\frac{c}{\alpha},\ 1\leq j\leq m.$$

Hence $\big(y+\mathcal{W}\big)\cap\mathcal{C}$ is bounded.

To finish the proof it is enough to show that if $\mathcal{W}\cap\mathcal{C}=\{0\}$, then $\mathcal{W}^{\perp}\cap\operatorname{int}(\mathcal{C})\neq\emptyset$. Suppose that $\mathcal{W}\cap\mathcal{C}=\{0\}$; we may assume that $\mathcal{W}\neq\{0\}$. I claim that in any finite dimensional real vector space $\mathcal{U}$, for any convex cone $S\subset\mathcal{U}$ with $0\notin S$ and any subspace $\mathcal{W}$ such that $\mathcal{W}\cap S=\emptyset$, there is a hyperplane $\mathcal{V}\subset\mathcal{U}$ such that $\mathcal{W}\subset\mathcal{V}$ and $\mathcal{V}\cap S=\emptyset$ also.

Assume for the moment that this claim holds. Then we have a hyperplane $\mathcal{V}$ in $\mathbb{R}^m$ such that $\mathcal{W}\subset\mathcal{V}$, and $\mathcal{V}\cap\mathcal{C}\setminus\{0\}=\emptyset$. There is $b\in\mathbb{R}^m$ such that

$$\sup_{z\in\mathcal{V}}\langle b,z\rangle\leq\inf_{z\in\operatorname{int}(\mathcal{C})}\langle b,z\rangle$$

(see for example [3, Chapter IV, Theorem 3.7]). Now since $\mathcal{V}$ is a subspace and 0 is a limit point of $\operatorname{int}\big(\mathcal{C}\big)$ we have $b\in\mathcal{V}^{\perp}\subset\mathcal{W}^{\perp}$ and $\langle b,z\rangle\geq 0$ holds for all

$z \in \mathcal{C}$. It follows that $b \in \operatorname{int}\big(\mathcal{C}\big)$: clearly $\epsilon_l b_l \geq 0$ holds for all $1 \leq l \leq m$, and if $b_l = 0$ for some l then $(0, 0, \ldots, 0, 1(l^{\text{-th}} \text{ position}), 0, \ldots, 0)$ belongs to $\{b\}^\perp = \mathcal{V}$, contradicting the claim.

Finally, we verify the claim by induction on m, the claim being obvious if $m = 1$. Suppose that the claim is true for $m', m' < m$, and let $Q : \mathcal{U} \to \mathcal{U}/\mathcal{W}$ be the canonical map. Then $Q(S)$ is a convex cone in $\mathcal{U}/\mathcal{W}$, and $0 \notin Q(S)$ since $\mathcal{W} \cap S = \emptyset$. By induction we have $\mathcal{V}_0$ a hyperplane in $\mathcal{U}/\mathcal{W}$ such that $\mathcal{V}_0 \cap Q(S) = \emptyset$. Then $\mathcal{V} = Q^{-1}(\mathcal{V}_0)$ is a hyperplane in $\mathcal{U}$ and $\mathcal{V} \cap S = \emptyset$. ∎

We are now ready to describe a precise criterion for finiteness of $m''_\epsilon(\zeta)$. Recall that we already know that a necessary condition for finiteness of $m''_\epsilon(\zeta)$ is that K'' acts with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$. Let $\mathcal{R}$ denote the row space of P. We shall state the criterion first in terms of the row space $\mathcal{R}$.

Lemma 5.9. *Fix a sign index ϵ and suppose that K'' acts on $\mathfrak{n}/\mathfrak{e}(\lambda)$ with full rank. Then $\mathcal{S}(h, P) \cap E^\epsilon$ is bounded if and only if $\mathcal{R} \cap int(\mathcal{C}^\epsilon) \neq \emptyset$.*

Proof. Denote the projection of $\mathcal{N}$ into $\mathcal{M}$ by $\mathcal{N}_\mathcal{M}$. Then the projection of $\mathcal{S}(P, h) \cap E^\epsilon$ is

$$\big(y + \mathcal{N}_\mathcal{M}\big) \cap \mathcal{C}^\epsilon$$

where y is the projection of $z^\circ(h)$. Now since $\operatorname{rank}(P_0) = q$, the projection of $\mathcal{N}$ into $\mathcal{M}$ is injective, whence the projection of $\mathcal{S}(P, h) \cap E^\epsilon$ into $\mathcal{M}$ is injective also. The image of $\mathcal{S}(P, h) \cap E^\epsilon$ under this projection is $\big(y + \mathcal{N}_\mathcal{M}\big) \cap \operatorname{int}(\mathcal{C}^\epsilon)$.

Suppose that $\mathcal{S}(P, h) \cap E^\epsilon$ is bounded. Then $\big(y + \mathcal{N}_\mathcal{M}\big) \cap \mathcal{C}^\epsilon$ is bounded, and so by Lemma 5.8, we have $\big(\mathcal{N}_\mathcal{M}\big)^\perp \cap \operatorname{int}(\mathcal{C}^\epsilon) \neq \emptyset$. But now

$$\big(\mathcal{N}_\mathcal{M}\big)^\perp \cap \mathcal{M} \cap \operatorname{int}(\mathcal{C}^\epsilon) \subset \mathcal{N}^\perp \cap \operatorname{int}(\mathcal{C}^\epsilon) = \mathcal{R} \cap \operatorname{int}(\mathcal{C}^\epsilon),$$

and hence $\mathcal{R} \cap \operatorname{int}(\mathcal{C}^\epsilon) \neq \emptyset$.

Suppose then that $\mathcal{R} \cap \operatorname{int}(\mathcal{C}^\epsilon) \neq \emptyset$. It is easily seen that

$$\mathcal{R} \cap \operatorname{int}(\mathcal{C}^\epsilon) = \mathcal{N}^\perp \cap \operatorname{int}(\mathcal{C}^\epsilon) \subset \mathcal{N}_\mathcal{M}^\perp \cap \operatorname{int}(\mathcal{C}^\epsilon).$$

Hence $\mathcal{N}_\mathcal{M}^\perp \cap \operatorname{int}(\mathcal{C}^\epsilon) \neq \emptyset$ and Lemma 5.8 says that $\big(y + \mathcal{N}_\mathcal{M}\big) \cap \mathcal{C}^\epsilon$ is bounded. Since the projection of $\mathcal{S}(h, P) \cap E^\epsilon$ onto $\big(y + \mathcal{N}_\mathcal{M}\big) \cap \mathcal{C}^\epsilon$ is a bijection of affine sets, then $\mathcal{S}(P, h) \cap E^\epsilon$ must be bounded as well. ∎

Lemma 5.10. *Suppose that K acts with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$, K'' does not act with full rank on $\mathfrak{e}(\lambda)/\mathfrak{d}(\lambda)$, and $\mathcal{R} \cap \operatorname{int}(\mathcal{C}^\epsilon) \neq \emptyset$. Then m''_ϵ is unbounded.*

Proof.

Here we have $\operatorname{rank}(P) < q+m$. Let $\zeta \in \hat{K}''$ such that $m''_\epsilon(\zeta) > 0$ and write $\zeta = \zeta_{g,h}$. Then

$$\mathcal{Z}(Q,g) \cap \mathcal{S}(P,h) \cap J^\epsilon \neq \emptyset.$$

We claim that

$$\sup_h \left| \mathcal{S}(P,h) \cap J^\epsilon \right| = \infty$$

Now for each h,

$$\mathcal{S}(P,h) \cap J^\epsilon = \bigcup_{g \in \widehat{K \cap F}} \mathcal{Z}(Q,g) \cap \mathcal{S}(P,h) \cap J^\epsilon$$

so that it is clear that the claim is sufficient. Now for each positive integer M, set $\mathcal{T}^M = \{t \in \mathcal{T} \mid Mt \in \mathbb{Z}^{q+m}\}$ and

$$\mathcal{S}(P,h)^M = \{v \in \mathcal{S}(P,h) \mid Mv \in \mathbb{Z}^{q+m}\}.$$

Then $\mathcal{S}(P,h)^M \supset z^\circ + R\left(\mathcal{T}^M\right)$ and so

$$\sup_M \left| \mathcal{S}(P,h)^M \cap E^\epsilon \right| = \infty.$$

But

$$\mathcal{S}(P,Mh) \cap J^\epsilon = \mathcal{S}(P,Mh)_{\mathbb{Z}} \cap E^\epsilon \supset M\mathcal{S}(P,h)^M \cap E^\epsilon$$

and the claim is proved. ∎

We have a natural map $r : \mathfrak{k}'' \to \mathcal{R}$ defined by

$$r(C) = i\mathbf{d}\varphi''(C) = [i\mathbf{d}\varphi''_1(C), i\mathbf{d}\varphi''_2(C), \ldots, i\mathbf{d}\varphi''_{q+m}(C)];$$

observe that this map is surjective. Let us say that an element $C \in \mathfrak{k}$ "acts on $\mathfrak{e}(\lambda)/\mathfrak{d}(\lambda)$ with sign ϵ" if $i\mathbf{d}\varphi''_l(C) = 0, 1 \leq l \leq q$ (that is, C acts trivially on $\mathfrak{n}/\mathfrak{e}(\lambda)$), and $\operatorname{sign}\left(i\mathbf{d}\varphi''_{q+l}(C)\right) = \epsilon_l, 1 \leq l \leq m$. Observe that $\mathfrak{k}$, hence $\mathfrak{k}''$, has an element that acts with sign ϵ if and only if $\mathcal{R} \cap \operatorname{int}\left(\mathcal{C}^\epsilon\right) \neq \emptyset$. We sum up the results of this section in these terms.

Theorem 5.4. *Let $G = N \rtimes H$ be an algebraic solvable Lie group with N simply connected nilpotent and H a connected Levi factor in G acting faithfully on N, and let K be the generic stabilizer in H. Let τ be the quasiregular representation of G induced from H, and let $\tau = \oplus_\epsilon \tau_\epsilon$ be the decomposition of Theorem 4.3. Then one of the following obtains.*

(1) If K acts with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$, then for each sign index ϵ, τ_ϵ has uniform multiplicity 2^r, where r is the split rank of K.

(2) If K does not act with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$, then for each sign index ϵ, τ_ϵ is infinite.

(3) If K acts with full rank on $\mathfrak{n}/\mathfrak{e}(\lambda)$, but not with full rank on $\mathfrak{n}/\mathfrak{d}(\lambda)$, then τ_ϵ has finite multiplicity if and only if $\mathfrak{k}$ contains an element that acts on $\mathfrak{e}(\lambda)/\mathfrak{d}(\lambda)$ with sign ϵ. Otherwise, τ_ϵ is infinite.

6. Examples

We conclude with several examples to illustrate the notations and conclusions of the preceding. We begin with the classical oscillator group.

Example 6.1. Let $N = \mathbb{C} \times R$ be the three-dimensional Heisenberg group: $(w,z)(w',z') = (w+w', z+z'+\Im(\overline{w}w'))$ and $H = \mathbb{T}$ acting by $a\cdot(w,z) = (a^{-1}w, z)$. The usual basis for $\mathfrak{n}$ is $\{Z, Y, X\}$ where $[X,Y] = Z$ and where the exponential mapping is just

$$zZ + yY + xX = zZ + \Re\big((x+iy)(X-iY)\big) \mapsto (x+iy, z)$$

An adaptable basis for $\mathfrak{l}$ consisting of eigenvectors is $Z_1 = Z, Z_2 = X + iY, Z_3 = X - iY$ and we have $\delta_1(a) = 1$, while $\delta_3(a) = \overline{\delta_2}(a) = a^{-1}$. The generic layer Ω consists of all $\ell \in \mathfrak{n}^*$ with $\ell(Z) \neq 0$, where for such ℓ we have $\mathbf{i} = \{2\}$ and $\mathbf{j} = \mathbf{j}'' = \{3\}$. Now $H = K = K''$ and $\Lambda = \Sigma = \Sigma_0$, and for $\lambda \in \Lambda$, $\epsilon(\lambda) = \operatorname{sign}(\lambda(Z))$ and $\Lambda = \Lambda^{+1} \cup \Lambda^{-1}$ accordingly. Put $\xi = \lambda(Z)$. The generic irreducible representations of N are $\pi_\xi := \pi_\lambda = \pi^\circ_\lambda$, realized in the space of holomorphic functions if $\xi > 0$ and anti-holomorphic functions if $\xi < 0$. Recall also that the Plancherel measure is (a constant multiple of) $|\xi| d\xi$.

Now $\varphi''(a)^{-1} = \delta_3(a)^{-1} = a$ and the action matrix P is given by $P = [1]$. For $\epsilon = 1$, $J^\epsilon = \{0, 1, 2, \dots\}$ and $Z^\epsilon = J^\epsilon$ with $m_\epsilon(\eta_h) = m''_\epsilon(\eta_h) = 1$ for $h = 0, 1, 2, \dots$. If $\epsilon = -1$, $J^\epsilon = \{0, -1, -2, \dots\}$ and $Z^\epsilon = J^\epsilon$ with $m_\epsilon(\eta_h) = m''_\epsilon(\eta_h) = 1$ on Z^ϵ also. Thus $\tau = \tau_{+1} \oplus \tau_{-1}$ where for $\epsilon = \pm 1$,

$$\tau_\epsilon \simeq \int_{\Lambda^\epsilon}^{\oplus} \oplus_{\epsilon h = 0}^{\infty} \tilde{\pi}_\xi \otimes \overline{\eta_h}\ |\xi| d\xi.$$

The next example exhibits a cross-section that is not flat.

Example 6.2. Let $N = \mathbb{C} \times \mathbb{R} \times \mathbb{C}$ with

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z' + \tfrac{1}{2}(xy' - x'y))$$

and $H = \mathbb{T}$ acting as $a \cdot (w,y,z) = (a^{-1}w, y, a^{-1}z)$. The natural basis for $\mathfrak{n}$ is $\{E_1, E_2, Y, X_1, X_2\}$ with $[X_j, Y] = E_j, j = 1,2$, and where the exponential mapping is

$$z_1E_1 + z_2E_2 + yY + x_1X_1 + x_2X_2 \mapsto (x_1 + ix_2, y, z_1 + iz_2).$$

Write $Z = E_1 + iE_2$ and $X = X_1 + iX_2$, and for $\ell \in \mathfrak{n}^*$ write $\xi = \ell(Z)$ and $\beta = \ell(X)$. The adaptable basis is $Z_1 = Z, Z_2 = \overline{Z}, Z_3 = Y, Z_4 = X, Z_5 = \overline{X}$ and $\delta_2(a) = \overline{\delta_1}(a) = \delta_5(a) = \overline{\delta_4}(a) = a^{-1}$. The generic layer here is $\Omega = \{\ell \in \mathfrak{n}^* \mid \ell(Z) \neq 0\}$, with index sequences $\mathbf{i} = \{3\}$ and $\mathbf{j} = \{4\}$. The H-invariant cross-section is determined by the conditions $\ell(Y) = 0$, and $\ell(Z_4(\ell)) = 0$ where $Z_4(\ell) = \frac{1}{2}\big(\ell[\overline{X}, Y]X + \ell[X,Y]\overline{X}\big)$. Precisely,

$$\Lambda = \{\ell \in \Omega \mid \beta \neq 0, \Re\big(\overline{\xi}\beta\big) = 0\}.$$

Now K and F are trivial here and $\Sigma = \{(\xi, 0, \beta) \mid \xi > 0, \beta \in i\mathbb{R}^*\}$. Each irreducible representation $\pi_{\xi,\beta} := \pi_\lambda$ of N is induced from the variable (but real) polarization

$$\mathfrak{p}(\lambda) = \mathbb{C}\text{-span}\{Z, \overline{Z}, Y, \frac{1}{2}\big(\ell[\overline{X}, Y]X - \ell[X, Y]\overline{X}\big)\}.$$

Note that the supplementary basis for $\mathfrak{p}(\lambda) \cap \mathfrak{n}$ in $\mathfrak{n}$ is $X(\lambda) = Z_4(\lambda)/|\xi|$, and $X(a \cdot \lambda) = a \cdot X(\lambda)$. Since the stabilizer K is trivial (while N is not abelian) the muliplicity is infinite, and (again up to a constant multiple) $d\tilde{\mu}(\lambda) = |\mathbf{Pf}(\lambda)|d\lambda$ where $\mathbf{Pf}(\lambda) = \xi$. Hence our formula reads

$$\tau \simeq \int_\Sigma \infty \cdot \rho_{\xi,\beta}\ d\tilde{\mu}(\xi, \beta) = \int_{-\infty}^{\infty} \int_0^{\infty} \infty \cdot \rho_{\xi,it}\ \xi d\xi dt$$

where $\rho_{\xi,\beta} = \mathrm{ind}_N^G(\pi_{\xi,\beta})$.

In the following the finite subgroup F used in the parametrization $\Lambda/H \simeq \Sigma/F$ is non-trivial.

Example 6.3. Let N be the 8-dimensional real Lie group realized as $N = \mathbb{C}^4$ with

$$(w, x, y, z)(w', x', y', z') = (w + w', x + x', y + y' - xw', z + z' + xy' - \frac{x^2 w'}{2})$$

and with $H = \mathbb{T}$ acting on $\mathfrak{n}$ by $a \cdot (w, x, y, z) = (aw, ax, a^2 y, a^3 z)$. A suitable adaptable basis (listed in the order of Z_1, Z_2, etc.) is $\{Z, \overline{Z}, Y, \overline{Y}, X, \overline{X}, W, \overline{W}\}$ with brackets $[W, X] = Y, [W, \overline{X}] = 0, [X, Y] = Z, [X, \overline{Y}] = 0$. (Note that the brackets of the real basis for $\mathfrak{n}$ consisting of real and imaginary parts of the preceding basis can be recovered from the above; the exponential mapping is exactly as in the preceding, for example $(w, 0, 0, 0) = \exp(\Re(w\overline{W}))$, etc..) The generic layer is $\{\ell \in \mathfrak{n}^* \mid \ell(Z) \neq 0\}$ with $\mathbf{i} = \{3, 4\}, \mathbf{j} = \{5, 6\}$. Writing $\ell(Z) = \xi, \ell(W) = \beta$, we have $\Lambda = \{\ell \in \Omega \mid \ell(Y) = \ell(X) = 0, \beta \neq 0\}$ and accordingly we write $\lambda = (\xi, \beta)$. Now $\chi_1(a) = \delta_1(a)^{-1} = a^3$, so H acts by rotations in the ξ-direction and $\Sigma = \{(\xi, \beta) \in \Lambda \mid \xi > 0\}$. On the other hand, $F = \ker(\chi_1) = \mathbb{F}(3)$, and for $t \in F, (\xi, \beta) \in \Sigma,\ t \cdot (\xi, \beta) = (\xi, t\beta)$. We put $\Sigma_0 = \{(\xi, \beta) \in \Sigma \mid \mathrm{sign}(\beta) = e^{i\theta} \text{ with } 0 \leq \theta < 2\pi/3\}$. Now as in Example 6.2, K is trivial and τ is infinite. Here $\mathbf{Pf}(\xi, \beta) = \xi^2$, so

$$\tau \simeq \int_{\Sigma_0} \infty \cdot \rho_{\xi,\beta}\ \ \xi^2 d\xi d\overline{\xi} d\beta d\overline{\beta}.$$

We close with an example where K acts on $\mathfrak{n}/\mathfrak{e}(\lambda)$ with complex roots, and where τ decomposes into finite unbounded and infinite subrepresentations.

Example 6.4. Let N be the 10-dimensional real Lie group realized as $N = \mathbb{C}^5$ with

$$(x, y, w_1, w_2, z)(x', y', w_1', w_2', z') =$$

$$(x + x', y + y', w_1 + w_1', w_2 + w_2', z + z' + \frac{1}{2}(xy' - x'y) + \frac{1}{2}(\Im(\overline{w_1} w_1') + i\Im(\overline{w_2} w_2')).$$

Let $H = S \times T_1 \times T_2$ where $S = \mathbb{R}_+^*$, $T_k = \mathbb{T}$, and so that for $a \in S, b_k \in T_k$,

$$ab_1b_2(x, y, w_1, w_2, z) = (ab_1x, a^{-1}b_1^{-1}y, b_2^{-1}w_1, b_2^{-1}w_2, z).$$

We choose the adaptable basis (listed in order): $\{Z, \overline{Z}, W_1, \overline{W_1}, W_2, \overline{W_2}, Y, \overline{Y}, X, \overline{X}\}$ with brackets $[X, Y] = Z, [X, \overline{Y}] = 0, [W_1, \overline{W_1}] = -2i\Re(Z), [W_2, \overline{W_2}] = -2i\Im(Z)$ and so that $\delta_1(ab_1b_2) = \delta_1(ab_1b_2) = 1$, $\delta_4(ab_1b_2) = \overline{\delta_3}(ab_1b_2) = \delta_6(ab_1b_2) = \overline{\delta_5}(ab_1b_2) = b_2^{-1}$, while $\delta_8 = \overline{\delta_7}(ab_1b_2) = a^{-1}b_1^{-1}$ and $\delta_{10} = \overline{\delta_9}(ab_1b_2) = ab_1$. (Definition of the exponential mapping follows the convention of the preceding.)

The generic layer is $\{\ell \in \mathfrak{n}^* \mid \ell(Z) \neq 0\}$ with jump sequences $\mathbf{i} = \{3, 5, 7, 8\}, \mathbf{j} = \{4, 6, 9, 10\}$. We have $\Lambda = \{\ell \in \Omega \mid \ell(W_1) = \ell(W_2) = \ell(Y) = \ell(X) = 0\}$ and for $\lambda \in \Lambda$ we write $\lambda = \xi$ where $\ell(Z) = \xi$. Hence $K = H$ in this example, so $\Sigma = \Lambda$ and $F = \{1\}$. Put $\xi_1(\lambda) = \xi_1 = \lambda(\Re(Z))$ and $\xi_2(\lambda) = \xi_2 = \lambda(\Im(Z))$ and $\epsilon_k(\lambda) = \text{sign}(\xi_k), \mathfrak{k} = 1, 2$. Note that $\Omega^\epsilon = \{\ell \in \Omega \mid \epsilon(\lambda) = \epsilon\}$ is non-empty for each sign index $\epsilon \in \{\pm 1\}^2$. The polarization $\mathfrak{p}(\lambda)$ for each $\lambda \in \Lambda$ obtained from the adaptable basis is a positive polarization only for those λ for which $\epsilon(\lambda) = (1, 1)$, and for sign indices $\epsilon = (\epsilon_1, \epsilon_2)$ we have

$$\mathfrak{p}^\epsilon(\lambda) = \mathbb{C}\text{-span}\{Z, \{Z, \overline{Z}, W_1^{\epsilon_1}, W_2^{\epsilon_2}, Y, \overline{Y}\}$$

is a positive polarization when $\epsilon = \epsilon(\lambda)$. Let $E \subset N$ be the subgroup

$$E = \{(0, y, w_1, w_2, z) \mid y, w_1, w_2, z \in \mathbb{C}\}.$$

Then $\pi_\lambda = \text{ind}_E^N(\pi_\lambda^\circ)$ where π_λ° acts in the Hilbert space $(\mathcal{A}^\epsilon(\mathbb{C}^2), \|\cdot\|_\lambda)$ of $\epsilon(\lambda)$-holomorphic functions in the variables w_1, w_2. Now $\mathcal{X} = \mathbb{C} = U \times \mathbb{T}$ where U is the set of positive reals and $\mathcal{H}_\lambda \simeq \mathcal{H}'_\lambda \otimes \mathcal{H}''_\lambda$ where $\mathcal{H}'_\lambda = L^2(U, sds)$ and $\mathcal{H}''_\lambda = L^2(\mathbb{T}) \otimes \mathcal{A}^\epsilon(\mathbb{C}^2)$. With regard to the action of K, we have $\mathbf{j}' = \mathbf{j}^c = \{9\}$ and K acts with full rank on $\mathfrak{n}/\mathfrak{e}$ (via S and T_1), but K'' acts with rank one on $\mathfrak{e}/\mathfrak{d}$ (via T_2). We have $\varphi''(b_1b_2)^{-1} = (b_1, b_2, b_2)$. so the action matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and we are in the situation (3) of Theorem 5.4. We have $\text{int}(\mathcal{C}^\epsilon) = \{(0, x_2, x_3) \mid \epsilon_1 x_2 > 0, \epsilon_2 x_3 > 0\}$ and the row space of P meets $\text{int}(\mathcal{C}^\epsilon)$ exactly when $\epsilon = (1, 1)$ or $\epsilon = (-1, -1)$. Hence we have $\tau = \oplus_\epsilon \tau_\epsilon$ where $\tau_{(1,1)}$ and $\tau_{(-1,-1)}$ have finite unbounded multiplicity, and τ_ϵ is infinite otherwise. We exhibit the finite unbounded subrepresentations $\tau_{(\pm 1, \pm 1)}$.

Since $\mathbf{j}' = \mathbf{j}^c$ and K' acts with full rank, then $m_\epsilon = m''_\epsilon$. For $\epsilon = (1, 1)$, and for $h \in \mathbb{Z}$, we find that $m''_\epsilon(\zeta_h) = 0$ if $h < 0$ while for $h \geq 0$,

$$m''_\epsilon(\zeta_h) = \left|\{(n_1, n_2) \mid n_k \in \{0, 1, 2, \dots\}, n_1 + n_2 = h\}\right| = h + 1.$$

Similarly, for $\epsilon = (-1,-1)$, $m''_\epsilon(\zeta_h) = 0$ if $h > 0$ while for $h \le 0$,

$$m''_\epsilon(\zeta_h) = \Big|\{(n_1,n_2) \mid n_k \in \{0,-1,-2,\dots\}, n_1+n_2 = h\}\Big| = h+1.$$

Hence

$$\tau_{(\pm1,\pm1)} \simeq \int_{\Lambda^\epsilon}^{\oplus} \oplus_{\pm h=0}^{\infty} \, (h+1) \; \tilde{\pi}_\xi \otimes \overline{\eta_h} \; |\mathbf{Pf}(\xi)| d\xi$$

and one computes that $\mathbf{Pf}(\xi) = \xi_1\xi_2(\xi_1^2 + \xi_2^2)$.

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