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Local and Global Aspects of Lie Superposition Theorem

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Abstract. In this paper we give the global conditions for an ordinary differential equation to admit a superposition law of solutions in the classical sense. This completes the well-known Lie superposition theorem. We introduce rigorous notions of pretransitive Lie group action and Lie-Vessiot systems. We prove that an ordinary differential equation admit a superposition law if and only if its enveloping algebra is spanned by fundamental fields of a pretransitive Lie group action. We discuss the relationship of superposition laws with differential Galois theory and review the classical result of Lie.
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1. Introduction and Main Results

Superposition laws of solutions of differential equations attracted the attention of researchers since the statement of the question by S. Lie at the 19th century. They appear widely, sometimes as reflect of certain linear phenomena, and sometimes as genuine non-linear superposition (see Example 6.8). The local characterization of ordinary differential equations admitting superposition laws was partially done by S. Lie and G. Sheffers in [16]. In the last quarter of the 20th century, superposition laws attracted again the attention of pure and applied mathematicians. Shnider and Winternitz, in [22], classified the germs of low dimension differential equations admitting superposition laws. This work can be seen as a classification of germs of homogeneous spaces. There is also a number of applied papers on the subject, including [1, 23, 21, 12, 9, 5, 6]. A good treatment of superposition laws can be seen in the textbook [11], and there is a monograph completely devoted to the subject [4].

However, the theoretical approach to superposition laws relies is Lie's characterization. This is a local result, and avoid the global problem of construction of the group of transformations hidden behind the superposition law. This problem was tackled first by Vessiot in [24], who analyzed the same subject that Lie, but under a slightly different point of view, more related with differential Galois

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theory. Instead of taking the Lie algebra of the group from the joint invariants, Vessiot extract the group structure directly from the superposition law. This idea is systematically used here along Section 6. It is the key points that allow us to relate the superposition of law with differential Galois theory.

In the original infinitesimal Lie's approach, it is necessary to integrate certain Lie algebra of autonomous vector fields. Thus, we obtain a number of arbitrary constants that correspond to joint invariants. By joint invariants we mean first integral of the differential equation lifted to the cartesian power. However, in the general case, there are not suitable joint invariants (see Example 4.2). Instead of this, we only get a foliation on the cartesian power of the phase space. Each leave of this foliation correspond to a determination of those theoretical arbitrary constants. This defect was noticed by Cariñena, Grabowsky and Marmo [7]. They substitute the notion of superposition law for the milder notion of local superposition; then the classical statement of Lie holds. Here we give the conditions for the existence of a superposition law in the global classical sense, that is, a map that allow us to recover the general solution from a finite number of particular solutions.

An ordinary differential equation admitting a *fundamental system of solutions* is, by definition, a system of non-autonomous differential equations,

$$\frac{dx_i}{dt} = F_i(t, x_1, \ldots, x_n) \quad i = 1, \ldots, n \tag{1}$$

for which there exists a set of formulae,

$$\varphi_i(x^{(1)}, \ldots, x^{(r)}; \lambda_1, \ldots, \lambda_n) \quad i = 1, \ldots, n \tag{2}$$

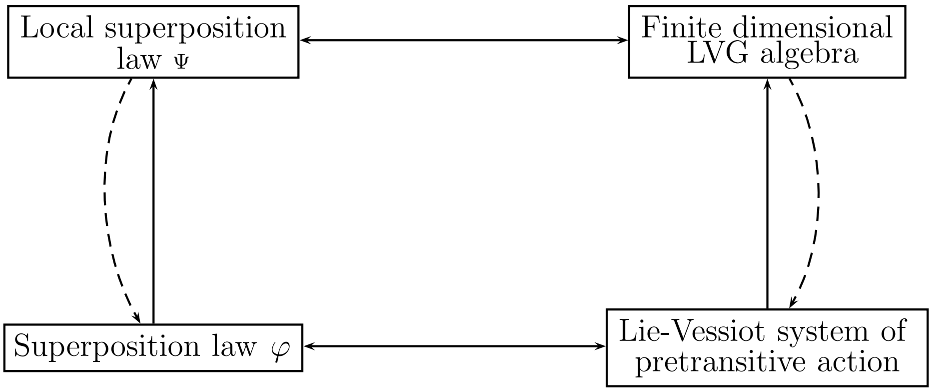
expressing the *general solution* as function of r particular solutions of (1) and some arbitrary constants λ_i. This means that for r particular solutions $(x_1^{(i)}(t), \ldots, x_n^{(i)}(t))$ of the equations, satisfying certain non-degeneracy condition,

$$x_i(t) = \varphi_i(x_1^{(1)}(t), \ldots, x_n^{(1)}(t), \ldots, x_n^{(r)}(t); \lambda_1, \ldots, \lambda_n) \tag{3}$$

is the general solution of the equation (1). In [17] Lie also stated that the arbitrary constants λ_i parametrize the solution space, in the sense that for different constants, we obtain different solutions: there are not functional relations between the arbitrary constants λ_i. The set of formulae φ_i is usually referred to as a *superposition law for solutions* of (1).

The class of ordinary differential equations admitting *fundamental systems of solutions*, or *superposition laws*, was introduced by S. Lie in 1885 [16], as certain class of auxiliary equations appearing in his integration methods for ordinary differential equations. Further development is due to Guldberg and mainly to Vessiot [24]. The characterization of such class of differential equations is given by S. Lie and G. Scheffers in 1893 [17], and it is know as *Lie's superposition theorem*, or *Lie-Scheffers theorem.* There is a Lie algebra canonically attached to any non-autonomous differential equation, the so called *Lie-Vessiot-Guldberg* or *enveloping* algebra. Lie superposition theorem states that a differential equation admit a superposition law if and only if its enveloping algebra is finite dimensional.

This result, as observed by some contemporary authors (see [4] and [7]), is not true in the general case. In fact, Lie theorem characterizes a bigger class of differential equations, whichever admit certain invariant foliation known as *local superposition law*. Here we find the characterization of differential equations admitting superposition laws in the classical sense. In order to do so, we introduce the concepts of *Lie-Vessiot systems* and *pretransitive Lie group actions*. Finally we obtain the whole picture of local and global superposition laws, expressed in the following diagram. Note that dashed lines mean integration or inversion processes that can no be done globally in the general case.



Throughout this paper we will examine the different links in the above diagram. The main result is the link between the bottom items, which is the characterization of the differential equations admitting global superposition laws, *i.e.* superposition laws in the classical sense.

Theorem 6.1 (Global Lie). *A non-autonomous complex analytic vector field $\vec{X}$ in a complex analytic manifold M admits a superposition law if and only if it is a Lie-Vessiot system related to a pretransitive Lie group action in M.*

Using this global philosophy it is easy to jump into the algebraic category. First, is follows easily that rational superposition laws lead to rational actions of algebraic groups.

Theorem 7.1 *Let $\vec{X}$ be a non-autonomous meromorphic vector field in M that admits a rational superposition law. Then, it is a Lie-Vessiot system related to a algebraic action of an algebraic group on M.*

This result allow us to connect Lie's result with the frame of differential Galois theory. The main purpose of differential Galois theory is to classify differential equations with respect to their integrability and reducibility, and of course, solve those that are integrable. The first step in differential Galois theory is the theory due to Picard and Vessiot that deals with linear differential equations. In this frame, there is a correspondence between linear differential equations and their differential Galois groups, which are linear algebraic groups. Being the Lie-Vessiot systems a natural generalization of linear equations, then it is expectable to generalize the differential Galois theory for them. In [13], E. Kolchin gives the theory of *strongly normal extensions*, a generalization of Picard-Vessiot theory in which the Galois groups are arbitrary algebraic groups. However, this theory deals with differential field extensions and not with differential equations. Thus, the connection with superposition laws remained hidden for long time. There is a first paper on the subject by K. Nishioka [20] relating strongly normal extensions and Lie-Vessiot systems. However this work relies on Lie's local characterization and does not consider superposition laws in an explicit way. For comparison between Nishioka's results and ours, see [3]. The connection between rational superposition laws and strongly normal extensions was first noticed by J. Kovacic, and exposed in a series of lectures. Here we give this connection in a explicit way.

Theorem 8.6 *Let $\vec{X}$ be a non-autonomous meromorphic vector field in an algebraic manifold M with coefficients in the Riemann surface S that admits a rational superposition law. Let L be the differential field spanned by the coordinates of any fundamental system of solutions of $\vec{X}$. Then, $\mathcal{M}(S) \subset L$ is a strongly normal extension in the sense of Kolchin. Moreover, if G is the algebraic group of transformations in M induced by the superposition law, each particular solution $\sigma(t)$ of the associated automorphic system $\vec{A}$ in G induce an injective map,*

$$\mathrm{Aut}_{\mathcal{M}(S)}(L) \to G.$$

which is an anti-morphism of groups.

There are also some related results [8] due to G. Casale connecting strongly normal extensions and non-linear differential Galois theory. From our results and those from him we can expect the following: differential equations admitting a rational superposition are those whose Galois D-grupoid is an algebraic group of finite dimension acting on some transversal structure. This point would require however a more detailed analysis.

2. Non-autonomous Complex Analytic Vector Fields

Notation. From now on, the phase space M is a complex analytic manifold, S is a Riemann surface together with a holomorphic derivation $\partial\colon \mathcal{O}_S \to \mathcal{O}_S$, being $\mathcal{O}_S$ the sheaf of holomorphic functions in S. By *the extended phase space* we mean the cartesian power $S \times M$. We shall denote t for the general point of S and x for the general point of M. We write $\bar{x}$ for a r-frame $(x^{(1)}, \ldots, x^{(r)}) \in M^r$. Under this rule, we shall write $f(x)$ for functions in M, $f(t)$ for functions in S and $f(t,x)$ for functions in the extended phase space. Whenever we need we take a local system of coordinates $x_1, \ldots, x_n$ for M. The induced system of coordinates in M^r, given component by component, is then $x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}$. The general point of the cartesian power M^{r+1} is seen as a pair $(\bar{x}, x)$ of an r-frame and an point of M. Thus, local coordinates in M^{r+1} are $x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(r)}, \ldots, x_n^{(r)}, x_1, \ldots, x_n$.

Definition 2.1. A non-autonomous complex analytic vector field $\vec{X}$ in M, depending on the Riemann surface S, is an autonomous vector field in $S \times M$

compatible with ∂ in the following sense:

$$\vec{X}f(t) = \partial f(t) \qquad \begin{array}{ccc} \mathcal{O}_{S\times M} & \xrightarrow{\vec{X}} & \mathcal{O}_{S\times M} \\ \uparrow & & \uparrow \\ \mathcal{O}_S & \xrightarrow{\partial} & \mathcal{O}_S \end{array}$$

In each cartesian power M^r of M we consider the *lifted* vector field $\vec{X}^r$. This is just the direct sum of copies of $\vec{X}$ acting in each component of the cartesian power M^r. We have the local expression for $\vec{X}$,

$$\vec{X} = \partial + \sum_{i=1}^{n} F_i(t,x)\frac{\partial}{\partial x_i},$$

and also the local expression for $\vec{X}^r$, which is a non-autonomous vector field in M^r,

$$\vec{X}^r = \partial + \sum_{i=1}^{n} F_i(t,x^{(1)})\frac{\partial}{\partial x_i^{(1)}} + \ldots + \sum_{i=1}^{n} F_i(t,x^{(r)})\frac{\partial}{\partial x_i^{(r)}}. \tag{4}$$

Let us consider $\vec{X}$ a non-autonomous vector field in M. We can see $\vec{X}$ as an holomorphic map $\vec{X}\colon S \to \mathfrak{X}(M)$, which assigns to each $t_0 \in S$ an autonomous vector field $\vec{X}_{t_0}$.

Definition 2.2. The *enveloping* algebra of $\vec{X}$ is the Lie algebra of vector fields in M spanned by the set vector fields $\{\vec{X}_{t_0}\}_{t_0\in S}$. The enveloping algebra of $\vec{X}$ is denoted $\mathfrak{g}(\vec{X})$.

Remark 2.3. (**Lie Test**) Assume that there exist $\vec{X}_1,\ldots,\vec{X}_s$ autonomous vector fields in M spanning a finite dimensional Lie algebra, and holomorphic functions $f_1(t),\ldots,f_s(t)$ in S such that,

$$\vec{X} = \partial + \sum_{i=1}^{s} f_i(t)\vec{X}_i, \qquad [\vec{X}_i,\vec{X}_j] = \sum_k c_{ij}^k \vec{X}_k. \tag{5}$$

It follows that the enveloping algebra of $\vec{X}$ is a subalgebra of the one spanned by $\vec{X}_1,\ldots,\vec{X}_s$. Reciprocally, if the enveloping algebra of $\vec{X}$ is of finite dimension, let us consider a basis $\{\vec{X}_i\}$. The coordinates of $\vec{X}_t$ in such basis, when t varies in S, define holomorphic functions $f_i(t)$. Thus, we obtain an expression as above for $\vec{X}$. Expression (5) is the condition expressed in [17] by S. Lie, and it is clear that it is equivalent to the finiteness of the dimension of the enveloping algebra.

3. Superposition Laws

Definition 3.1. A superposition law for $\vec{X}$ is an analytic map

$$\phi\colon U \times P \to M,$$

where U is analytic open subset of M^r and P is an n-dimensional manifold, satisfying:

(a) U is union of integral curves of $\vec{X}^r$

(b) If $\bar{x}(t)$ is a solution of $\vec{X}^r$, defined for t in some open subset $S' \subset S$, then $x_\lambda(t) = \phi(\bar{x}(t); \lambda)$, is the general solution of $\vec{X}$ for t varying in S'.

The problem of dependence on parameters can be reduced to the dependence on initial conditions. The parameter space is then replaced by the phase space. In virtue of the following proposition, from now on we will assume that the space of parameters of any superposition law is the phase space itself.

Proposition 3.2. *If $\vec{X}$ admits a superposition law,*

$$\phi\colon U \times P \to M, \qquad U \subset M^r,$$

then it admits another superposition law

$$\varphi\colon U \times M \to M, \qquad U \subset M^r,$$

whose space of parameters is the space of the initial conditions in the following sense: for a given $t_0 \in S$ there is a solution $\bar{x}(t)$ of $\vec{X}^r$ defined in a neighborhood of t_0 such that for all $x \in M$, $\varphi(\bar{x}(t_0); x) = x$.

Proof. Consider a superposition law ϕ as in the statement. Let us choose $t_0 \in S$ and certain local solution $\bar{x}_0(t)$ of $\vec{X}^r$. Then,

$$x(t; \lambda) = \phi(\bar{x}_0(t); \lambda),$$

is the general solution. Define:

$$\xi\colon P \to M, \quad \lambda \mapsto x(t_0; \lambda),$$

then, by local existence and uniqueness of solutions for differential equations, for each $x_0 \in M$ there exist a unique local solution $x(t)$ such that $x(t_0) = x_0$. Hence, there exist a map φ,

$$\begin{tikzcd} U \times P \arrow[r, "\phi"] \arrow[d, "Id\times\xi"'] & M \\ U \times M \arrow[ur, "\varphi"'] & \end{tikzcd}$$

which is a superposition law for $\vec{X}$ satisfying the assumptions of the statement. ∎

Example 3.3 (Linear systems). Let us consider a linear system of ordinary differential equations,

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j, \quad i = 1, \ldots, n$$

as we should know, a linear combination of solutions of this system is also a solution. Thus, the solution of the system is a n dimensional vector space, and we can express the global solution as linear combinations of n linearly independent solutions. The superposition law is written down as follows,

$$\mathbb{C}^{n\times n}_{x_i^{(j)}} \times \mathbb{C}^n_{\lambda_j} \to \mathbb{C}^n, \quad (x_i^{(j)}, \lambda_j) \mapsto (y_i) \quad y_i = \sum_{j=1}^n \lambda_j x_i^{(j)}.$$

Example 3.4 (Riccati equations)**.** Let us consider the ordinary differential equation,

$$\frac{dx}{dt} = a(t) + b(t)x + c(t)x^2,$$

let us consider four different solutions $x_1(t), x_2(t), x_3(t), x_4(t)$. A direct computation gives that the anharmonic ratio is constant,

$$\frac{d}{dt}\frac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)} = 0.$$

If x_1, x_2, x_3 represent three known solutions, we can obtain the unknown solution x from the expression,

$$\lambda = \frac{(x_1-x_2)(x_3-x)}{(x_1-x)(x_3-x_2)}$$

obtaining,

$$x = \frac{x_3(x_1-x_2) - \lambda x_1(x_3-x_2)}{(x_1-x_2) - \lambda(x_3-x_2)}$$

which is the general solution as function of the constant λ, and then a superposition law for the Riccati equation.

Let us consider $\mathcal{O}_M$ the sheaf of holomorphic functions in M, and for each r, the sheaf $\mathcal{O}_{M^r}$ of holomorphic functions in the cartesian power M^r. We denote by $\mathcal{O}_{M^r}^{\vec{X}^r}$ the subsheaf of first integrals of the lifted vector field $\vec{X}^r$ (4) in $\mathcal{O}_{M^r}$. For the notion of *regular ring* we follow the definition of [19].

Definition 3.5. Let x be a point of M and $\mathcal{O}_{M,x}$ be the ring of germs of complex analytic functions in x. A subring $\mathcal{R} \subset \mathcal{O}_{M,x}$ is a regular ring if it is the ring of first integrals of k germs in x of vector fields $\vec{Y}_1, \ldots, \vec{Y}_k$, which are $\mathbb{C}$-linearly independent at x and in involution:

$$[\vec{Y}_i, \vec{Y}_j] = 0.$$

The dimension $\dim \mathcal{R}$ is the number $\dim_{\mathbb{C}} M - k$.

Consider a sheaf of rings $\Psi \subset \mathcal{O}_{M^{r+1}}$ of complex analytic functions in the cartesian power M^{r+1}. We say that Ψ is a sheaf of *generically regular rings* if for a generic point (*i.e.* outside a closed analytic set) $(\bar{x}, x) \in M^{r+1}$ the stalk $\Psi_{(\bar{x},x)}$ is a regular ring. A sheaf Ψ of generically regular rings is the sheaf of first integrals of a generically regular Frobenius integrable foliation. We denote this foliation by $\mathcal{F}_\Psi$.

Definition 3.6. A *local* superposition law for $\vec{X}$ is a sheaf of rings $\Psi \subset \mathcal{O}_{M^{r+1}}$, for some $r \in \mathbb{N}$, verifying:

(1) Ψ is a sheaf of generically regular rings of dimension $\geq n$.

(2) $\vec{X}^{r+1}\Psi = 0$, or equivalently, $\vec{X}^{r+1}$ is tangent to $\mathcal{F}_\Psi$.

(3) $\mathcal{F}_\Psi$ is generically transversal to the fibers of the projection $M^{r+1} \to M^r$.

This notion is found for first time in [7], exposed in the language of foliations. They prove that the existence of a local superposition law of $\vec{X}$ is equivalent to having a finite dimensional enveloping algebra, as we also are going to see.

Proposition 3.7. *If $\vec{X}$ admits a superposition law, then it admits a local superposition law.*

Proof. Let us consider a superposition law for $\vec{X}$,

$$\varphi\colon U \times M \to M, \qquad U \subset M^r, \qquad \varphi = (\varphi_1, \ldots, \varphi_n).$$

We write the general solution,

$$x(t) = \varphi(x^{(1)}(t), \ldots, x^{(r)}(t); \lambda). \tag{6}$$

The local analytic dependency with respect to initial conditions ensures that the partial jacobian $\frac{\partial(\varphi_1, \ldots \varphi_n)}{\partial(\lambda_1, \ldots, \lambda_n)}$ does not vanish. Therefore, at least locally, we can invert with respect to the last variables,

$$\lambda = \psi(x^{(1)}(t), \ldots, x^{(r)}(t), x(t)).$$

From that, the components ψ_i of ψ are first integrals of the lifted vector field $\vec{X}^{r+1}$. We consider the sheaf of rings Ψ generated by these functions ψ_i. This is a sheaf of regular rings of dimension n. By construction $\mathcal{F}_\Psi$ is transversal to the fibers of the projection $M^r \times M \to M^r$. We conclude that this sheaf is a local superposition law for $\vec{X}$. ■

4. Local Superposition Theorem

Theorem 4.1 ((Local Superposition Theorem [7])). *The non-autonomous vector field $\vec{X}$ in M admits a local superposition law if and only its enveloping algebra is finite dimensional.*

Once we understand the relation between superposition law and local superposition law, we see that Lie's original proof is still valid. Here we follow [17]. However we explain some points that were not explicitly detailed and remain obscure in the original proof.

It is clear that the finiteness of the enveloping algebra is not a sufficient condition for the existence of a superposition law. It is necessary to integrate the enveloping algebra into a Lie group action.

For instance, any non-linear *autonomous* vector field has a one dimensional enveloping Lie algebra, but it is widely know that the knowledge of a number of integral curves of a generic non-linear vector field do not lead to the general solution. Some people can argue that this case is too degenerated because the enveloping algebra of an autonomous vector field is too small in dimension. However, we can construct a differential equation in dimension 4, having an enveloping Lie algebra of dimension 4 for which no superposition Law is expectable.

Example 4.2. Let us consider the direct product of the well known Lorentz system and a Riccati equation

$$\begin{array}{rcl} \dot{x} & = & \sigma(y-x) \\ \dot{y} & = & x(\rho - z) - y \\ \dot{z} & = & xy - \beta z \\ \dot{u} & = & \frac{2u}{t} - \frac{2}{t^2} - u^2 \end{array} \tag{7}$$

being σ, β, and ρ parameters. We can apply Lie test and we arrive to de conclusion that the enveloping algebra is of dimension 4 and have the following system of generators.

$$\begin{aligned} \vec{X}_1 &= \sigma(y-x)\frac{\partial}{\partial x} + (x(\rho - z) - y)\frac{\partial}{\partial y} + (xy - \beta z)\frac{\partial}{\partial z} \\ \vec{X}_2 &= u^2\frac{\partial}{\partial u} \\ \vec{X}_3 &= u\frac{\partial}{\partial u} \\ \vec{X}_4 &= \frac{\partial}{\partial u} \end{aligned}$$

Our original vector field is $\vec{X} = \vec{X}_1 - \vec{X}_2 + \frac{2}{t}\vec{X}_3 - \frac{2}{t^2}\vec{X}_4$. Let us assume that this system admits a superposition Law,

$$\varphi\colon (\mathbb{C}^4)^r \times \mathbb{C}^4 \to \mathbb{C}^4.$$

The solutions of the Riccati equation in (7) can be easily found, and they turn to be of the form $u(t) = \frac{1+2at}{t(1+at)}$ with a constant. Therefore, the solutions of (7) are of the form $\left(x(t), y(t), z(t), \frac{1+2at}{t(1+at)}\right)$ where $(x(t), y(t), z(t))$ is a solution of the Lorentz system. By taking suitable values for $a_1, \ldots, a_r$, we can put the corresponding solutions of the Riccati equation inside the superposition law, project $\mathbb{C}^4$ onto $\mathbb{C}^3$, and we will get a map $\bar{\varphi}$ of the same kind that φ,

$$\bar{\varphi}\colon (\mathbb{C}^3)^r \times \mathbb{C}^4 \to \mathbb{C}^3, \quad (x_i, y_i, z_i, \lambda) \mapsto \pi_{x,y,z}\left(\varphi\left(x_i, y_i, z_i, \frac{1+2a_it}{t(1+a_it)}; \lambda\right)\right)$$

that gives us the general solution of the Lorentz system in function of r known solutions and 4 arbitrary constants. Of course, the existence of such a map for the Lorenz attractor equations is not expectable at all.

Local Superposition Law Implies Finite Dimensional Enveloping Algebra

First, let us assume that $\vec{X}$ admits a local superposition law Ψ in $\mathcal{O}_{M^{r+1}}$. Let us consider the cartesian power $\vec{X}^{r+1}$ as a non-autonomous vector field in M^{r+1}. Any section ψ of Ψ is a first integral of $\vec{X}^{r+1}$.

Consider the family of vector fields $\{\vec{X}_t^{r+1}\}_{t\in S} \subset \mathfrak{X}(M^{r+1})$. Let us take a maximal subfamily $\{\vec{X}_t^{r+1}\}_{t\in\Lambda}$ of $\mathcal{O}_{M^{r+1}}$-linearly independent vector fields; here, Λ is some subset of S. The cardinal of a set $\mathcal{O}_{M^{r+1}}$-linearly independent vector fields is bounded by the dimension of M^{r+1}. Hence,

$$\Lambda = \{t_1, \ldots, t_m\},$$

is a finite subset. We denote the corresponding vector fields as follows:

$$\vec{X}_i^{r+1} = \vec{X}_{t_i}^{r+1} \quad i = 1, \ldots, m.$$

We consider the following notation; in the cartesian power M^{r+1} we denote the different copies of M as follows:

$$M^{r+1} = M^{(1)} \times \ldots \times M^{(r)} \times M.$$

We recall that the *lifted* vector field $\vec{X}_i^{r+1}$ is the sum,

$$\vec{X}_i^{r+1} = \vec{X}_i^{(1)} + \ldots + \vec{X}_i^{(r)} + \vec{X}_i,$$

of $r+1$ copies of the same vector field acting in the different copies of M.

The sheaf Ψ consists of first integrals of the fields $\vec{X}_i^{r+1}$. Thus, for all i, j, the Lie bracket $[\vec{X}_i^{r+1}, \vec{X}_j^{r+1}]$ also annihilates the sheaf Ψ. We can write the Lie bracket as a sum of $r+1$ copies of the same vector field in M. Let us note that the Lie bracket of two lifted vector fields is again a lifted vector field.

$$[\vec{X}_i^{r+1}, \vec{X}_j^{r+1}] = [\vec{X}_i^{(1)}, \vec{X}_j^{(1)}] + \ldots + [\vec{X}_i^{(r)}, \vec{X}_j^{(r)}] + [\vec{X}_i, \vec{X}_j].$$

We recall that a regular distribution of vector fields is a distribution of constant rank, and a regular distribution is said Frobenius integrable if it is closed by Lie brackets.

The m vector fields $\vec{X}_i^{r+1}$ span a distribution which is generically regular of rank m. However, in the general case, it is not Frobenius integrable. We add all the feasible Lie brackets, obtaining an infinite family

$$\mathcal{X} = \bigcup_{k=0}^{\infty} \mathcal{X}_k$$

where,

$$\mathcal{X}_0 = \{\vec{X}_1^{r+1}, \ldots, \vec{X}_m^{r+1}\},$$

and,

$$\mathcal{X}_k = \{[\vec{Y}, \vec{Z}] \mid \vec{Y} \in \mathcal{X}_i, \vec{Z} \in \mathcal{X}_j \text{ for } i < k \text{ and } j < k\}.$$

The family $\mathcal{X}$ is a set of *lifted* vector fields. They span a generically regular distribution in M^{r+1} which is, by construction, Frobenius integrable. We extract of this family a maximal subset of $\mathcal{O}_{M^{r+1}}$-linearly independent vector fields,

$$\{\vec{Y}_1^{r+1}, \ldots, \vec{Y}_s^{r+1}\}.$$

They span the same generically regular Frobenius integrable distribution of rank $s \geq m$. Without any lose of generality we can assume that the m vector fields $\vec{X}_i^{r+1}$ are within this family. These vector fields $\vec{Y}_i^{r+1}$ annihilate the sheaf Ψ. Thus, we also obtain the following *Lie's inequality*,

$$s \leq nr \tag{8}$$

because the distribution of vector fields annihilating Ψ has rank $nr+n-\dim(\Psi)$, and by hypothesis $\dim(\Psi)$ is greater than n.

The vector fields $\vec{Y}_i^{r+1}$ in M^{r+1} are *lifted*. We write them as a sum of different copies of a vector field in M.

$$\vec{Y}_i^{r+1} = \vec{Y}_i^{(1)} + \ldots + \vec{Y}_i^{(r)} + \vec{Y}_i. \tag{9}$$

They span an integrable distribution, so that there exist analytic functions λ_{ij}^k in M^{r+1} such that,

$$[\vec{Y}_i^{r+1}, \vec{Y}_j^{r+1}] = \sum_{k=1}^{s} \lambda_{ij}^k(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_k^{r+1}, \tag{10}$$

Let us prove that these functions λ_{ij}^k are, in fact, constants. From (9) we have

$$[\vec{Y}_i^{r+1}, \vec{Y}_j^{r+1}] = [\vec{Y}_i^{(1)}, \vec{Y}_j^{(1)}] + \ldots + [\vec{Y}_i^{(r)}, \vec{Y}_j^{(r)}] + [\vec{Y}_i, \vec{Y}_j]$$

and then substituting again (9) and (10) we obtain for all $a = 1, \ldots, r$

$$[\vec{Y}_i^{(a)}, \vec{Y}_j^{(a)}] = \sum_{k=1}^{s} \lambda_{ij}^k(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_k^{(a)} \tag{11}$$

and

$$[\vec{Y}_i, \vec{Y}_j] = \sum_{k=1}^{s} \lambda_{ij}^k(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_k \tag{12}$$

These vector fields act exclusively in their respective copies of M. Then $\sum_{k=1}^{s} \lambda_{ij}^k \vec{Y}_k$ is a vector field in M, and $\sum_{k=1}^{s} \lambda_{ij}^k \vec{Y}_k^{(a)}$ is also a vector field in $M^{(a)}$, the a-th component in the cartesian power M^{r+1}. There is an expression in local coordinates for the vector fields $\vec{Y}_i$,

$$\vec{Y}_i = \sum_{l=1}^{n} \xi_{il}(x)\frac{\partial}{\partial x_l},$$

and

$$\vec{Y}_i^{(a)} = \sum_{l=1}^{n} \xi_{il}(x^{(a)})\frac{\partial}{\partial x_l}.$$

Thus,

$$\sum_{k=1}^{s}\lambda_{ij}^{k}(x^{(1)},\ldots,x^{(r)},x)\vec{Y}_k=\sum_{k=1}^{s}\sum_{l=1}^{n}\lambda_{ij}^{k}(x^{(1)},\ldots,x^{(r)},x)\xi_{kl}(x)\frac{\partial}{\partial x_l}$$

is a vector field in M, and for each a,

$$\sum_{k=1}^{s}\lambda_{ij}^{k}(x^{(1)},\ldots,x^{(r)},x)\vec{Y}_k^{(a)}=\sum_{k=1}^{s}\sum_{l=1}^{n}\lambda_{ij}^{k}(x^{(1)},\ldots,x^{(r)},x)\xi_{kl}(x^{(a)})\frac{\partial}{\partial x_l^{(a)}}$$

is a vector field in $M^{(a)}$. Therefore, the coefficients of $[\vec{Y}_i,\vec{Y}_j]$,

$$\sum_{k=1}^{s}\lambda_{ij}^{k}(x^{(1)},\ldots,x^{(r)},x)\xi_{kl}(x); \tag{13}$$

depend only on $x_1,\ldots,x_n$; and analogously for each a varying from 1 to r the coefficients of $[\vec{Y}_i^{(a)},\vec{Y}_j^{(a)}]$,

$$\sum_{k=1}^{s}\lambda_{ij}^{k}(x^{(1)},\ldots,x^{(r)},x)\xi_{kl}(x^{(a)}) \tag{14}$$

depend only on the coordinate functions $x_1^{(a)},\ldots,x_n^{(a)}$. Fix $1\leq\alpha\leq n$. Let us prove that $\frac{\partial\lambda_{ij}^k}{\partial x_\alpha}=0$ for all $i=1,\ldots,s$, $j=1,\ldots,s$ and $k=1,\ldots,s$. The same argument is valid for $\frac{\partial\lambda_{ij}^k}{\partial x_\alpha^\beta}$, just interchanging the factors of the cartesian power M^{r+1}.

The expressions (14) do not depend on x_α, and then their partial derivative vanish,

$$\sum_{k=1}^{s}\frac{\partial\lambda_{ij}^k}{\partial x_\alpha}\xi_{kl}(x^{(a)})=0. \tag{15}$$

We can write these expressions together in matrix form,

$$\begin{pmatrix}\xi_{11}(x^{(1)}) & \xi_{21}(x^{(1)}) & \ldots & \xi_{s1}(x^{(1)})\\ \xi_{12}(x^{(1)}) & \xi_{22}(x^{(1)}) & \ldots & \xi_{s2}(x^{(1)})\\ \vdots & \vdots & \ddots & \vdots\\ \xi_{1n}(x^{(1)}) & \xi_{2n}(x^{(1)}) & \ldots & \xi_{sn}(x^{(1)})\\ \xi_{11}(x^{(2)}) & \xi_{21}(x^{(2)}) & \ldots & \xi_{s1}(x^{(2)})\\ \vdots & \vdots & & \vdots\\ \vdots & \vdots & & \vdots\\ \xi_{1n}(x^{(r)}) & \xi_{2n}(x^{(r)}) & \ldots & \xi_{sn}(x^{(r)})\end{pmatrix}\begin{pmatrix}\frac{\partial\lambda_{ij}^1}{\partial x_\alpha}\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \frac{\partial\lambda_{ij}^s}{\partial x_\alpha}\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ \vdots\\ 0\end{pmatrix}; \tag{16}$$

for all $i=1,\ldots,s$ and $j=1,\ldots,s$. Then, the vector $\left(\frac{\partial\lambda_{ij}^1}{\partial x_\alpha},\ldots,\frac{\partial\lambda_{ij}^s}{\partial x_\alpha}\right)$ is a solution of a linear system of $n\cdot r$ equations. If we prove that the matrix of coefficients above is of maximal rank, then from Lie's inequality (8) we know that this system has only a trivial solution and $\frac{\partial\lambda_{ij}^k}{\partial x_\alpha}=0$.

In order to do that, let us consider the natural projections,

$$\pi\colon M^{(1)}\times\ldots\times M^{(r)}\times M\to M^{(1)}\times\ldots\times M^{(r)},$$

onto the first r factors and

$$\pi_1\colon M^{(1)}\times\ldots\times M^{(r)}\times M\to M,$$

onto the last factor.

Vector fields $\vec{Y}_k^{r+1}$ are projectable by π, and their projection is

$$\pi_*(\vec{Y}_k^{r+1})=\vec{Y}_k^{(1)}+\ldots+\vec{Y}_k^{(r)},$$

The expression of $\pi_*(\vec{Y}_k^{r+1})$ in local coordinates is precisely,

$$\pi_*(\vec{Y}_k^{r+1})=\sum_{l=1}^{n}\sum_{m=1}^{r}\xi_{kl}(x^{(m)})\frac{\partial}{\partial x_l^{(m)}},$$

the k-th column of the matrix of coefficients of (16). We can state that $\frac{\partial\lambda_{ij}^k}{\partial x_\alpha}$ vanish if and only if the vector fields $\pi_*(\vec{Y}_1^{r+1}),\ldots,\pi_*(\vec{Y}_s^{r+1})$ are generically linearly independents. Assume that there is a non-trivial linear combination equal to zero,

$$\sum_{k=1}^{s}F_k(x^{(1)},\ldots x^{(r)})\pi_*(\vec{Y}_k^{r+1})=\sum_{k=1}^{s}\sum_{m=1}^{r}F_k(x^{(1)},\ldots x^{(r)})Y_k^{(m)}=0$$

the fields $\vec{Y}_k^{r+1}$ are generically linearly independent, and then the vector field

$$\vec{Z}=\sum_{k=1}^{s}F_k(x^{(1)},\ldots x^{(r)})\vec{Y}_k^{r+1}=\sum_{k=1}^{s}F_k(x^{(1)},\ldots x^{(r)})\vec{Y}_k$$

is different from zero. The vector field $\vec{Z}$ annihilates simultaneously the sheaves Ψ and $\pi_1^*(\mathcal{O}_M)$. It annihilates Ψ because it is a linear combination of the $\vec{Y}_k^{r+1}$. Moreover it annihilates $\pi_1^*(\mathcal{O}_M)$, because it is a linear combination of the $\vec{Y}_k$. This is in contradiction with the transversality condition of Definition 3.6.

We have proven that the $\pi_*(\vec{Y}_n^{r+1}),\ldots\pi_*(\vec{Y}_n^{r+1})$ are linearly independent, and then that the partial derivatives $\frac{\partial\lambda_{ij}^k}{\partial x_\alpha}$ vanish. If we fix a superindex (a) between 1 and r, the same argument is valid for the partial derivatives $\frac{\partial\lambda_{ij}^k}{\partial x_\alpha^{(a)}}$. Hence, the λ_{ij}^k are constants $c_{ij}^k\in\mathbb{C}$,

$$[\vec{Y}_i,\vec{Y}_j]=c_{ij}^k\vec{Y}_k, \tag{17}$$

and the vector fields $\vec{Y}_1,\ldots,\vec{Y}_s$ span a s-dimensional Lie algebra in M, the enveloping algebra of $\vec{X}$.

Finite Dimensional Enveloping Algebra Implies Local Superposition

Reciprocally, let us assume that $\vec{X}$ has a finite dimensional enveloping algebra. Throughout this section we assume that this enveloping algebra acts *transitively* in M. It means that for a generic point x of M the values of elements of $\mathfrak{g}(\vec{X})$ at x span the whole tangent space:

$$\mathfrak{g}(\vec{X})_x = T_xM.$$

This assumption of transitivity can be done in the local case without any lose of generality. If the $\mathfrak{g}(\vec{X})$ is not transitive, it gives a non-trivial foliation of M. We can then substitute the integral leaves of the foliation for the phase space M.

Lemma 4.3. *Let us consider $\vec{Y}_1, \ldots, \vec{Y}_m$ vector fields in M, $\mathbb{C}$-linearly independent but generating a distribution which is generically of rank less than m. Then there is a natural number r such that the lifted vector fields $\vec{Y}_1^r$, $, \vec{Y}_m^r$ generate a generically regular distribution of rank m of vector fields of M^r.*

Proof. We use an induction argument on m. The initial case is true, because a non null vector field generate a distribution which is generically regular of rank 1. Let us assume that the lemma is proven for the case of m vector fields. Consider $m+1$ vector fields $\vec{Y}_1, \ldots, \vec{Y}_m, \vec{Y}$. We can substitute some cartesian power M^r for M and apply the induction hypothesis on $\vec{Y}_1, \ldots, \vec{Y}_m$. Thus, we can assume that $Y_1^r, \ldots, Y_m^r$ span a generically regular distribution of vector fields in M^r. From now on, we denote N for the cartesian power M^r, and $\vec{Z}_i$ for the lifted vector field $\vec{Y}_i^r$.

If the distribution spanned by $\vec{Z}_1, \ldots, \vec{Z}_m, \vec{Z}$ is not of rank $m+1$, then $\vec{Z}$ can be written as a linear combination of the others with coefficients functions in N,

$$\vec{Z} = \sum_{i=1}^m f_i(x)\vec{Z}_i.$$

Consider $N^2 = N^{(1)} \times N^{(2)}$. Let us prove that the distribution generated by $\vec{Z}_1^2, \ldots, \vec{Z}_m^2, \vec{Z}^2$ is generically regular of rank $m+1$. Using *reductio ad absurdum* let us assume that the rank is less that $m+1$. Then Z^2 is a linear combination of $\vec{Z}_1^2, \ldots, \vec{Z}_m^2$ with coefficients functions in N^2,

$$\vec{Z}^2 = \sum_{i=1}^m g_i(x^{(1)}, x^{(2)})(\vec{Z}_i^{(1)} + Z_i^{(2)}), \tag{18}$$

on the other hand,

$$\vec{Z}^2 = \vec{Z}^{(1)} + \vec{Z}^{(2)} = \sum_{i=1}^m f_i(x^{(1)})\vec{Z}_i^{(1)} + f_i(x^{(2)})\vec{Y}_i^{(2)}. \tag{19}$$

Equating (18) and (19) we obtain,

$$g_i(x^{(1)}, x^{(2)}) = f_i(x^{(1)}) = f_i(x^{(2)}), \quad i = 1, \ldots, m.$$

Hence, the functions g_i are constants $c_i \in \mathbb{C}$ but in such case,

$$\vec{Z} = \sum_{k=1}^{m} c_i \vec{Z}_i,$$

$\vec{Z}$ is $\mathbb{C}$-linear combination of $\vec{Z}_1, \ldots \vec{Z}_m$, in contradiction with the hypothesis of the lemma of $\mathbb{C}$-linear independence of these vector fields. Then, the rank of the distribution spanned by $\vec{Y}_1^{2r}, \ldots, \vec{Y}_m^{2r}, \vec{Y}^{2r}$ is $m+1$. ∎

Let us take a basis $\{\vec{Y}_1, \ldots \vec{Y}_s\}$ of $\mathfrak{g}(\vec{X})$. The previous lemma says that there exist r such that the distribution generated by $\vec{Y}_1^r, \ldots, \vec{Y}_s^r$ is generically regular of rank s in M^r, the dimension of M^r is nr so that we have again $s \leq nr$. We take one additional factor in the cartesian power, and hence we consider the complex analytic manifold M^{r+1}. We define Ψ as the sheaf of first integrals of the lifted vector fields $\vec{Y}_1^{r+1}, \ldots, Y_s^{r+1}$ in M^{r+1}. These vector fields span a generically Frobenius integrable distribution of rank s, and then Ψ is a sheaf of regular rings of dimension $n + nr - s$, which is greater or equal to n.

Let us see that the foliation $\mathcal{F}_\Psi$ is generically transversal to the fibers of the projection $\pi\colon M^{r+1} \to M^r$. By the *transitivity* hypothesis on $\mathfrak{g}(\vec{X})$, the tangent space to these fibers is spanned by the s vector fields $\vec{Y}_k$, where k varies from 1 to s. Let $\vec{Z}$ be a vector field tangent to the fibers of π. If this vector field is tangent to $\mathcal{F}_\Psi$ then it is a linear combination of the $\vec{Y}_i^{r+1}$ with coefficients in $\mathcal{O}_{M^{r+1}}$,

$$\vec{Z} = \sum_{i=1}^{s} F_i(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_i^{r+1} = \sum_{i=1}^{s} F_i Y_i^r + \sum_{i=1}^{s} F_i Y_i,$$

and from this we obtain that,

$$\sum_{i=1}^{s} F_i(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_i^r = 0, \tag{20}$$

$$\sum_{i=1}^{s} F_i(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_i^{r+1} = \sum_{i=1}^{s} F_i(x^{(1)}, \ldots, x^{(r)}, x)\vec{Y}_i \tag{21}$$

where $\vec{Y}_i^r = \vec{Y}_i^{(1)} + \ldots + \vec{Y}_i^{(r)}$. The vector fields $\vec{Y}_1^r, \ldots, \vec{Y}_s^r$ span a distribution which is generically of rank s by construction. And, if considered as vector fields in M^{r+1}, they do not depend of the last factor in the cartesian power. Then we can specialize the functions F_i to some fixed value $x \in M$ in the last component, for which the linear combination (20) is not trivial. We obtain an expression,

$$\sum_{i=1}^{s} G_i(x^{(1)}, \ldots, x^{(r)})\vec{Y}_i = 0, \tag{22}$$

that gives us is a linear combination of the $\vec{Y}_i$ with coefficients functions if M^r. Vector fields $\vec{Y}_i$ are $\mathbb{C}$-linearly independent; therefore this linear combination (22) is trivial and the functions G_i vanish. These functions are the restriction of the functions F_i to arbitrary values in the last factor of M^{r+1}, then the functions F_i also vanish. We conclude that the vector field $\vec{Z}$ is zero, and $\mathcal{F}_\Psi$ intersect transversally the fibers of the projection π.

5. Actions of complex analytic Lie groups

5.1. Complex Analytic Lie Groups.

Definition 5.1. A complex analytic Lie Group G is a group endowed with an structure of complex analytic manifold. This structure is compatible with the group law in the sense of that the map:

$$G \times G \to G, \quad (\sigma, \tau) \mapsto \sigma\tau^{-1}$$

is complex analytic.

From now on, let us consider a complex analytic Lie group G. We say that a subset H is a complex analytic subgroup of G if it is both a subgroup and a complex analytic submanifold of G. In the usual topology, the closure $\bar{H}$ of a subgroup of G is just a Lie subgroup, and it is not in general complex analytic. Instead of $\bar{H}$ we consider the *complex analytic closure*,

$$\tilde{H} = \bigcap_{H \subset G'} G' \qquad G' \text{ complex analytic.}$$

This is the smaller complex analytic subgroup containing H.

Invariant Vector Fields

Each element $\sigma \in G$ defines two biholomorphisms of G as complex analytic manifold. They are the right translation R_σ and the left translation L_σ:

$$R_\sigma(\tau) = \tau\sigma, \quad L_\sigma(\tau) = \sigma\tau.$$

The following result is true in the general theory of groups. It is quite interesting that right translations are the symmetries of left translations and viceversa. This simple fact has amazing consequences in the theory of differential equations.

Remark 5.2. Let $f\colon G \to G$ be a map. Then, there is an element σ such that f is the right translation R_σ if and only if f commutes with all left translations.

Let us consider $\mathfrak{X}(G)$ the space of all regular complex analytic vector fields in G. It is, in general, an infinite dimensional Lie algebra over $\mathbb{C}$ – with the Lie bracket of derivations –. A biholomorphism f of G transform regular vector field into regular vector fields. By abuse on notation we denote by the same symbol f the automorphism of the Lie algebra of regular vector fields.

$$f\colon \mathfrak{X}(G) \to \mathfrak{X}(G), \qquad \vec{X} \mapsto f(\vec{X}) \quad f(\vec{X})_\sigma = f'(\vec{X}_{f^{-1}(\sigma)})$$

Definition 5.3. A regular vector field $\vec{A} \in \mathfrak{X}(G)$ is called a right invariant vector field if for all $\sigma \in G$ the right translation R_σ transform $\vec{A}$ into itself. A regular vector field $\vec{A} \in \mathfrak{X}(G)$ is called a left invariant vector field if for all $\sigma \in G$ the right translation L_σ transforms $\vec{A}$ into itself.

For any tangent vector $\vec{A}_e \in T_eG$ in the tangent space to the identity element of G there is an only right invariant vector field $\vec{A}$ which takes the value $\vec{A}_e$ at e. Then the space of right invariants vector fields is a $\mathbb{C}$-vector space of the same dimension as G. The Lie bracket of right invariant vector fields is a right invariant vector field. *We denote by $\mathcal{R}(G)$ the Lie algebra of right invariants vector fields in G.* The same discussion holds for left invariants vector fields. *We denote by $\mathcal{L}(G)$ the Lie algebra of left invariants vector fields in G.*

Note that the inversion morphism that sends σ to σ^{-1} is an involution that conjugates right translations with left translations. Then, it sends right invariant vector fields to left invariant vector fields. Thus, $\mathcal{R}(G)$ and $\mathcal{L}(G)$ are isomorphic Lie algebras.

Exponential Map

Let us consider $\vec{A} \in \mathcal{R}(G)$. There is a germ of solution curve of $\vec{A}$ that sends the origin to the identity element $e \in G$. By the composition law, it extend to a globally defined curve:

$$\sigma\colon \mathbb{C} \to G, \quad t \mapsto \sigma_t.$$

This map is a group morphism and $\{\sigma_t\}_{t\in\mathbb{C}}$ is a monoparametric subgroup of G.

Definition 5.4. We call exponential map to the application that assigns to each right invariant vector field $\vec{A}$ the element σ_1 of the solution curve σ_t of $\vec{A}$:

$$\exp\colon \mathcal{R}(G) \to G, \quad \vec{A} \mapsto \exp(\vec{A}) = \sigma_1.$$

The exponential map is defined in the same way for left invariant vector fields. If $\vec{A} \in \mathcal{R}(G)$ and $\vec{B} \in \mathcal{L}(G)$ take the same value at e, then they share the same solution curve $\{\sigma_t(e)\}_{t\in\mathbb{C}}$, and $\exp(t\vec{A}) = \exp(t\vec{B})$.

If $\vec{A}$ is a right invariant vector field, then the exponential of $\vec{A}$ codifies the flow of $\vec{A}$ as vector field. This flow is given by the formula:

$$\Phi^{\vec{A}}\colon \mathbb{C} \times G \to G, \quad \Phi^{\vec{A}}\colon (t,\sigma) \mapsto \exp(t\vec{A}) \cdot \sigma.$$

The same is valid for left invariant vector fields. If $\vec{B}$ is a left invariant vector field then the flow of $\vec{B}$ is:

$$\Phi^{\vec{B}}\colon \mathbb{C} \times G \to G, \quad \Phi^{\vec{B}}\colon (t,\sigma) \mapsto \sigma \cdot \exp(t\vec{B}).$$

Finally we see that *right invariant vector fields are infinitesimal generators of monoparametric groups of left translations.* Reciprocally left invariant vector fields are infinitesimal generators of monoparametric groups of right translations. The following statement is the infinitesimal version of the commutation of right and left translations:

Remark 5.5. Assume G is connected. Let $\vec{X}$ be a vector field in G. It is a right invariant vector field if and only if for all left invariant vector field $\vec{B}$ in G the Lie bracket, $[\vec{B},\vec{X}]$, vanish.

The same is true if we change the roles of right and left in the previous statement. We have seen that the Lie algebra of symmetries of right invariant vector fields is the algebra of left invariant vector fields and viceversa:

$$[\mathcal{R}(G), \mathcal{L}(G)] = 0.$$

5.2. Complex Analytic Homogeneous Spaces.

Definition 5.6. A G-space M is a complex analytic manifold M endowed with a complex analytic action of G,

$$G \times M \xrightarrow{a} M, \quad (\sigma, x) \mapsto \sigma \cdot x.$$

Let M be a G-space. For a point $x \in M$ we denote by H_x the *isotropy subgroup* of elements of G that let x fixed, and by O_x the *orbit* $G \cdot x$ of x. Let us remind that an action is said to be *transitive* if there is a unique orbit; it is said *free* if for any point x the isotropy H_x consist of the identity element $e \in G$. The kernel K of the action is the intersection of all isotropy subgroups $\cap_{x \in M} H_x$. It is a normal subgroup of G. The action of G in M is said to be *faithful* if its kernel consists only of the identity element.

Definition 5.7. A G-space M is a homogeneous G-space if the action of G in M is transitive. A principal homogeneous G-space M is a G-space such that the action of G in M is free and transitive.

By a morphism of G-spaces we mean a morphism f of complex analytic manifolds which commutes with the action of G, $f(\sigma \cdot x) = \sigma \dot{f}(x)$. Whenever it does not lead to confusion we write homogeneous space instead of G-homogeneous space.

Let M be a G-space. For any x in M the natural map,

$$G \to O_x \quad \sigma \mapsto \sigma \cdot x$$

induces an isomorphism of O_x with the homogeneous space of cosets G/H_x; it follows that any homogeneous space is isomorphic to a quotient of G.

A point $x \in M$ is called a *principal point* if and only if $H_x = \{e\}$, if and only if O_x is a principal homogeneous space. In such case we say that O_x is a *principal orbit.*

Definition 5.8. For a G-space M we denote by M/G the space orbits of the action of G in M.

The main problem of the *invariant theory* is the study of the structure of such quotient spaces. In general this space of orbits M/G can have a very complicated structure, and then it should not exist within the category of complex analytic manifolds.

Fundamental Vector Fields

Let M be a G-space. An element $\vec{A}$ of the Lie algebra $\mathcal{R}(G)$ can be considered as a vector field in the cartesian product $G \times M$ acting only on the first component. This vector field $\vec{A}$ is projectable by the action,

$$a\colon G \times M \to M,$$

and its projection in M is a vector field that we denote by $\vec{A}^M$, the *fundamental vector field* in M induced by $\vec{A}$. Namely, for any $x \in M$ we have:

$$(\vec{A}^M)_x = \frac{d}{dt}(\exp(t\vec{A}) \cdot x).$$

Proposition 5.9. *The map that assign fundamental vector fields to right invariant vector fields,*

$$\mathcal{R}(G) \to \mathfrak{X}(M), \quad \vec{A} \to \vec{A}^M,$$

is a Lie algebra morphism.

The kernel of this map is the Lie algebra of the kernel subgroup of the action, *i.e.* $\cap_{x\in M} H_x$. its image is isomorphic to the quotient $\mathcal{R}(G)/\mathcal{R}\left(\cap_{x\in M} H_x\right)$. Thus, if the action is faithful then it is injective.

Definition 5.10. We denote by $\mathcal{R}(G,M)$ the Lie algebra of fundamental vector fields of the action of G on M.

The flow of fundamental vector fields is given by the exponential map in the group G:

$$\Phi^{\vec{A}^M}\colon \mathbb{C} \times M \to M, \quad (t,x) \mapsto \exp(t\vec{A}) \cdot x.$$

Basis of an Homogeneous Space

Here we introduce the concept of *basis* and *rank* of homogeneous spaces. This concept has been implicitly used by S. Lie and E. Vessiot in their research on superposition laws for differential equations. However, we have not found any modern reference on this point. From now on, let us consider an homogeneous space M.

Definition 5.11. Let S be any subset of M. We denote by H_S to the isotropy subgroup of S:

$$H_S = \bigcap_{x\in S} H_x = \{\sigma \mid \sigma \cdot x = x \ \ \forall x \in S\}.$$

Definition 5.12. We call $\langle S\rangle$, space spanned by $S \subset M$, to the space of invariants of H_S:

$$\langle S\rangle = \{x \mid \sigma \cdot x = x \ \ \forall \sigma \in H_S\}.$$

Definition 5.13. A subset $S \subset M$ is called a system of generators of M as G-space if $\langle S\rangle = M$.

S is a system of generators if and only if the isotropy H_S is coincides with the kernel of the action H_M,

$$\bigcap_{x\in S} H_x = \bigcap_{x\in M} H_x.$$

In particular, for a faithful action, S is a system of generators of M if an only if $H_S = \{e\}$.

Definition 5.14. We call a *basis* of M to a minimal system of generators of M. We say that an homogeneous space is of finite rank if there exist a finite basis of M. The minimum cardinal of basis of M is called the rank of M.

Let us consider an homogeneous space M. The group G acts in each cartesian power M^r component by component, so M^r is a G-space. The following proposition characterizing basis of M is elemental:

Proposition 5.15. *Let us assume that M is a faithful homogeneous space of finite rank r. Let us consider $\bar{x} = (x^{(1)}, \ldots, x^{(r)}) \in M^r$. The following statements are equivalent:*

(1) $\{x^{(1)}, \ldots, x^{(r)}\}$ is a basis of M.

(2) $\bar{x}$ is a principal point of M^r.

(3) $H_{\bar{x}} = \{e\}$.

Corollary 5.16 (Lie's inequality)**.** *If M is a faithful homogeneous space of finite rank r then,*

$$r \cdot \dim M \geq \dim G.$$

Proof. The dimension of M^r is $r\cdot \dim M$; but M^r contains principal orbits $O_{\bar{x}}$ which are isomorphic to G, and then of the same dimension. ■

Proposition 5.17. *Let M be a faithful homogeneous space of finite rank r. The subset $B \subset M^r$ of all principal points of M^r is an analytic open subset.*

Proof. Consider $\{\vec{A}_1, \ldots, \vec{A}_s\}$ a basis of the Lie algebra $\mathcal{R}(G, M^r)$ of fundamental fields in M^r. For each $\bar{x} \in M^r$, those vector fields span the tangent space to $O_{\bar{x}}$,

$$\langle \vec{A}_{1,\bar{x}}, \ldots, \vec{A}_{s,\bar{x}}\rangle = T_{\bar{x}} O_{\bar{x}} \subset T_{\bar{x}} M^r.$$

By Proposition 5.15 there is a principal point $\bar{y} \in M^r$. The dimension of $O_{\bar{y}}$ is s, so that $\vec{A}_{1,\bar{y}}$, $\ldots$, $\vec{A}_{s,\bar{y}}$ are linearly independent. The set

$$B_0 = \{\bar{x}\in M^r \mid \vec{A}_{1,\bar{x}}, \ldots, \vec{A}_{s,\bar{x}} \text{ are linearly independent } \},$$

is the complementary of the analytic closed subset defined by the minors of rank s of the matrix formed by the vectors $\vec{A}_i$. It is non-void, because it contains $\bar{y}$. Thus, B_0 is a non-void *analytic open subset* of M.

By definition, a r-frame $\bar{x}$ is in B_0 if and only if its orbit $O_{\bar{x}}$ is of dimension s, if and only if the isotropy subgroup of $\bar{x}$ is a discrete subgroup of G; therefore $B \subset B_0 \subset M^r$.

The function in B_0 that assigns to each $\bar{x}$ the cardinal $\#H_{\bar{x}}$ of its isotropy subgroup is upper semi-continuous; thus, it reach its minimum 1, along an analytic open subset $B \subset B_0$. This B is the *set of all principal points* of M^r, it is an analytic open subset of B_0 and then also of M^r. ■

5.3. Pretransitive actions and Lie-Vessiot systems.

From now on, let us consider a complex analytic Lie group G, and a *faithful* analytic action of G on M,

$$G \times M \to M, \quad (\sigma, x) \to \sigma \cdot x.$$

$\mathcal{R}(G)$ is the Lie algebra of *right-invariant* vector fields in G. We denote $\mathcal{R}(G, M)$ the Lie algebra of fundamental vector fields of the action of G on M (Definition 5.10).

For each $r \in \mathbb{N}$, G acts in the cartesian power M^r component by component. There is a minimum r such that there are principal orbits in M^r. If M is an homogeneous space then this number r is the *rank* of M (Definition 5.14).

Definition 5.18. We say that the action of G on M is pretransitive if there exists $r \geq 1$ and an analytic open subset $U \subset M^r$ such that:

(a) U is union of principal orbits.

(b) The space of orbits U/G is a complex analytic manifold.

Proposition 5.19. *If M is a G-homogeneous space of finite rank, then the action of G in M is pretransitive.*

Proof. Let r be the rank of M. The set $B \subset M^r$ of *principal points* of M^r is an analytic open subset (Proposition 5.17). Let us see that the space of orbits B/G is a complex analytic manifold. In order to do so we will construct an atlas for B/G. Let us consider the natural projection:

$$\pi \colon B \to B/G, \quad \bar{x} \mapsto \pi(\bar{x}) = O_{\bar{x}}.$$

Let us consider $\bar{x} \in B$; let us construct a coordinate open subset for $\pi(\bar{x})$. $O_{\bar{x}}$ is a submanifold of B. We take a submanifold L of B such that $O_{\bar{x}}$ and L intersect transversally in $\bar{x}$,

$$T_{\bar{x}}U = T_{\bar{x}}O_{\bar{x}} \oplus T_{\bar{x}}L.$$

Let us prove that there is an polydisc $U_{\bar{x}}$, open in L and centered on $\bar{x}$, such that $O_{\bar{x}} \cap U_{\bar{x}} = \bar{x}$. The action of G on B is continuous and free. So that, if the required statement statement holds, then there exists a maybe smaller polydisc $V_{\bar{x}}$ centered in $\bar{x}$ and open in L such that for all $\bar{y} \in V_{\bar{x}}$ the intersection of $O_{\bar{y}}$ with $V_{\bar{x}}$ is reduced to the only point $\bar{y}$. Thus, $V_{\bar{x}}$ projects one-to-one by π.

$$\pi \colon V_{\bar{x}} \xrightarrow{\sim} V_{\pi(\bar{x})} \qquad \bar{y} \mapsto \pi(\bar{y}) = O_{\bar{y}}$$

$V_{\pi(\bar{x})}$ is an open neighborhood of $\pi(x)$. The inverse of π is an homeomorphism of such open subset of B/G with a complex polydisc. In this way we obtain an open covering of the space of orbits. Transitions functions are the holonomy maps of the foliation whose leaves are the orbits. Thus, transition functions are complex analytic and B/G is a complex analytic manifold.

Reasoning by *reductio ad absurdum* let us assume that there is not a polydisc verifying the required hypothesis. Each neighborhood of x in L intersects then O_x in more than one points. Let us take a topological basis,

$$U_1 \supset U_2 \supset \ldots \supset \ldots,$$

of open neighborhoods of $\bar{x}$ in L.

$$\bigcap_{i=1}^{\infty} U_i = \{\bar{x}\}, \quad U_i \cap O_{\bar{x}} = \{\bar{x}, \bar{x}_i, \ldots\}.$$

In this way we construct a sequence $\{\bar{x}_i\}_{i\in\mathbb{N}}$ in $L \cap O_{\bar{x}}$ such that,

$$\lim_{i\to\infty} \bar{x}_i = \bar{x},$$

where $\bar{x}_i$ is different from $\bar{x}$ for all $i \in \mathbb{N}$. The action of G is free, so that, for each $i \in \mathbb{N}$ there is an unique $\sigma_i \in G$ such that $\sigma_i(\bar{x}_i) = \bar{x}$. We write these r-frames in components,

$$\bar{x} = (x^{(1)}, \ldots, x^{(r)}), \quad \bar{x}_i = (x_i^{(1)}, \ldots, x_i^{(r)}),$$

we have that for all $i \in \mathbb{N}$ and $k = 1, \ldots, r$,

$$x^{(k)} = \sigma_i(x_i^{(k)}).$$

For all $i \in \mathbb{N}$ and $k = 1, \ldots, r$, let us denote,

$$H_{x_i^{(k)}, x^{(k)}} = \{\sigma \in G \,|\, \sigma \cdot x_i^{(k)} = x^{(k)}\},$$

the set of all elements of G sending $x_i^{(k)}$ to $x^{(k)}$. This set is the image of the isotropy group $H_{x^{(k)}}$ by a right translation in G. When $x_i^{(k)}$ is closer to $x^{(k)}$ this translation is closer to the identity. We have,

$$\sigma_i \subset \bigcap_{k=1}^{r} H_{x_i^{(k)}, x^{(k)}},$$

and from that,

$$\lim_{i\to\infty} \sigma_i \in \bigcap_{k=1}^{r} H_{x^{(k)}},$$

but this intersection in precisely the isotropy group of $\bar{x}$ and then $\sigma_i \xrightarrow{i\to\infty} e$.

On the other hand there is a neighborhood of $\bar{x}$ in $O_{\bar{x}}$ that intersects L only in $\bar{x}$. So that there is a neighborhood U_e of the identity in G such that $U_e \cdot \bar{x}$ intersects L only in $\bar{x}$. From certain i_0 on, σ_i is in U_e, and then $\sigma_i = e$. Then $\bar{x}_i = \bar{x}$, in contradiction with the hypothesis. ∎

Remark 5.20. Pretransitive actions are not uncommon. For example, if M and N are G homogeneous spaces of finite rank, then $M \times N$ is not in general an homogeneous space, but it is clear that it is a pretransitive space. Also the linear case provides useful examples. For a vector space E it is clear that the action of the linear group $GL(E)$ on any tensor construction on E is pretransitive.

Definition 5.21. A non-autonomous analytic vector field $\vec{X}$ in M is called a *Lie-Vessiot system*, relative to the action of G, if its enveloping algebra is spanned by fundamental fields of the action of G on M; *i.e.* $\mathfrak{g}(\vec{X}) \subset \mathcal{R}(G, M)$.

6. Global Superposition Theorem

In this section we prove our first main result. As mentioned, it is inspired in Vessiot [24] philosophy of extracting the group action from the superposition law.

Theorem 6.1. *A non-autonomous complex analytic vector field $\vec{X}$ in a complex analytic manifold M admits a superposition law if and only if it is a Lie-Vessiot system related to a pretransitive Lie group action in M.*

Pretransitive Lie-Vessiot Systems admit Superposition Laws

Consider a faithful pretransitive Lie group action,

$$G \times M \to M, \quad (\sigma, x) \mapsto \sigma \cdot x,$$

and assume that $\vec{X}$ is a Lie-Vessiot system relative to the action of G. Let us consider the application,

$$\mathcal{R}(G) \to \mathfrak{X}(M), \quad \vec{A} \mapsto \vec{A}^M,$$

that send a right-invariant vector field in G to its corresponding fundamental vector field in M. Let $\{A_1, \ldots, A_s\}$ be a basis of $\mathcal{R}(G)$, and denote by $\vec{X}_i$ their corresponding fundamental vector fields $\vec{A}_i^M$. Then,

$$\vec{X} = \partial + \sum_{i=1}^{s} f_i(t)\vec{X}_i, \quad f_i(t) \in \mathcal{O}_S.$$

Definition 6.2. The following vector field in $S \times G$, $\vec{A} = \partial + \sum_{i=1}^{s} f_i(t)\vec{A}_i$, is called the *automorphic system* associated to $\vec{X}$.

From the definition of fundamental vector fields we know that if $\sigma(t)$ is a solution of $\vec{A}$, then $\sigma(t) \cdot x_0$ is a solution of $\vec{X}$ for any $x_0 \in M$. The automorphic system is a particular case of a Lie-Vessiot system. The relation between solutions of the Lie-Vessiot system $\vec{X}$ and its associated automorphic system $\vec{A}$ leads to Lie's reduction method and representation formulas as exposed in [2].

Proposition 6.3. *The automorphic system satisfies:*

(i) *Let $\vec{A}$ be the automorphic system associated to $\vec{X}$, and $\sigma(t)$ a solution of $\vec{A}$ defined in $S' \subset S$. Then, $\sigma(t) \cdot x_0$ is a solution of $\vec{X}$.*

(ii) *The automorphic system associated to $\vec{A}$ is $\vec{A}$ itself.*

(iii) *The group law $G \times G \to G$ is a superposition law admitted by the automorphic system; thus, $\varphi(\sigma; \lambda) = \sigma \cdot \lambda$.*

(iv) *If $\sigma(t)$ and $\tau(t)$ are two solutions of $\vec{A}$, then $\sigma(t)^{-1}\tau(t)$ is a constant element of G.*

Proof. (i) The tangent vector a $t = t_0$ to the curve $\sigma(t) \cdot x_0$ is precisely the projection by the action of the tangent vector to $\sigma(t)$ at $t = t_0$. This is by definition, $\vec{A}^M_{\sigma(t_0)\cdot x_0}$.
(ii) When we consider the case of a Lie group acting on itself, its algebra of fundamental fields is its algebra of right invariant fields. We have $\vec{A}^G_i = \vec{A}_i$.
(iii) It is a direct consequence of (i).
(iv) Let $\sigma(t)$ and $\tau(t)$ be two solutions of $\vec{A}$. By (iii) we know that the group law is a superposition law. Thus, there is a constant $\lambda \in G$ such that $\sigma(t) \cdot \lambda = \tau(t)$. Then, we have $\sigma(t)^{-1}\tau(t) = \lambda$. ■

There exists an analytic open subset $W \subset M^r$ which is union of principal orbits, and there exist the quotient W/G as an analytic manifold. In such case the bundle,

$$\pi \colon W \to W/G,$$

is a *principal bundle* modeled over the group G. Consider a section s of π defined in some analytic open subset $V \subset W/G$. Let U be the preimage of V for π. The section s allow us to define a surjective function,

$$g \colon U \to G, \quad \bar{x} \to g(\bar{x}) \quad g(\bar{x}) \cdot s(\pi(\bar{x})) = \bar{x}$$

which is nothing but the usual trivialization function. We define the following maps φ and ψ,

$$\varphi \colon U \times M \to M, \quad (\bar{x}, y) \mapsto g(\bar{x}) \cdot y$$
$$\psi \colon U \times M \to M, \quad (\bar{x}, y) \mapsto g(\bar{x})^{-1} \cdot y.$$

Lemma 6.4. *The section φ is a superposition law for $\vec{X}$, and $\Psi = \psi^*\mathcal{O}_M$ is a local superposition law for $\vec{X}$.*

Proof. Let us consider the map $g \colon U \to G$. By construction it is a morphism of G-spaces: it verifies $g(\sigma \cdot \bar{x}) = \sigma \cdot g(\bar{x})$. Therefore, it maps fundamental vector fields in U to fundamental vector fields in G. Let us remember that fundamental vector fields in G are right invariant vector fields. We have that $\vec{X}^r$ is projectable by $Id \times g$,

$$Id \times g \colon S \times U \to S \times G, \quad \vec{X}^r \mapsto \vec{A},$$

and the image of $\vec{X}^r$ is the automorphic vector field $\vec{A}$.

Finally, let us consider r particular solutions of $\vec{X}$, $x^{(1)}(t),\ldots,x^{(r)}(t)$. We denote by $\bar{x}(t)$ the curve $(x^{(1)}(t),\ldots,x^{(r)}(t))$ in M^r. Therefore, $\bar{x}(t)$ is a solution of $\vec{X}^r$ in M^r. Hence, $g(\bar{x}(t))$ is a solution of the automorphic system $\vec{A}$ in G, and for $x_0\in M$ we obtain by composition $g(\bar{x}(t))\cdot x_0$, which is a solution of $\vec{X}$. By the uniqueness of local solutions we know that when x_0 varies in M we obtain the general solution. Then,

$$\varphi(x^{(1)}(t),\ldots,x^{(r)}(t),x_0)=g(\bar{x}(t))\cdot x_0$$

is the general solution and φ is a superposition law for $\vec{X}$.

With respect to ψ, let us note that it is a partial inverse for φ in the sense,

$$\psi(\bar{x},\varphi(\bar{x},x))=x,\quad \varphi(\bar{x},\psi(\bar{x},x))=x.$$

Hence, if $(\bar{x}(t),x(t))$ is a solution of $\vec{X}^{r+1}$ then $\varphi(\bar{x}(t),\psi(\bar{x}(t),x(t))=x(t)$ is a solution of $\vec{X}$ and then $\psi(\bar{x}(t),x(t))$ is a constant point x_0 of M, and $\Psi=\psi^*\mathcal{O}_M$ is a sheaf of first integrals of $\vec{X}^r$. ■

Superposition Law implies Pretransitive Lie-Vessiot

Consider a superposition law for $\vec{X}$,

$$\varphi\colon U\times M\to M.$$

First, let us make some consideration on the open subset $U\subset M^r$ appearing as a factor in the domain of definition of φ. Consider the family $\{(\varphi_\lambda,U_\lambda)\}_{\lambda\in\Lambda}$ of different superposition laws admitted by $\vec{X}$, φ_λ defined in $U_\lambda\times M$. There is a natural partial order in this family: we write $\lambda<\gamma$ if $U_\lambda\subset U_\gamma$ and φ_λ is obtained from φ_γ by restriction: $\varphi_\gamma|_{U_\lambda\times M}=\varphi_\lambda$. For a totally ordered subset $\Gamma\subset\Lambda$ we construct a supreme element by setting,

$$U_\Gamma=\bigcup_{\gamma\in\Gamma}U_\gamma,$$

and defining φ_Γ as the unique function defined in U_Γ compatible with the restrictions. By Zorn lemma, we can assure that there exist a superposition law φ defined in some $U\times M$ which is maximal with respect to this order. From now on *we assume that the considered superposition law is maximal.* It is clear that

$$\bar{\varphi}\colon U\times M^r\to M^r,\quad (\bar{x},\bar{y})\mapsto(\varphi(\bar{x},y^{(1)}),\varphi(\bar{x},y^{(2)}),\ldots,\varphi(\bar{x},y^{(r)})),$$

is a superposition formula for the lifted Lie-Vessiot system $\vec{X}^r$ in M^r. For each $\bar{x}\in U$ we consider the map

$$\sigma_{\bar{x}}\colon M\to M,\quad x\mapsto\varphi(\bar{x},x).$$

It is clear that $\sigma_{\bar{x}}$ is a complex analytic automorphism of M. We denote by $\bar{\sigma}_{\bar{x}}$ for the cartesian power of $\sigma_{\bar{x}}$ acting component by component,

$$\bar{\sigma}_{\bar{x}}\colon M^r\to M^r,\quad \bar{y}\mapsto\bar{\varphi}(\bar{x},\bar{y}).$$

The map $\bar{\sigma}_{\bar{x}}$ is a complex analytic automorphism of M^r.

Lemma 6.5 (Extension Lemma). *If $\bar{x}$ and $\bar{y}$ are in U and there exist $\bar{z} \in M^r$ such that $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$, then $\bar{z}$ is also in U.*

Proof. Let us consider $\bar{x}$, $\bar{y}$, $\bar{z}$ in U such that $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$. Then we have a commutative diagram

$$\begin{tikzcd}
M \arrow[dr, "\sigma_{\bar{x}}"] & \\
M \arrow[u, "\sigma_{\bar{z}}"] \arrow[r, "\sigma_{\bar{y}}"] & M
\end{tikzcd}$$

and it is clear that $\sigma_{\bar{z}} = \sigma_{\bar{y}} \sigma_{\bar{x}}^{-1}$, or equivalently

$$\varphi(\bar{z}, \xi) = \varphi(\bar{y}, \sigma_{\bar{x}}^{-1}(\xi)) \tag{23}$$

for all ξ in M. Now let us consider $\bar{x}$ and $\bar{y}$ in U and $\bar{z} \in M^r$ verifying the same relation $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$. We can define $\varphi(\bar{z}, \xi)$ as in equation (23).

Let V be the set of all r-frames $\bar{z}$ satisfying $\varphi(\bar{x}, \bar{z}) = \bar{y}$ for certain $\bar{x}$ and $\bar{y}$ in U. Let us see that V is an open subset containing U. Relation $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$ is equivalent to $\bar{z} = \bar{\sigma}_{\bar{x}}^{-1}(\bar{y})$, and then it is clear that

$$V = \bigcup_{\bar{x} \in U} \bar{\sigma}_{\bar{x}}^{-1}(U),$$

which is union of open subsets, and then it is an open subset. The superposition law φ naturally extends to $V \times M$, by means of formula (23). Because of the maximality of this superposition law, we conclude that U coincides with V and then such a $\bar{z}$ is in U. ∎

Lemma 6.6. *Let G be following set of automorphisms of M,*

$$G = \{\sigma_{\bar{x}} \mid \bar{x} \in U\}.$$

Then G is a group of automorphisms of M.

Proof. Let us consider two elements σ_1, σ_2 of G. They are respectively of the form $\sigma_{\bar{x}}$, $\sigma_{\bar{y}}$ with $\bar{x}$ and $\bar{y}$ in U. $\bar{\varphi}$ is a superposition law for $\vec{X}^r$. From the local existence of solutions for differential equations we know that there exist $\bar{z} \in M^r$ such that $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$. From the previous lemma we know that $\bar{z} \in U$. We have that $\sigma_{\bar{y}} \sigma_{\bar{x}}^{-1} = \sigma_{\bar{z}}$. Then $\sigma_2 \sigma_1^{-1}$ is in G and it is a subgroup of the group of automorphisms of M. ∎

Now let us see that G is endowed with an structure of complex analytic Lie group and that the Lie algebra $\mathcal{R}(G, M)$ of its fundamental fields in M contains the Lie-Vessiot-Guldberg algebra $\mathfrak{g}(\vec{X})$ of $\vec{X}$. The main idea is to translate infinitesimal deformations in U to infinitesimal deformations in G. This approach goes back to Vessiot [24].

Consider $\bar{x}$ in U, and a tangent vector $\vec{v}_x \in T_{\bar{x}} U$. We can project this vector to M by means of the superposition principle $\varphi(\bar{x}, x)$. Letting x as a free variable we define a vector field $\vec{V}$ in M,

$$\vec{V}_{\varphi(\bar{x}, x)} = \varphi'_{(\bar{x}, x)}(\vec{v}_x);$$

where $\varphi'_{(\bar{x},x)}$ is the tangent map to φ at $(\bar{x}, x)$:

$$\varphi'_{(\bar{x},x)}\colon T_{(\bar{x},x)}(U \times M) \to T_{\varphi(\bar{x},x)}M.$$

This assignation is linear, and then we obtain an injective map

$$T_{\bar{x}}U \to \mathfrak{X}(M), \quad \vec{v}_x \mapsto \vec{V}$$

that identifies $T_{\bar{x}}U$ with a finite dimensional space of vector fields in M which we denote by $\mathfrak{g}_{\bar{x}}$. Let us see that this space does not depends on $\bar{x}$ in U. Consider another r-frame $\bar{y} \in U$. There exist a unique $\bar{z} \in U$ such that $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$. Consider the map,

$$R_{\bar{z}}\colon U \to M^r, \quad \xi \mapsto \bar{\varphi}(\xi, \bar{z}), \quad R_{\bar{z}}(\bar{x}).$$

We have $\bar{\varphi}(\bar{y}, \xi) = \bar{\varphi}(R_{\bar{z}}\bar{x}, \xi)$, and then the vector field $\vec{V}$ induced in M by the tangent vector $\vec{v}_x \in T_{\bar{x}}U$ is the same vector field that the induced by the tangent vector $R'_{\bar{z}}(\vec{v}) \in T_{\bar{y}}U$. Then we conclude that $\mathfrak{g}_{\bar{x}} \subseteq \mathfrak{g}_{\bar{y}}$. We can gave the same argument for the reciprocal by taking $\bar{w}$ such that $\bar{\varphi}(\bar{y}, \bar{w}) = \bar{x}$. Then $\mathfrak{g}_{\bar{x}} = \mathfrak{g}_{\bar{y}}$. We denote by $\mathfrak{g}$ this finite dimensional space of vector fields in M and s its complex dimension. This space $\mathfrak{g}$ is a quotient of $T_{\bar{x}}U$, so that Lie's inequality $s \leq nr$ holds.

Let us see that $\mathfrak{g}$ is a Lie algebra. We can invert, at least locally, the superposition law with respect to the last component,

$$x = \varphi(\bar{x}, \lambda), \quad \lambda = \psi(\bar{x}, x).$$

Denote by Ψ the sheaf generated by the components ψ_i of these local inversions; Ψ is a local superposition law. Consider $\vec{v} \in T_{\bar{x}}U$ and $\vec{V}$ the induced vector field $\vec{V} \in \mathfrak{g}$. It is just an observation that $\vec{V}^r_{\bar{x}}$, the value at $\bar{x}$ of the r-th cartesian power of $\vec{V}$ also induces the same vector field $\vec{V}$. Then, in the language of small displacements ,

$$x + \varepsilon\vec{V}_x = \varphi(\bar{x} + \varepsilon\vec{V}^r_{\bar{x}}, \lambda), \quad \lambda = \psi(\bar{x} + \varepsilon\vec{V}^r_{\bar{x}}, x + \varepsilon\vec{V}_x).$$

Then $\vec{V}^{r+1}\psi_i = 0$, and Ψ is a sheaf of first integrals of $r+1$ cartesian powers of the fields of $\mathfrak{g}$. By using the the same argument that in the proof of Theorem 4.1 we conclude that $\mathfrak{g}$ spans a finite dimensional Lie algebra.

Now let us consider the surjective map $\pi\colon U \to G$. For $\sigma \in G$ the preimage $\pi^{-1}(\sigma)$ is defined by analytic equations,

$$\pi^{-1}(\sigma) = \{\bar{x} \in U \mid \forall x \in M \ \varphi_i(\bar{x}, x) = \sigma(x)\}. \tag{24}$$

Let us see that $\pi^{-1}(\sigma)$ is a closed sub-manifold of U. Let us compute the tangent space to $\pi^{-1}(\sigma)$ at $\bar{x}$. A tangent vector $\vec{v} \in T_{\bar{x}}U$ is tangent to the fiber of σ if and only if $\vec{v}$ is into the kernel of the canonical map $T_{\bar{x}}U \to \mathfrak{g}$. Therefore, $\pi^{-1}(\sigma)$ has constant dimension, so that it is defined by a finite subset of the equations (24). Hence, the stalk $\pi^{-1}(\sigma)$ is a closed submanifold of U of dimension $nr - s$.

In such case there is a unique analytic structure on G such that $\pi\colon U \to G$ is a fiber bundle. Consider $\mathfrak{g}^r$ the Lie algebra spanned by cartesian power vector fields,

$$\vec{V}^r = \vec{V}^{(1)} + \ldots + \vec{V}^{(r)},$$

with $\vec{V} \in \mathfrak{g}$. This Lie algebra $\mathfrak{g}^r$ is canonically isomorphic to $\mathfrak{g}$. The fields of $\mathfrak{g}^r$ are projectable by π. The Lie algebra $\mathfrak{g}$ is then identified with the Lie algebra $\mathcal{R}(G)$ of right invariant vector fields in G. We deduce that $\mathfrak{g}$ is the algebra $\mathcal{R}(G, M)$ of fundamental fields of G in M.

Finally, let us see that the elements of $\mathcal{R}(G, M)$ span the Lie-Vessiot-Guldberg algebra of $\vec{X}$. For all $t_0 \in S$ we have,

$$\varphi(\bar{x} + \varepsilon(\vec{X}^r_{t_0})_{\bar{x}}, \lambda) = x + (\vec{X}_{t_0})_x,$$

and then $\vec{X}_{t_0}$ is the vector field of $\mathfrak{g}$ induced by the tangent vector $(\vec{X}^r_{t_0})_{\bar{x}}$ at any $\bar{x}$ in U. Then $\vec{X}_t \in \mathfrak{g}$ for all $t \in S$ and therefore there are $\vec{V}_i \in \mathfrak{g}$ and analytic functions $f_i(t) \in \mathcal{O}_S$ for $i = 1, \ldots, s$ such that,

$$\vec{X} = \partial + \sum_{i=1}^{s} f_i(t)\vec{V}_i,$$

and $\vec{X}$ is a Lie-Vessiot system in M related to the action of G.

Lemma 6.7. *The action of G on M is pretransitive.*

Proof. First, by Lemma 6.5 the open subset U is union of principal orbits. Let us prove that the space of orbits U/G in a complex analytic manifold. Consider $\pi\colon U \to G$, $\bar{x} \mapsto \sigma_{\bar{x}}$ as in the previous lemma. Let U_0 be preimage of Id, which is a closed submanifold of U. Consider the map,

$$\pi_2\colon U \to U, \quad \bar{x} \mapsto \sigma_{\bar{x}}^{-1}\bar{x},$$

then $\pi(\pi_2(\bar{x})) = Id$, and the image of π_2 is U_0. The two projections $\pi\colon U \to G$ and $\pi_2\colon U \to U_0$ give a decomposition $U = G \times U_0$ and then the quotient $U/G \simeq U_0$ is a complex analytic manifold. We conclude that the action of G on M is pretransitive. ■

We conclude that $\vec{X}$ is a Lie-Vessiot system associated to the pretransitive action of G on M, this ends the proof of theorem 6.1.

Example 6.8. Let us consider Weierstrass equation for the $\wp$ function,

$$\dot{\wp}^2 = 4(\wp^3 + g_2\wp + g_3),$$

the classical addition formula for Weierstrass $\wp$ function,

$$\wp(a+b) = -\wp(a) - \wp(b) - \frac{1}{4}\frac{\dot{\wp}(a) - \dot{\wp}(b)}{\wp(a) - \wp(b)}$$

can be understood as a superposition law for the differential equation,

$$\dot{x} = 2f(t)\sqrt{(x^3 + g_2 x + g_3)}. \tag{25}$$

The general solution of (25) is:

$$x = \wp\left(\int_{t_0}^{t} f(\xi)d\xi + \lambda\right). \tag{26}$$

By taking off the constant λ of the formula (26) we obtain the mentioned superposition law,

$$\varphi(x;\lambda) = -x - \lambda - \frac{1}{2}\frac{\sqrt{x^3 - g_2 x - g_3} - \sqrt{\lambda^3 - g_2\lambda - g_3}}{x - \lambda}.$$

Where the square roots are defined in a double-sheet ramified covering $\mathbb{C}$. If we apply Vessiot global method here, it is clear that we recover the elliptic curve group structure. However, infinitesimal Lie approach gives us no information about this non-linear algebraic group, but just about its Lie algebra. Using Lie's infinitesimal approach, equation (25) is just equivalent to,

$$\dot{x} = f(t).$$

This example can be generalized to addition formulas of abelian functions in several variables. In those cases, non-linear and non-globally linearizable superposition laws appear.

7. Rational Superposition Laws

Using this global philosophy it is easy to jump into the algebraic category. First, it follows easily that rational superposition laws lead to rational actions of algebraic groups. From now on we will assume some background of algebraic geometry. We will assume also that all the algebraic varieties will be over the complex field.

We will denote by M to a fixed algebraic variety (over the complex). We recall that the standard topology in algebraic geometry is the Zariski topology and that a rational map is a map from an open set to an algebraic variety. Furthermore an algebraic group G is a group that is at the same time an algebraic variety such that both structures are compatible; *i.e.*, the group law and taking of inverses are morphisms of algebraic varieties.

Theorem 7.1. *Let $\vec{X}$ be a non-autonomous meromorphic vector field in M that admits a rational superposition law. Then, it is a Lie-Vessiot system related to a algebraic action of an algebraic group on M.*

Proof. First, note that the extension lemma 6.5 also works in the algebraic case. So that we can assume that the superposition law

$$\varphi\colon U \times M \to M$$

is defined in a Zariski open subset $U \subset M^r$ such that, if $\bar{x}, \bar{y}$ are in U and there exist $\bar{z}$ such that $\bar{\varphi}(\bar{x}, \bar{z}) = \bar{y}$ then $\bar{z}$ is also in U. Then, by by the proof of Lemma 6.7 we have a decomposition, $U = G \times U_0$ being U_0 the fiber of the identity in U. Let us see that we can identify G with an algebraic submanifold of U. Let us fix $\bar{\lambda} \in U_0$, then G is identified with the set $\varphi(\bar{x}, \bar{\lambda})$ where $\bar{x}$ varies in U, which is the image of the map:

$$\psi \colon U \to U, \quad \psi(\bar{x}) = \varphi(\bar{x}, \bar{\lambda}),$$

This map ψ is clearly rational. The image of U by a rational map is a constructible set by Chevalley's projection theorem (see [10], Theorem 4.4). That is, a boolean combination of algebraic subsets of U. But, in the other hand, it has been proved that the image of ψ is analytically isomorphic to G (paragraph below Lemma 6.6). Then, it follows that ψ is an embedding of G into U whose image is an algebraic submanifold of U. Then, G inherits this algebraic structure. It is clear that the composition law, taking of inverses and the action are then algebraic. ■

8. Strongly Normal Extensions

From now on, let $\vec{X}$ be a non-autonomous *meromorphic* vector field in an algebraic manifold M with coefficients in the Riemann surface S that admits a rational superposition law. Applying Theorem 7.1, we also consider the algebraic group G that realizes $\vec{X}$ as an algebraic Lie-Vessiot system.

The objective of this section is to prove that a rational differential equation with meromorphic coefficients in a Riemann surface S, admitting a rational superposition law, have its general solution in a differential field extension of the field of meromorphic functions in S which is an *strongly normal extension* in the sense of Kolchin. Then, superposition laws fall into the differential Galois theory developed in [13]. This relation of superposition laws and strongly normal extensions is implicit in the geometric characterization given by Kovacic [14, 15].

The field $\mathcal{M}(S)$ of meromorphic functions in S is a differential field, endowed of the derivation ∂, its field of constants is clearly $\mathbb{C}$. For a differential field K we denote its field of constants by C_K. Let K, L be two differential fields. By a differential field morphism we mean a field morphism $\varphi \colon K \to L$ which commutes with the derivation. When we fix a non-trivial (and then injective) differential field morphism, $K \subset L$ we speak of a differential field extension. By a K-isomorphism of L we mean a differential field morphism from L into a differential field extension of K that fix K pointwise. We will denote by $\mathrm{Aut}_K(L)$ the set of differential automorphisms of L that fix K pointwise.

The following lemmas tell us that the vector field $\vec{X}$ cannot have movable singularities, *i.e.* its satisfy the so called Painlevé property.

Lemma 8.1. *There is a discrete set R in S such that $\vec{X}$ is holomorphic in $M \times S^{\times}$ being $S^{\times} = S \setminus R$. Thus*

$$\vec{X} = \partial + \sum_{i=1}^{s} f_i(t) \vec{X}_i,$$

being the $f_i(t)$ meromorphic functions in S with poles in R.

Proof. Let t_0 be a point of S such that $\vec{X}$ is defined at least at one point of the fiber $t = t_0$ in $M \times S$. Then, it is defined in at least an open subset of the fiber $t = t_0$. It implies that there exist then r local holomorphic solutions $x_1(t), \ldots, x_r(t)$, defined in an in $S' \subset S$ around t_0 that form a fundamental system of solutions. Then, by the superposition law, its follows that for any particular initial condition $x_0 \in M$ there exist $\lambda \in M$ such that $\varphi(x_1(t), \ldots, x_r(t), \lambda)$ is the solution of $\vec{X}$ with initial condition $x(t_0) = x_0$. It follows that the fiber $t = t_0$ does not meet the singular locus of $\vec{X}$.

Thus, the singular locus of $\vec{X}$ should be an union of fibers of the projection $S \times M \to S$. Those fibers should be isolated, because the vector field $\vec{X}$ is meromorphic. ∎

From now on we consider $S^{\times}$ as above, and we set $\widetilde{S^{\times}}$ to be the universal covering of $S^{\times}$.

Lemma 8.2. *Any germ of solution $x(t)$ defined in $S' \subset S^{\times}$ can be prolonged into a solution defined in the universal covering $\widetilde{S^{\times}}$ of $S^{\times}$.*

Proof. It is enough to see that any solution can be analytically prolonged along any differentiable path. Let $\gamma\colon [0,1] \to S^{\times}$ be a differentiable path. For any point $\bar{t}$ in the image of γ we consider a simply connected open neighborhood $U_{\bar{t}}$ such that there exist a fundamental system of solutions $x_1^{\bar{t}}, \ldots x_r^{\bar{t}}$ defined in $U_{\bar{t}}$. Now, we apply that the interval $[0,1]$ is compact, so we can take just a finite number of open subsets $U_1, \ldots, U_n$. In each one of this open subset the general solution exist for any initial condition. So that, we can continue any solution defined around $\gamma(0)$ along γ to $\gamma(1)$. ∎

Let $x(t)$ be a particular solution of $\vec{X}$ defined in $\widetilde{S^{\times}}$. Then, we can consider $\mathcal{M}(S) \subset \mathcal{M}(S)\langle x(t)\rangle$, the differential field extension spanned by the coordinates of $x(t)$ in any affine chart.

Lemma 8.3. *Let $\bar{x}(t) = (x_1(t), \ldots, x_r(t))$ be a fundamental system of solutions of $\vec{X}$, and $\sigma(t)$ a solution of the associated automorphic system $\vec{A}$ in G. Then $\mathcal{M}(S)\langle \bar{x}(t)\rangle = \mathcal{M}(S)\langle \sigma(t)\rangle$.*

Proof. First, there exist a r-tuple $\bar{\lambda}$ in U such that $\bar{x}(t) = \sigma(t) \cdot \bar{\lambda}$. Then, it is clear that the coordinates of $\bar{x}(t)$ are rational functions on the coordinates of $\sigma(t)$, *i.e.* $\mathcal{M}(S)\langle \bar{x}(t)\rangle \subset \mathcal{M}(S)\langle \sigma(t)\rangle$. Let us see the converse. Consider the decomposition $U \simeq G \times U_0$ given in the proof of Lemma 6.7 and let π the projection from U to G. This projection π sends $\bar{x}(t)$ to a solution $\tau(t)$ of $\vec{A}$. Then the coordinates of $\tau(t)$ are, by this projection, rational functions of the coordinates of $\bar{x}(t)$ and we have that $\mathcal{M}(S)\langle \tau(t)\rangle \subset \mathcal{M}(S)\langle x(t)\rangle$. Finally, by Proposition 6.3 (iii), we know that $\tau(t)$ and $\sigma(t)$ are related by a left translation in G, so that $\mathcal{M}(S)\langle \sigma(t)\rangle$ coincides with $\mathcal{M}(S)\langle \tau(t)\rangle$. ∎

By the above Lemma 8.3 any fundamental system of solutions of $\vec{X}$, or equivalently any solution of $\vec{A}$, span the same differential field extension of $\mathcal{M}(S)$. Then, we can define:

Definition 8.4. We call the Galois differential field of $\vec{X}$ to the differential field $\mathcal{M}(S)\langle\bar{x}(t)\rangle$ spanned by the coordinates of a fundamental system of solutions $\bar{x}(t)$, defined in $\widetilde{S^\times}$.

From now on we denote by L to the Galois differential field of $\vec{X}$. We have that $\mathcal{M}(S) \subset L \subset \mathcal{M}(\widetilde{S^\times})$. It is clear that if $x(t)$ is a particular solution of $\vec{X}$ then $\mathcal{M}(S)\langle x(t)\rangle$ is contained in L.

Definition 8.5. Let K be a differential field whose field of constants is $\mathbb{C}$. A strongly normal extension is a differential field extension $K \subset L$ such that,

(i) the field of constants of L is $\mathbb{C}$,

(ii) for any K-isomorphism $\sigma\colon L \to \mathcal{U}$ into a differential field extension $\mathcal{U}$ of L we have that $L \cdot \sigma L \subset L \cdot C_{\mathcal{U}}$.

Theorem 8.6. *Let $\vec{X}$ be a non-autonomous meromorphic vector field in an algebraic manifold M with coefficients in the Riemann surface S that admits a rational superposition law. Let L be the Galois differential field of $\vec{X}$. Then, $\mathcal{M}(S) \subset L$ is a strongly normal extension in the sense of Kolchin. Moreover, each particular solution $\sigma(t)$ of the associated automorphic system induce an injective map,*

$$\mathrm{Aut}_{\mathcal{M}(S)}(L) \to G.$$

which is an anti-morphism of groups.

Proof. We consider, as above, $\sigma(t)$ a solution of the associated automorphic system, so that $L = \mathcal{M}(S)\langle\sigma(t)\rangle$. Let $F\colon L \to \mathcal{U}$ be a $\mathcal{M}(S)$-isomorphism. Then, acting coordinate by coordinate $F(\sigma(t))$ is an element of G with coordinates in $\mathcal{U}$. Let us recall F fixes $\mathcal{M}(S)$ pointwise, we have that the coordinates $F(\sigma(t))$ satisfy all the differential equations satisfied by the coordinates of $\sigma(t)$ with coefficients in $\mathcal{M}(S)$. Then $F(\sigma(t))$ is a solution of $\vec{A}$. Let us denote $F(\sigma(t)) = \tau(t)$.

By the properties of automorphic systems, we have that $\sigma(t)^{-1} \cdot \tau(t)$ which is an element of G with coordinates in $\mathcal{U}$ is constant, that is, its coordinates are in $C_{\mathcal{U}}$. Let us call it $\lambda = \sigma(t)^{-1} \cdot \tau(t)$. Then, we have $L \cdot \sigma L \subset L(\lambda) \subset L \cdot C_{\mathcal{U}}$, and we conclude that $\mathcal{M}(S) \subset L$ is a strongly normal extension.

Let us construct the anti-morphism of groups above stated. We fix the solution $\sigma(t)$ of $\vec{A}$. Let F be a K-automorphism of L. By the above argument $F(\sigma(t))$ is also a solution of $\vec{A}$, so that there exist a unique λ_F such that $F(\sigma(t)) = \sigma(t) \cdot \lambda_F$. Then we define:

$$\Phi_\sigma\colon \mathrm{Aut}_{\mathcal{M}(S)}(L) \to G, \quad F \to \lambda_F.$$

If we consider a pair of automorphisms F, and G, we have:

$$F(G(\sigma(t)) = F(\sigma(t)) \cdot \lambda_G) = \sigma(t) \cdot \lambda_F\lambda_G,$$

so that it is clear that it is anti-morphism. Finally, for proving that this map is injective it suffices to recall that $\sigma(t)$ spans L. Thus, if an automorphism let $\sigma(t)$ fixed then it is the identity. ■

Remark 8.7. In the above construction we have constructed a special kind of strongly normal extension, called a G-extension, associated with an automorphic system in an algebraic group. Not every strongly normal extension is of this kind, it depends on some obstructions related with Galois cohomology. In the general case, an strongly normal extension is related with a principal homogeneous space structure. This problem has been approached in a purely algebraic way, with no explicit relation with superposition laws. See, for instance [3] Theorem 4.40., or [13] Chapter VI, Section 7.

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References

[1] Anderson, R. L., J. Harnad, and P. Winternitz, *Group theoretical approach to superposition rules for systems of Riccati equations*, Lett. Math. Phys. **5** (1981), 143–148.

[2] Blázquez Sanz, D., "Differential Galois Theory and Lie-Vessiot Systems", PhD. Dissertation (advisor J. J. Morales-Ruiz). Universitat Politècnica de Catalunya, Barcelona 2008.

[3] Blázquez Sanz, D., and J. J. Morales Ruiz, *Differential Galois Theory of Algebraic Lie-Vessiot Systems*, Differential Algebra, Complex Analysis and Orthogonal Polynomials (P. B. Acosta-Humánez, F. Marcellán, Editors) Contemporary Mathematics **509**, Amer. Math. Soc.; IMA, 2010, 231 pp., 1–58.

[4] Cariñena, J. F., J. Grabowski, and G. Marmo, "Lie-Scheffers systems: a geometric approach", Napoli Series on Physics and Astrophysics. Bibliopolis, Naples, 2000.

[5] Cariñena, J. F., J. Grabowski, and A. Ramos, *Reduction of time-dependent systems admitting a superposition law*, Acta Appl. Math. **66** (2001), 67–87.

[6] Cariñena, J. F., and A. Ramos, *A new geometric approach to Lie systems and physical applications. Symmetry and perturbation theory*, Acta Appl. Math. **70** (2002), 49-63.

[7] —, *Superposition rules, Lie theorem, and Partial Differential equations*, Rep. Math. Phys. **60** (2007), 237–258.

[8] Casale, G., *Feuilletages singuliers de codimension un, groupoïde de Galois et intégrales premières* Ann. Inst. Fourier (Grenoble) **56** (2006), 735–779.

[9] Havlíček, H., S. Pošta, and P. Winternitz, *Nonlinear superposition formulas based on imprimitive group action*, J. Math. Phys. **40** (1999), 3104–3122.

[10] Humphreys, J. E., "Linear Algebraic Groups", Graduate Texts in Mathematics, Springer Verlag, 4th printing, 1995.

[11] Ibragimov, N. H., "Elementary Lie Group Analysis and Ordinary Differential Equations", John Wiley & Sons 1999, 347pp.

[12] Lafortune, S., and P. Winternitz, *Superposition formulas for pseudounitary matrix Riccati equations*, J. Math. Phys. **37** (1996), 1539–1550.

[13] Kolchin, E., "Differential Algebra and Algebraic Groups", Academic Press, 1973.

[14] Kovacic, J., *The Differential Galois Theory of Strongly Normal Extensions*, Trans. Amer. Math. Soc. **355** (2003), 4475–4522.

[15] —, *Geometric Characterization of Strongly Normal Extensions*, Trans. Amer. Math. Soc. **358** (2009), 4135–4157.

[16] Lie, S., *Allgemeine Untersuchungen über Differentialgleichungen die eine kontinuerliche endliche Gruppe gestatten*, Math. Ann. **25** (1885).

[17] —, „Vorlesungen über continuerliche Gruppen mit geometrischen und anderen Andwendungen", Herausgegeben und bearbeitet durch G. Scheffers, Teubner, Leizpig, 1893.

[18] —, *Verwertung des Gruppenbegriffes für Differentialgleichungen I* Leipziger Berichte, 1895.

[19] Muñoz-Díaz, J. "Ecuaciones Diferenciales I" Ediciones Universidad de Salamanca, 1982 (Spanish).

[20] Nishioka, K. *Lie extensions* Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 82–85.

[21] del Olmo, M. A., M. A. Rodríguez, and P. Winternitz, *Superposition formulas for the rectangular matrix Riccati equations*, J. Math. Phys. **28** (1987), 530–535.

[22] Shnider, S., and P. Winternitz, *Classification of systems of non-linear ordinary differential equations with superposition laws* J. Math. Phys. **25** (1984), 3155–3165.

[23] Shorine, M., and P. Winternitz, *Superposition Laws for Solutions of Differential Matrix Riccati Equations Arising in Control Theory* IEEE Transactions on automatic control, AC-**30** (1985), 266–272.

[24] Vessiot, E., *Sur les systèmes d'equations differentielles du premier ordre qui ont des systèmes fondamentaux d'integrales* Annales de la faculté de sciences de Toulouse Sr. 1, **8** (1894) H1-H33.

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