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On the Index of the Quotient of a Borel Subalgebra by an ad-Nilpotent Ideal

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Abstract. In this paper, we give upper bounds for the index of the quotient of a Borel subalgebra of a simple Lie algebra or its nilpotent radical by an ad-nilpotent ideal. For the nilpotent radical quotient, our bound is a generalization of the formula for the index given by Panov in the type A case. In general, this bound is not exact. Using results of Panov [*On the index of certain nilpotent Lie algebras*, J. of Math. Sci. **161** (2009), 122–129], we show that the upper bound for the Borel quotient is exact in the type A case, and we conjecture that it is exact in general.
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1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Recall that the index of a finite-dimensional Lie algebra $\mathfrak{a}$ over $\Bbbk$ is the integer

$$\chi(\mathfrak{a}) = \min_{f\in\mathfrak{a}^*} \dim \mathfrak{a}^f$$

where for $f \in \mathfrak{a}^*$, we denote by $\mathfrak{a}^f = \{X \in \mathfrak{a}; f([X,Y]) = 0 \text{ for all } Y \in \mathfrak{a}\}$, the annihilator of f for the coadjoint representation of $\mathfrak{a}$. It is well-known that when $\mathfrak{a}$ is the Lie algebra of an algebraic group A, $\chi(\mathfrak{a})$ is the transcendence degree of the field of A-invariant rational functions on $\mathfrak{a}^*$.

There are quite a lot of recent works on the computation of the index of certain classes of Lie subalgebras of a semisimple Lie algebra: parabolic subalgebras and related subalgebras ([2], [4], [8], [11], [6]), centralizers of elements and related subalgebras ([9], [1], [13], [5], [3]).

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra defined over $\Bbbk$ and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}$, Δ the associated root system, Δ^+ the set of positive roots relative to $\mathfrak{b}$ and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ the corresponding set of simple roots. For each $\alpha \in \Delta$, let $\mathfrak{g}^\alpha$

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be the root subspace of $\mathfrak{g}$ relative to α. Denote by $\mathfrak{n} = \mathfrak{g}^{\Delta^+}$ the nilpotent radical of $\mathfrak{b}$ where for a subset P of Δ^+, we set

$$\mathfrak{g}^P = \bigoplus_{\alpha \in P} \mathfrak{g}^\alpha.$$

An ideal $\mathfrak{i}$ of $\mathfrak{b}$ is ad-nilpotent if and only if for all $x \in \mathfrak{i}$, $\mathrm{ad}_{\mathfrak{b}} x$ is nilpotent. Since any ideal of $\mathfrak{b}$ is $\mathfrak{h}$-stable, we deduce easily that an ideal is ad-nilpotent if and only if it is nilpotent, and there exists a subset $\Phi \subset \Delta^+$ such that $\mathfrak{i} = \mathfrak{g}^\Phi$. We set $\mathfrak{q}_\Phi = \mathfrak{b}/\mathfrak{i}$ and $\mathfrak{m}_\Phi = \mathfrak{n}/\mathfrak{i}$.

In [7], Panov determined the index of $\mathfrak{m}_\Phi$, when $\mathfrak{g}$ is simple of type A. His results are very explicit, and the index is completely determined by $\Delta^+ \setminus \Phi$ in a combinatorial way. A similar consideration of roots was used for the index of seaweed subalgebras in [11]. In this paper, we generalize these root combinatorial approaches to give upper bounds for the index of $\mathfrak{q}_\Phi$ and $\mathfrak{m}_\Phi$ in all types. Our upper bound for $\mathfrak{m}_\Phi$ is not exact when $\mathfrak{g}$ is not of type A. However, using the results of Panov for $\mathfrak{m}_\Phi$, we prove that our upper bound for $\mathfrak{q}_\Phi$ is exact when $\mathfrak{g}$ is of type A, and we have not found so far any counter-examples in the other types. We give also a short discussion on the existence of stable linear forms.

Our work is partly motivated by some work in progress of P. Damianou, H. Sabourin and P. Vanhaecke. Let $\mathfrak{i}$ be an ad-nilpotent ideal of $\mathfrak{b}$. They define a Poisson structure on $(\mathfrak{b}/\mathfrak{i})^*$ whose Poisson rank L is equal to the dimension of $\mathfrak{b}/\mathfrak{i}$ minus the index of $\mathfrak{b}/\mathfrak{i}$. Their problem is to construct an integrable system, that is to say a set of $\dim(\mathfrak{b}/\mathfrak{i})^* - L/2$ independent functions in involution. In view of the number of equations required, the calculation of the index of $\mathfrak{b}/\mathfrak{i}$ is involved in this problem.

We shall recall a more general definition of the index which is used in the paper. Let $\mathfrak{a}$ be the Lie algebra of an algebraic group A and V a rational A-module of finite dimension. The *index* of V is the integer

$$\begin{aligned}\chi(\mathfrak{a}, V) &= \dim V - \max_{h \in V^*} \dim \mathfrak{a}.h = \dim V - \max_{h \in V^*} \mathrm{codim}_{\mathfrak{a}} \mathfrak{a}^h \\ &= \mathrm{tr}\deg_{\Bbbk}(\Bbbk(V^*)^A)\end{aligned}$$

where for $f \in V^*$, $\mathfrak{a}^f = \{X \in \mathfrak{a}; X.f = 0\}$ and $\mathfrak{a}.f = \{X.f; X \in \mathfrak{a}\}$. When $f \in V^*$ is such that $\dim V - \dim \mathfrak{a}.f = \chi(\mathfrak{a}, V)$, we say that f is *regular*. The set of regular elements of V^* is a non-empty Zariski-open subset.

2. H-sequences

In this section, we introduce the combinatorial tools used to describe the upper bounds for the index of the quotients. This is a generalization of the "cascade" construction of Kostant (see for example [11, 12]) and the construction of Panov in type A in [7].

Recall the following standard partial order on Δ^+. For $\alpha, \beta \in \Delta^+$, we have $\alpha \leqslant \beta$ if and only if $\beta - \alpha$ is a sum of positive roots. Let $E \subset \Delta^+$ and $\gamma \in E$. We set:

$$H(E, \gamma) = \{\alpha \in E; \gamma - \alpha \in E \cup \{0\}\}.$$

Definition 2.1. Let $E \subset \Delta^+$ and $\theta_1 \in E$. We say that (θ_1) is an *H-sequence of length* 1 in E if $E = H(E, \theta_1)$.

By induction, for $\theta_1, \theta_2, \ldots, \theta_r \in E$, we say that $(\theta_1, \theta_2, \ldots, \theta_r)$ is an *H-sequence of length* r in E if and only if:

(i) $(\theta_2, \theta_3, \ldots, \theta_r)$ is an H-sequence of length $r-1$ in $E \setminus H(E, \theta_1)$,

(ii) θ_1 is a maximal element for $\leqslant$ in E.

Let $(\theta_1, \theta_2, \ldots, \theta_r)$ be an H-sequence of length r in E. Set:

$$E_1 = E \setminus H(E, \theta_1) \quad , \quad \Gamma_1 = H(E, \theta_1).$$

For $i = 1, \ldots, r-1$, we set

$$E_{i+1} = E_i \setminus H(E_i, \theta_{i+1}) \quad , \quad \Gamma_{i+1} = H(E_i, \theta_{i+1}).$$

It is clear from the definition that E is the disjoint union of $\Gamma_1, \ldots, \Gamma_r$, and we have $E_i = E_{i+1} \cup \Gamma_{i+1}$ for $i = 0, \ldots, r-1$, with the convention that $E_0 = E$.

Let $\mathbf{h}$ be an H-sequence. We denote by $\ell(\mathbf{h})$ its length, $D(\mathbf{h})$ the vector space in $\mathfrak{h}^*$ spanned by the elements in $\mathbf{h}$, and $d(\mathbf{h}) = \dim D(\mathbf{h})$.

Example 2.2. (i) Let $E = \Delta^+$. We recover (an ordered) Kostant's cascade construction of pairwise strongly orthogonal roots in Δ^+.

(ii) Let $\mathfrak{g}$ be of type A_6. Using the numbering of simple roots in [12], set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$. In this notation, we have, for $1 \leqslant i \leqslant j \leqslant 6$:

$$H(\Delta^+, \alpha_{i,j}) = \{\alpha_{i,k}; i \leqslant k \leqslant j\} \cup \{\alpha_{k,j}; i \leqslant k \leqslant j\}.$$

Take $\Phi = \{\alpha \in \Delta^+; \alpha \geqslant \alpha_{1,4} \text{ or } \alpha \geqslant \alpha_{2,6}\}$, and $E = \Delta^+ \setminus \Phi$. Then $\alpha_{1,3}$ is a maximal element in E, and $\mathbf{h} = (\alpha_{1,3}, \alpha_{2,5}, \alpha_{3,6}, \alpha_{4,6}, \alpha_{4,4})$ is an H-sequence of length 5 in E, where

$$\begin{aligned} &\Gamma_1 = \{\alpha_{1,3}, \alpha_{1,1}, \alpha_{2,3}, \alpha_{1,2}, \alpha_{3,3}\}, \quad \Gamma_4 = \{\alpha_{4,6}, \alpha_{4,5}, \alpha_{6,6}\}, \\ &\Gamma_2 = \{\alpha_{2,5}, \alpha_{2,2}, \alpha_{3,5}, \alpha_{2,4}, \alpha_{5,5}\}, \quad \Gamma_5 = \{\alpha_{4,4}\}, \\ &\Gamma_3 = \{\alpha_{3,6}, \alpha_{3,4}, \alpha_{5,6}\}. \end{aligned}$$

Let us illustrate the construction of H-sequences via diagram filling in this type A case. We display the set of positive roots into the following diagram:

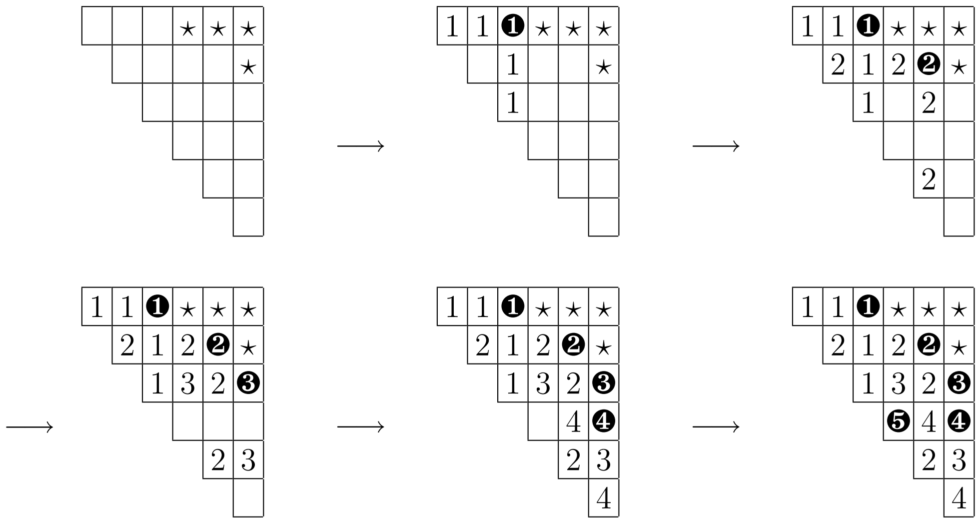
$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$	$\alpha_{1,5}$	$\alpha_{1,6}$
	$\alpha_{2,2}$	$\alpha_{2,3}$	$\alpha_{2,4}$	$\alpha_{2,5}$	$\alpha_{2,6}$
		$\alpha_{3,3}$	$\alpha_{3,4}$	$\alpha_{3,5}$	$\alpha_{3,6}$
			$\alpha_{4,4}$	$\alpha_{4,5}$	$\alpha_{4,6}$
				$\alpha_{5,5}$	$\alpha_{5,6}$
					$\alpha_{6,6}$

So the box in the i-th row and the j-th column corresponds to the positive root $\alpha_{i,j}$ for $1 \leqslant i \leqslant j \leqslant 6$. Observe that two roots are comparable if and only if they are in the same row or in the same column. Further, the roots are increasing from left to right, and decreasing from top to bottom with respect to the standard partial order on Δ^+.

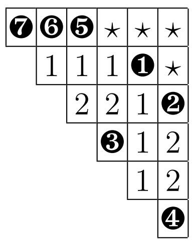
To construct an H-sequence in E, we start with the same diagram but unfilled. We fill the boxes corresponding to Φ by the symbol $\star$. Denote this new diagram by T (see the chain of diagrams below).

Now, maximal elements in E correspond to unfilled north-east corner boxes of T. Pick a maximal element θ_1 in E, and fill the corresponding box B_1 by the symbol ❶. Then the elements of $\Gamma_1 \setminus \{\theta_1\}$ correspond to certain unfilled boxes on the left or below the box B_1. We fill these boxes by 1. Denote the new diagram T_1. We start again with T_1, E_1 and symbols ❷ and 2 to obtain T_2. We iterate this process until the diagram is completely filled. The sequence of starred boxes in the final diagram gives an H-sequence.

For the construction of the H-sequence $\mathbf{h}$ above, the successive diagrams are as follows:



With another choice of maximal elements, we obtain another H-sequence $\mathbf{h}' = (\alpha_{2,5}, \alpha_{3,6}, \alpha_{4,4}, \alpha_{6,6}, \alpha_{1,3}, \alpha_{1,2}, \alpha_{1,1})$
which is of length 7. The final filled diagram corresponding to $\mathbf{h}'$ is:



Observe that $d(\mathbf{h}) = 5$ and $d(\mathbf{h}') = 6$.

Lemma 2.3. *Let $E \subset \Delta^+$ and $\mathbf{h} = (\theta_1, \ldots, \theta_r)$ be an H-sequence of length r in E. Let $i, j, k \in \{1, \ldots, r\}$.*

(i) Let $\alpha \in \Gamma_i$ *and* $\beta \in \Gamma_j$ *be such that* $\alpha + \beta = \theta_k$*. Then* $k \geqslant \min(i,j)$*.*

(ii) There do not exist i, j, k *such that* $\theta_i + \theta_j = \theta_k$*.*

Proof. (i) If $k < \min(i,j)$, then $\alpha, \beta \in E_k$. It follows that $\alpha, \beta \in \Gamma_k$ and $k = j = i$, which contradicts the hypothesis.

(ii) Assume that there exist i, j, k such that $\theta_i + \theta_j = \theta_k$. Then $\theta_k > \theta_i$ and $\theta_k > \theta_j$ and therefore by construction $k < \min(i,j)$, which contradicts the first point. ■

3. Upper bounds for the index

We give in this section upper bounds for the index of the quotients. The proof follows closely the one for the index of seaweed Lie algebras in [11] even though we do not have the nice properties on the roots from the "cascade construction".

Recall that if $\mathfrak{a}$ is a finite-dimensional Lie algebra over $\Bbbk$ and $f \in \mathfrak{a}^*$, we can define an alternating bilinear form Φ_f on $\mathfrak{a}$ by setting

$$\Phi_f(X,Y) = f([X,Y]),$$

for $X, Y \in \mathfrak{a}$. Then $\mathfrak{a}^f = \{X \in \mathfrak{a}; \Phi_f(X,Y) = 0, \text{ for all } Y \in \mathfrak{a}\}$ is the kernel of Φ_f. Therefore we have

$$\chi(\mathfrak{a}) = \min\{\operatorname{corank} \Phi_f; f \in \mathfrak{a}^*\}.$$

Let $\{H_1, \dots, H_\ell\}$ be a basis of $\mathfrak{h}$. For $\alpha \in \Delta$, we denote by X_α a non-zero element of $\mathfrak{g}^\alpha$. Then $\{H_i; 1 \leqslant i \leqslant \ell\} \cup \{X_\alpha; \alpha \in \Delta\}$ is a basis of $\mathfrak{g}$ and we shall denote by $\{H_i^*; 1 \leqslant i \leqslant \ell\} \cup \{X_\alpha^*; \alpha \in \Delta\}$ the corresponding dual basis.

Let Φ be a subset of Δ^+ such that $\mathfrak{i} = \mathfrak{g}^\Phi$ is an ad-nilpotent ideal of $\mathfrak{b}$. Suppose that $\mathbf{h} = (\theta_1, \dots, \theta_s)$ is an H-sequence of length s of $E = \Delta^+ \setminus \Phi$. We have the following $\mathfrak{h}$-module isomorphisms:

$$\mathfrak{q}_\Phi \simeq \mathfrak{h} \oplus \mathfrak{g}^{\Delta^+ \setminus \Phi} \quad , \quad \mathfrak{m}_\Phi \simeq \mathfrak{g}^{\Delta^+ \setminus \Phi}.$$

Let $\mathbf{a} = (a_1, \dots, a_s)$ be an element of $(\Bbbk^*)^s$. Identifying $\mathfrak{q}_\Phi^*$ with $\mathfrak{h}^* \oplus \sum_{\alpha \in \Delta^+ \setminus \Phi} \Bbbk X_\alpha^*$, we define the following element of $\mathfrak{q}_\Phi^*$:

$$f_{\mathbf{a}} = \sum_{i=1}^{s} a_i X_{\theta_i}^*.$$

We fix a total order $<$ on Δ^+ compatible with the partial order $\leqslant$.

For $i \in \{1, \dots, s\}$, set

$$\mathcal{G}_i = \{(\alpha, \beta) \in \Gamma_i \times \Gamma_i; \alpha + \beta = \theta_i \text{ and } \alpha < \beta\} \quad , \quad t_i = \sharp \mathcal{G}_i \quad ,$$

and

$$\mathcal{G} = \bigcup_{i=1}^{s} \mathcal{G}_i \quad , \quad t = \sharp \mathcal{G}.$$

Denote by $\mathcal{Z}$ the set of pairs (α, β) of E^2 such that $\alpha < \beta$ and there exists $k \in \{1, \dots, s\}$ satisfying $\alpha + \beta = \theta_k$.

For $z = (\alpha, \beta) \in \mathcal{Z}$, we set

$$v_z = X_\alpha^* \wedge X_\beta^* \in \bigwedge^2 \mathfrak{q}_\Phi^*.$$

Identifying $\Phi_{f_{\mathbf{a}}}$ with an element of $\bigwedge^2 \mathfrak{q}_\Phi^*$, we have

$$\Phi_{f_{\mathbf{a}}} = \Psi_{f_{\mathbf{a}}} + \Theta_{f_{\mathbf{a}}}$$

where

$$\Theta_{f_{\mathbf{a}}} = \sum_{i=1}^{s} K_i \wedge X_{\theta_i}^* \quad , \quad \Psi_{f_{\mathbf{a}}} = \sum_{z \in \mathcal{Z}} \lambda_z v_z \quad ,$$

with $K_i \in D(\mathbf{h})$ for $i = 1, \dots, s$ and $\lambda_z \in \mathbb{k}$ for all $z \in \mathcal{Z}$.

If $z = (\alpha, \beta) \in \mathcal{Z}$, then $\Theta_{f_{\mathbf{a}}}(X_\alpha, X_\beta) = 0$. Moreover, we have $[X_\alpha, X_\beta] = \mu_z X_{\theta_i}$, for some $i \in \{1, \dots, s\}$ and a non-zero scalar μ_z. Consequently,

$$\lambda_z = \Phi_{f_{\mathbf{a}}}(X_\alpha, X_\beta) = f_{\mathbf{a}}([X_\alpha, X_\beta]) = \mu_z a_i. \tag{1}$$

Thus λ_z is non-zero.

Lemma 3.1. *In the above notations:*

(i) $\mathfrak{q}_\Phi^{f_{\mathbf{a}}}$ *contains a commutative subalgebra of* $\mathfrak{q}_\Phi$*, consisting of semi-simple elements, of dimension* $\ell - d(\mathbf{h})$.

(ii) We have $\bigwedge^{d(\mathbf{h})} \Theta_{f_{\mathbf{a}}} \neq 0$ *and* $\bigwedge^{d(\mathbf{h})+1} \Theta_{f_{\mathbf{a}}} = 0$.

(iii) There exists a non-empty open subset U *of* $(\mathbb{k}^*)^s$ *such that we have* $\bigwedge^t \Psi_{f_{\mathbf{a}}} \neq 0$ *and* $\bigwedge^{d(\mathbf{h})+t} \Phi_{f_{\mathbf{a}}} \neq 0$ *whenever* $\mathbf{a} \in U$.

Proof. The proof is similar to the one of the lemme in [11, §3.9].

(i) For simplicity, we write $\mathfrak{q} = \mathfrak{q}_\Phi$. Let $\mathfrak{t} = \{x \in \mathfrak{h}; \theta_i(x) = 0 \text{ for } i = 1, \dots, s\}$ be the annihilator of $D(\mathbf{h})$ in $\mathfrak{h}$. Then:

$$\dim \mathfrak{t} = \dim \mathfrak{h} - \dim D(\mathbf{h}) = \ell - d(\mathbf{h}).$$

We also have that $[\mathfrak{t}, \mathfrak{q}] \subset \bigoplus_{\alpha \in E \setminus \{\theta_1, \dots, \theta_s\}} \mathfrak{g}^\alpha$. It follows that $\mathfrak{t}$ is contained in $\mathfrak{q}^{f_{\mathbf{a}}}$, and therefore, we obtain the result.

(ii) Set $r = d(\mathbf{h})$. Let $\mathcal{I} = \{i_1, \dots, i_r\} \subset \{1, \dots, s\}$ be such that $(\theta_{i_1}, \dots, \theta_{i_r})$ is a basis of $D(\mathbf{h})$ and complete it to a basis $\mathcal{B}' = (\beta_1, \dots, \beta_\ell)$ of $\mathfrak{h}^*$ such that $\beta_k = \theta_{i_k}$ for $k = 1, \dots, r$. Denote by $\mathcal{B} = (h_1, \dots, h_\ell)$ the basis of $\mathfrak{h}$ dual to $\mathcal{B}'$. Then we have,

$$f_{\mathbf{a}}([h_k, X_{\theta_j}]) = \begin{cases} a_{i_k} & \text{if } j = i_k, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce from this that

$$\Theta_{f_{\mathbf{a}}} = \sum_{i\in\mathcal{I}} a_i\theta_i \wedge X^*_{\theta_i} + \sum_{j\notin\mathcal{I}} K_j \wedge X^*_{\theta_j},$$

where $K_j \in D(\mathbf{h})$. The result follows easily because $\sharp\mathcal{I} = d(\mathbf{h})$.

(iii) If $z, z' \in \mathcal{Z}$, we have $v_z \wedge v_{z'} = v_{z'} \wedge v_z$ and $v_z \wedge v_z = 0$. Let $z_1, \dots, z_n$ be the elements of $\mathcal{Z}$ such that $z_1, \dots, z_t$ are the elements of $\mathcal{G}$. For simplicity, let us write $\lambda_i v_i$ for $\lambda_{z_i} v_{z_i}$ and μ_i for μ_{z_i}. Consequently:

$$\bigwedge^t \Psi_{f_{\mathbf{a}}} = t! \sum_{1\leqslant i_1<\cdots<i_t\leqslant n} \lambda_{i_1}\cdots\lambda_{i_t} v_{i_1}\wedge\cdots\wedge v_{i_t}.$$

In the previous sum, the coefficient of $v_1 \wedge \cdots \wedge v_t$ is by (1)

$$\prod_{i=1}^{s} a_i^{t_i}\Big(\prod_{z\in\mathcal{G}} \mu_z\Big).$$

Now assume that $v_{i_1} \wedge \cdots \wedge v_{i_t} = \lambda v_1 \wedge \cdots \wedge v_t$, with $\lambda \in \Bbbk^*$, where $i_1 < \cdots < i_t$ and $(i_1, \dots, i_t) \neq (1, \dots, t)$.

If $z = (\alpha, \beta) \in \mathcal{Z}$, we denote by $\widetilde{z} = \{\alpha, \beta\}$ the underlying set of z. Then the set $\mathcal{S} = \widetilde{z}_1 \cup \cdots \cup \widetilde{z}_t$ is the disjoint union of the sets $\widetilde{z_{i_k}}$ for $1 \leqslant k \leqslant t$. It follows that if $z_{i_k} \notin \mathcal{G}$, then by Lemma 2.3 there exist $i, j \in \{1, \dots, s\}$ such that $i \neq j$ and $z_{i_k} = (\alpha, \beta)$ where $(\alpha, \beta) \in (\Gamma_i \setminus \{\theta_i\}) \times (\Gamma_j \setminus \{\theta_j\})$.

Let $\mathcal{I} = \{k; z_{i_k} \notin \mathcal{G}\}$. Let i_0 be minimal among the elements $j \in \{1, \dots, s\}$ satisfying:

$$(\Gamma_j \setminus \{\theta_j\}) \cap \left(\bigcup_{k\in\mathcal{I}} \widetilde{z_{i_k}}\right) \neq \emptyset.$$

Then there exist $\alpha \in \Gamma_{i_0}$, $k \in \mathcal{I}$, and $\beta \in \Delta^+$ such that $\widetilde{z_{i_k}} = \{\alpha, \beta\}$. By our choice of i_0 and since $z_{i_k} \notin \mathcal{G}$, there exist $j \geqslant i_0$ and $l \in \{1, \dots, s\}$ such that $\beta \in \Gamma_j$ and $\alpha + \beta = \theta_l$. Then, by Lemma 2.3, we have $l > \min(i_0, j) = i_0$. It follows that $\lambda_{z_{i_k}} = \mu_{z_{i_k}} a_l$, where $l \neq i_0$.

We deduce that the coefficient of $v_{i_1} \wedge \cdots \wedge v_{i_t}$ in the sum giving $\bigwedge^t \Psi_{f_{\mathbf{a}}}$ is of the form

$$\mu_{i_1}\cdots\mu_{i_t} \prod_{i=1}^{s} a_i^{m_i},$$

with $m_{i_0} < t_{i_0}$.

It is now clear that there exists a non-empty open subset U of $(\Bbbk^*)^s$ satisfying $\bigwedge^t \Psi_{f_{\mathbf{a}}} \neq 0$ if $\mathbf{a} \in U$.

Finally, we have

$$\bigwedge^{r+t} \Phi_{f_{\mathbf{a}}} = \sum_{k=0}^{r+t} \binom{r+t}{k} \Big(\bigwedge^k \Psi_{f_{\mathbf{a}}}\Big) \wedge \Big(\bigwedge^{r+t-k} \Theta_{f_{\mathbf{a}}}\Big).$$

Set $\mathfrak{u} = \sum_{i=1}^{s} \Bbbk X_{\theta_i}$. Since $\bigwedge^j \Theta_{f_{\mathbf{a}}} \in (\bigwedge^j D(\mathbf{h})) \wedge (\bigwedge^j \mathfrak{u}^*)$, to show that $\bigwedge^{r+t} \Phi_{f_{\mathbf{a}}} \neq 0$, it suffices to prove that $(\bigwedge^t \Psi_{f_{\mathbf{a}}}) \wedge (\bigwedge^r \Theta_{f_{\mathbf{a}}}) \neq 0$.

If $\mathbf{a} \in U$, then we deduce from the preceding paragraphs that

$$\bigwedge^t \Psi_{f_{\mathbf{a}}} = \lambda v_1 \wedge \cdots \wedge v_t + w,$$

where $\lambda \in \Bbbk^*$ and w is a linear combination of elements of the form $v_{z_{i_1}} \wedge \cdots \wedge v_{z_{i_t}}$, with $\widetilde{z_{i_1}} \cup \cdots \cup \widetilde{z_{i_t}} \neq \mathcal{S}$. It is therefore clear that $(\bigwedge^t \Psi_{f_{\mathbf{a}}}) \wedge (\bigwedge^r \Theta_{f_{\mathbf{a}}}) \neq 0$ if $\mathbf{a} \in U$. ■

Theorem 3.2. *Let Φ be a subset of Δ^+ such that $\mathfrak{g}^\Phi$ is an ad-nilpotent ideal of $\mathfrak{g}$. Denote by $\mathcal{H}$ the set of H-sequences of $\Delta^+ \setminus \Phi$. Then, we have*

$$\begin{aligned} \chi(\mathfrak{q}_\Phi) &\leqslant \min\{\ell + \ell(\mathbf{h}) - 2d(\mathbf{h}); \mathbf{h} \in \mathcal{H}\}, \\ \chi(\mathfrak{m}_\Phi) &\leqslant \min\{\ell(\mathbf{h}); \mathbf{h} \in \mathcal{H}\}. \end{aligned}$$

Proof. Let $\mathbf{h} \in \mathcal{H}$ with $\ell(\mathbf{h}) = s$ and t as defined in the beginning of this section. By definition, we have

$$\begin{aligned} \dim \mathfrak{q}_\Phi &= \dim \mathfrak{h} + \ell(\mathbf{h}) + 2t, \\ \dim \mathfrak{m}_\Phi &= \ell(\mathbf{h}) + 2t. \end{aligned}$$

Let U be a non-empty open subset of $(\Bbbk^*)^s$ satisfying part (iii) of Lemma 3.1. If $\mathbf{a} \in U$, then the fact that $\bigwedge^{d(\mathbf{h})+t} \Phi_{f_{\mathbf{a}}} \neq 0$ implies that $\mathrm{rk}(\Phi_{f_{\mathbf{a}}}) \geqslant 2(d(\mathbf{h}) + t)$. Thus

$$\dim \mathfrak{q}_\Phi^{f_{\mathbf{a}}} \leqslant \dim \mathfrak{q}_\Phi - 2(d(\mathbf{h}) + t).$$

Hence

$$\dim \mathfrak{q}_\Phi^{f_{\mathbf{a}}} \leqslant \dim \mathfrak{h} + \ell(\mathbf{h}) - 2d(\mathbf{h}).$$

In the same manner, if $\mathbf{a} \in U$, then the fact that $\bigwedge^t \Psi_{f_{\mathbf{a}}} \neq 0$ implies that $\mathrm{rk}(\Psi_{f_{\mathbf{a}}}) \geqslant 2t$. Thus

$$\dim \mathfrak{m}_\Phi^{f_{\mathbf{a}}} \leqslant \dim \mathfrak{m}_\Phi - 2t = \ell(\mathbf{h}).$$

So we are done. ■

For an H-sequence $\mathbf{h}$, we define

$$c(\mathbf{h}) = \ell + \ell(\mathbf{h}) - 2d(\mathbf{h}).$$

Proposition 3.3. *Let us keep the notations of Theorem 3.2. If $\mathbf{h} \in \mathcal{H}$ verifies $c(\mathbf{h}) \in \{0, 1\}$, then $\chi(\mathfrak{q}_\Phi) = c(\mathbf{h})$.*

Proof. The case $c(\mathbf{h}) = 0$ is clear by Theorem 3.2. So let us suppose that $c(\mathbf{h}) = 1$.

We have $\dim \mathfrak{q}_\Phi - c(\mathbf{h}) = 2(d(\mathbf{h}) + t)$. Since $\dim \mathfrak{q}_\Phi - \chi(\mathfrak{q}_\Phi)$ is an even integer (it is the rank of an alternating bilinear form on $\mathfrak{q}_\Phi$), we deduce that $c(\mathbf{h})$ and $\chi(\mathfrak{q}_\Phi)$ are of the same parity. So $\chi(\mathfrak{q}_\Phi) = 1$. ■

4. Type A

Let us assume in this section that $\mathfrak{g}$ is of type A_ℓ. We shall show that the upper bound for the index of $\mathfrak{q}_\Phi$ is exact in this special case.

We fix a subset Φ of Δ^+ such that $\mathfrak{i} = \mathfrak{g}^\Phi$ is an ad-nilpotent ideal of $\mathfrak{g}$. As in example 2.2, we use the numbering of simple roots in [12], and we set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$ when $i \leqslant j$.

We fix the following total order $\prec$ on Δ^+ compatible with the partial order $\leqslant$:

$$\alpha_{1,\ell} \succ \alpha_{1,\ell-1} \succ \cdots \succ \alpha_{1,2} \succ \alpha_{1,1} \succ \alpha_{2,\ell} \succ \alpha_{2,\ell-1} \succ \cdots \succ \alpha_{\ell-1,\ell} \succ \alpha_{\ell,\ell}.$$

It is clear that there is a unique H-sequence $\mathbf{h} = (\theta_1, \ldots, \theta_s)$ of $E = \Delta^+ \setminus \Phi$, satisfying $\theta_1 \succ \theta_2 \succ \cdots \succ \theta_s$. This H-sequence is considered by Panov in [7], and we shall call this H-sequence the *Panov H-sequence* of E.

Using the notation of section 2, for $j = 1, \ldots, s$, set:

$$\mathfrak{n}_j = \bigoplus_{\alpha \in E_j \cup \Phi} \mathfrak{g}^\alpha \quad , \quad \mathfrak{m}_j = \mathfrak{n}_j / \mathfrak{i}.$$

In [7], Panov proved that for $j = 1, \ldots, s$, $\mathfrak{n}_j$ and $\mathfrak{m}_j$ are Lie subalgebras of $\mathfrak{n}$ and $\mathfrak{n}/\mathfrak{i}$ respectively. Consider the localization $S(\mathfrak{m}_{j-1})_{X_{\theta_j}}$ of the algebra $S(\mathfrak{m}_{j-1})$ with respect to the multiplicative subset generated by X_{θ_j}. He defined an embedding of Poisson algebras $\Psi_{j-1} : S(\mathfrak{m}_j) \to S(\mathfrak{m}_{j-1})_{X_{\theta_j}}$. Moreover, one observes directly from the definition of Ψ_{j-1} that it is $\mathfrak{h}$-equivariant.

Extending the Ψ_j with the appropriate localizations, we set

$$f_1 = X_{\theta_1} \quad , \quad f_j = \Psi_0 \circ \cdots \circ \Psi_{j-2}(X_{\theta_j}) \text{ for } 2 \leqslant j \leqslant s.$$

Panov proved that $f_1, \ldots, f_s$ are algebraically independent elements of $\Bbbk(\mathfrak{m}_\Phi^*)^{\mathfrak{m}_\Phi}$. More precisely, we have $\Bbbk(\mathfrak{m}_\Phi^*)^{\mathfrak{m}_\Phi} = \Bbbk(f_1, \ldots, f_s)$, and hence $\chi(\mathfrak{m}_\Phi) = s$. Furthermore, using the fact that the embeddings Ψ_{j-1} are $\mathfrak{h}$-equivariant, the element f_j is of weight θ_j for $j = 1, \ldots, s$.

Let $\mathcal{I} \subset \{1, \ldots, s\}$ be such that $\{\theta_i ; i \in \mathcal{I}\}$ is a basis of $D(\mathbf{h})$. For $j \in \{1, \ldots, s\} \setminus \mathcal{I}$, we have

$$\lambda_j \theta_j = \sum_{i \in \mathcal{I}} \lambda_i \theta_i$$

where $\lambda_i \in \mathbb{Z}$ for $i \in \mathcal{I}$ and $\lambda_j \in \mathbb{Z}^*$.

Set

$$g_j = \left(\prod_{i \in \mathcal{I}} f_i^{\lambda_i} \right) f_j^{-\lambda_j} \in \Bbbk(\mathfrak{m}_\Phi^*)^{\mathfrak{m}_\Phi}.$$

By construction, the elements g_j are of weight zero. Hence $g_j \in \Bbbk(\mathfrak{m}_\Phi^*)^{\mathfrak{q}_\Phi}$. Since the elements $f_1, \ldots, f_s$ are algebraically independent, it follows that

$$\chi(\mathfrak{q}_\Phi, \mathfrak{m}_\Phi) = \operatorname{tr}\deg_\Bbbk(\Bbbk(\mathfrak{m}_\Phi^*)^{\mathfrak{q}_\Phi}) \geqslant \ell(\mathbf{h}) - d(\mathbf{h}). \tag{2}$$

Theorem 4.1. *Let* $\mathbf{h} = (\theta_1, \ldots, \theta_s)$ *be the Panov H-sequence of* E*. There exists a non-empty open subset* U *of* $(\Bbbk^*)^s$ *such that* $\dim \mathfrak{q}_\Phi^{f_\mathbf{a}} = c(\mathbf{h})$ *whenever* $\mathbf{a} \in U$*. Moreover, we have*

$$\chi(\mathfrak{q}_\Phi, \mathfrak{m}_\Phi) = \ell(\mathbf{h}) - d(\mathbf{h}).$$

Proof. Let U be a non-empty open subset of $(\Bbbk^*)^s$ satisfying part (iii) of Lemma 3.1. Set $r = d(\mathbf{h})$. Let $\mathcal{I} = \{i_1, \ldots, i_r\} \subset \{1, \ldots, s\}$ be such that $(\theta_{i_1}, \ldots, \theta_{i_r})$ is a basis of $D(\mathbf{h})$ and complete to a basis $\mathcal{B}' = (\beta_1, \ldots, \beta_\ell)$ of $\mathfrak{h}^*$ such that $\beta_k = \theta_{i_k}$ for $k = 1, \ldots, r$. Denote by $\mathcal{B} = (h_1, \ldots, h_\ell)$ the basis of $\mathfrak{h}$ dual to $\mathcal{B}'$.

Let $m = \dim \mathfrak{m}_\Phi$ and $\mathcal{C}$ be a basis of $\mathfrak{m}_\Phi$. Then the matrix of $\Phi_{f_\mathbf{a}}$ in the basis $\mathcal{B}' \cup \mathcal{C}$ is

$$M = \begin{pmatrix} 0_{\ell,\ell} & A \\ -{}^tA & B \end{pmatrix},$$

where A is an element of rank $d(\mathbf{h})$ in the set of $\ell \times m$ matrices $\mathcal{M}_{\ell,m}(\Bbbk)$, and $B \in \mathcal{M}_{m,m}(\Bbbk)$ the set of $m \times m$ matrices. Set

$$M' = \begin{pmatrix} A \\ B \end{pmatrix}.$$

Then by (2), we have

$$\dim \mathfrak{m}_\Phi - \mathrm{rk}(M') \geqslant \chi(\mathfrak{q}_\Phi, \mathfrak{m}_\Phi) \geqslant \ell(\mathbf{h}) - d(\mathbf{h}). \tag{3}$$

It follows that

$$\mathrm{rk}(M') \leqslant \dim \mathfrak{m}_\Phi - \ell(\mathbf{h}) + d(\mathbf{h}) \quad , \tag{4}$$

and since $\mathrm{rk}(M) \leqslant \mathrm{rk}(A) + \mathrm{rk}(M')$, we deduce that

$$\mathrm{rk}(M) \leqslant \dim \mathfrak{m}_\Phi - \ell(\mathbf{h}) + 2d(\mathbf{h}). \tag{5}$$

Hence,

$$\dim \mathfrak{q}_\Phi^{f_\mathbf{a}} = \dim \mathfrak{q}_\Phi - \mathrm{rk}(M) \geqslant c(\mathbf{h}).$$

It follows by Theorem 3.2 that $\dim \mathfrak{q}_\Phi^{f_\mathbf{a}} = c(\mathbf{h})$ and we have equalities in (3) and (4), (5). Consequently $\mathrm{rk}(M') = \dim \mathfrak{m}_\Phi - \ell(\mathbf{h}) + d(\mathbf{h})$ and $\chi(\mathfrak{q}_\Phi, \mathfrak{m}_\Phi) = \ell(\mathbf{h}) - d(\mathbf{h})$. ■

Theorem 4.2. *Let* $\mathbf{h} = (\theta_1, \ldots, \theta_s)$ *be the Panov H-sequence of* E*. Then we have* $\chi(\mathfrak{q}_\Phi) = \ell + \ell(\mathbf{h}) - 2d(\mathbf{h})$*.*

Proof. Let U be a non-empty open subset of $(\Bbbk^*)^s$ satisfying part (iii) of Lemma 3.1. Let S be the subset of $\mathfrak{m}_\Phi^*$ consisting of elements of the form

$$f_{\boldsymbol{\lambda}} = \sum_{i=1}^{s} \lambda_i X_{\theta_i}^*$$

for $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_s) \in \Bbbk^s$. Then $\Omega = \{f_\mathbf{a}; \mathbf{a} \in U\}$ is an open subset of S.

Let M be the algebraic adjoint group of $\mathfrak{m}_\Phi$. Consider the elements $z_1, \ldots, z_s$ of $\Bbbk[\mathfrak{m}_\Phi^*]^M$ constructed by Panov in [7] such that $\Bbbk(\mathfrak{m}_\Phi^*)^M = \Bbbk(z_1, \ldots, z_s)$, and

$$z_i = X_{\theta_i} P + R, \tag{6}$$

where P is some product of powers of $z_1, \ldots, z_{i-1}$ and R is a polynomial in X_α for $\alpha \succ \theta_i$.

For $i = 1, \ldots, s$, denote by $U_i = \{f \in \mathfrak{q}_\Phi^*; z_i(f) \neq 0\}$ the standard open subset of $\mathfrak{q}_\Phi^*$ associated to z_i. By (6), $z_1 = X_{\theta_1}$ so we clearly have $U_1 \cap \Omega \neq \emptyset$. Next, for $i > 0$, we have

$$z_{i+1}(f_{\boldsymbol{\lambda}}) = \lambda_{i+1} P(f_{\boldsymbol{\lambda}}) + R(f_{\boldsymbol{\lambda}}).$$

By the properties of P and R from the preceding paragraph, P depends only on $z_1, \ldots, z_i$, and $R(f_{\boldsymbol{\lambda}})$ depends only on $\lambda_1, \ldots, \lambda_i$. By induction, we obtain

$$\Omega \cap \left(\bigcap_{j=1}^{i+1} U_j \right) \neq \emptyset.$$

Hence $\Omega' = \Omega \cap \left(\bigcap_{i=1}^{s} U_i \right)$ is a non-empty open subset of Ω.

Consider the map

$$\begin{array}{rccc} \Psi & : & M \times \Omega' & \to & \mathfrak{m}_\Phi^* \\ & & (m, f) & \mapsto & m.f \end{array}$$

Assume that $f_{\boldsymbol{\lambda}}$ and $f_{\boldsymbol{\mu}}$ are two elements of Ω' which are in the same M-orbit. Then we have $z_i(f_{\boldsymbol{\lambda}}) = z_i(f_{\boldsymbol{\mu}})$ for $i = 1, \ldots, s$. In particular, we have $z_1(f_{\boldsymbol{\lambda}}) = z_1(f_{\boldsymbol{\mu}})$ so $\lambda_1 = \mu_1$.

Let us proceed by induction. Suppose that $i > 0$ and that $\lambda_j = \mu_j$ for $1 \leqslant j \leqslant i$. We have

$$\begin{aligned} z_{i+1}(f_{\boldsymbol{\lambda}}) &= \lambda_{i+1} P(f_{\boldsymbol{\lambda}}) + R(f_{\boldsymbol{\lambda}}), \\ z_{i+1}(f_{\boldsymbol{\mu}}) &= \mu_{i+1} P(f_{\boldsymbol{\mu}}) + R(f_{\boldsymbol{\mu}}). \end{aligned}$$

Since $f_{\boldsymbol{\lambda}}, f_{\boldsymbol{\mu}} \in \Omega'$, we deduce from the properties of P and R described above that $\lambda_{i+1} = \mu_{i+1}$. Hence $\boldsymbol{\lambda} = \boldsymbol{\mu}$.

It follows that for any $f_{\boldsymbol{\lambda}} \in \Omega'$, we have

$$\Psi^{-1}(f_{\boldsymbol{\lambda}}) = \{(m, g) \in M \times \Omega'; m.g = f_{\boldsymbol{\lambda}}\} \simeq \mathrm{Stab}_M(f_{\boldsymbol{\lambda}}).$$

By [7] and Theorem 3.2, we have $\dim \mathfrak{m}_\Phi^{f_{\boldsymbol{\lambda}}} = \ell(\mathbf{h}) = s$. Since $\dim \mathfrak{m}_\Phi^{f_{\boldsymbol{\lambda}}} = \dim \mathrm{Stab}_M(f_{\boldsymbol{\lambda}})$ and $\dim \mathrm{Im}\Psi = \dim(M \times \Omega') - \dim \Psi^{-1}(f_{\boldsymbol{\lambda}})$, we obtain that $\dim \mathrm{Im}\Psi = \dim M$. Therefore $M.\Omega'$ contains an open subset $\mathcal{O}$ of $\mathfrak{m}_\Phi^*$.

Let p be the projection $\mathfrak{q}_\Phi^* \to \mathfrak{m}_\Phi^*$ via restriction. Then p is M-equivariant. Since the set of regular elements of $\mathfrak{q}_\Phi^*$ is an open subset, we deduce that there exist $\varphi \in \mathfrak{q}_\Phi^*$ regular and $f_{\boldsymbol{\lambda}} \in \Omega'$ such that

$$\varphi_{|\mathfrak{m}_\Phi} = f_{\boldsymbol{\lambda}|\mathfrak{m}_\Phi}.$$

For all $X, Y \in \mathfrak{q}_\Phi$, we have $\varphi([X, Y]) = f_{\boldsymbol{\lambda}}([X, Y])$, and therefore

$$\mathrm{Mat}(\Phi_\varphi) = \begin{pmatrix} 0_{\ell,\ell} & A \\ -{}^tA & B \end{pmatrix} = \mathrm{Mat}(\Phi_{f_{\boldsymbol{\lambda}}}).$$

Hence by Theorem 4.1, we have

$$\chi(\mathfrak{q}_\Phi) = \dim \mathfrak{q}_\Phi^{f_\lambda} = \ell + \ell(\mathbf{h}) - 2d(\mathbf{h}).$$

■

Remark 4.3. According to the previous theorem, $c(\mathbf{h})$ is minimal when $\mathbf{h}$ is the Panov H-sequence.

Let $\mathfrak{g}$ be of type A_6. Set $\Phi = \{\alpha \in \Delta^+; \alpha \geqslant \alpha_{2,5}\}$, and $E = \Delta^+ \setminus \Phi$. Then the Panov H-sequence of E is $\mathbf{h} = \{\alpha_{1,4}, \alpha_{2,3}, \alpha_{3,5}, \alpha_{4,6}, \alpha_{5,6}, \alpha_{5,5}\}$ and we have $\ell(\mathbf{h}) = d(\mathbf{h}) = 6$.

We have another H-sequence $\mathbf{h}' = (\alpha_{3,5}, \alpha_{4,6}, \alpha_{6,6}, \alpha_{1,4}, \alpha_{1,3}, \alpha_{1,2}, \alpha_{2,3}, \alpha_{2,2})$ such that $\ell(\mathbf{h}') = 8$ and $d(\mathbf{h}') = 6$. Observe that $c(\mathbf{h}) = 0$ and $c(\mathbf{h}') = 2$.

5. Stability

Let $\mathfrak{a}$ be an algebraic Lie algebra and let A be its adjoint algebraic group. Recall that $g \in \mathfrak{a}^*$ is *stable* if there exists an open subset U of $\mathfrak{a}^*$ containing g such that $\mathfrak{a}^g$ and $\mathfrak{a}^h$ are A-conjugate for all $h \in U$.

The following result is proved in [10]:

Proposition 5.1. *Let $\mathfrak{a}$ be an algebraic Lie algebra and $f \in \mathfrak{a}^*$.*

(i) If f is stable, then it is a regular element of $\mathfrak{a}^*$.

(ii) The linear form f is stable if and only if $[\mathfrak{a}, \mathfrak{a}^f] \cap \mathfrak{a}^f = \{0\}$.

In this section, we return to the general case, that is $\mathfrak{g}$ is not necessarily of type A.

Proposition 5.2. *Let Φ be a subset of Δ^+ such that $\mathfrak{g}^\Phi$ is an ad-nilpotent ideal of $\mathfrak{g}$. Let $\mathbf{h} = (\theta_1, \dots, \theta_s)$ be an H-sequence of $\Delta^+ \setminus \Phi$ consisting of linear independent elements. Then there exists $f \in \mathfrak{q}_\Phi^*$ which is stable and $\chi(\mathfrak{q}_\Phi) = c(\mathbf{h})$.*

Proof. Our hypothesis implies that $\ell(\mathbf{h}) = d(\mathbf{h})$. So

$$c(\mathbf{h}) = \ell - d(\mathbf{h}).$$

Let U be a non-empty open subset of $(\mathbb{k}^*)^s$ satisfying part (iii) of Lemma 3.1. By Theorem 3.2 and Lemma 3.1, if $\mathbf{a} \in U$, we have $\dim \mathfrak{q}_\Phi^{f_\mathbf{a}} = c(\mathbf{h})$. It follows from Lemma 3.1 that $\mathfrak{q}_\Phi^{f_\mathbf{a}}$ is a commutative Lie subalgebra of $\mathfrak{q}_\Phi$ consisting of semi-simple elements. Therefore, there exists a vector subspace $\mathfrak{r}$ of $\mathfrak{q}_\Phi$ such that $\mathfrak{q}_\Phi = \mathfrak{q}_\Phi^{f_\mathbf{a}} \oplus \mathfrak{r}$ and $[\mathfrak{q}_\Phi^{f_\mathbf{a}}, \mathfrak{r}] \subset \mathfrak{r}$. So $[\mathfrak{q}_\Phi, \mathfrak{q}_\Phi^{f_\mathbf{a}}] \subset \mathfrak{r}$ and the result follows by Theorem 5.1. ■

We shall now show that $\mathfrak{q}_\Phi$ does not necessarily contain a stable linear form in general.

Let $\mathfrak{gl}_6(\Bbbk)$ be the set of 6×6 matrices and let $\{E_{i,j}, 1\leqslant i,j\leqslant 6\}$ be its canonical basis. Set $\mathfrak{g}=\mathfrak{sl}_6(\Bbbk)$, the set of 6×6 matrices of trace zero, $\mathfrak{h}$ the set of diagonal matrices in $\mathfrak{g}$ and $\mathfrak{b}$ the set of upper triangular matrices in $\mathfrak{g}$.

Therefore, we can choose $X_{\alpha_i+\cdots+\alpha_j}=E_{i,j+1}$, for $1\leqslant i\leqslant j\leqslant 5$. Set

$$\Phi=\{\alpha\in\Delta^+;\alpha\geqslant\alpha_1+\alpha_2+\alpha_3 \text{ or } \alpha\geqslant\alpha_3+\alpha_4+\alpha_5\}.$$

The Panov H-sequence of $\Delta^+\setminus\Phi$ is

$$\mathbf{h}=(\alpha_1+\alpha_2,\alpha_2+\alpha_3+\alpha_4,\alpha_3+\alpha_4,\alpha_3,\alpha_4+\alpha_5,\alpha_5).$$

We have $\ell(\mathbf{h})=6$, $d(\mathbf{h})=5$ and $c(\mathbf{h})=1$.

By definition, $\mathfrak{i}=\mathfrak{g}^\Phi$ is an ad-nilpotent ideal of $\mathfrak{g}$.

Proposition 5.3. *The Lie algebra $\mathfrak{q}_\Phi^*$ does not possess any stable linear form.*

Proof. By Theorem 4.2 or Proposition 3.3, we have $\chi(\mathfrak{q}_\Phi)=1$. Let $t\in\Bbbk^*$ and $\boldsymbol{\lambda}_t=(1,\ldots,1,t)\in\Bbbk^6$. Set $f_{\boldsymbol{\lambda}_t}=\sum_{i=1}^5 X_{\theta_i}+tX_{\theta_6}$ and

$$Z_t=X_{\alpha_1}-X_{\alpha_3}+\frac{1}{t}X_{\alpha_5}+(1+\frac{1}{t})X_{\alpha_2+\alpha_3}+X_{\alpha_3+\alpha_4}-X_{\alpha_4+\alpha_5}.$$

A simple calculation gives $\mathfrak{q}_\Phi^{f_{\boldsymbol{\lambda}_t}}=\mathrm{Vect}(Z_t)$.

Let $H\in\mathfrak{h}$ be such that $\alpha_1(H)=\alpha_3(H)=\alpha_5(H)=1$ and $\alpha_2(H)=\alpha_4(H)=0$. Then $[H,Z_t]=Z_t$, so $[\mathfrak{q}_\Phi,\mathfrak{q}_\Phi^{f_{\boldsymbol{\lambda}_t}}]\cap\mathfrak{q}_\Phi^{f_{\boldsymbol{\lambda}_t}}\neq\{0\}$. By Theorem 5.1, $f_{\boldsymbol{\lambda}_t}$ is not stable.

Denote by Q the algebraic adjoint group of $\mathfrak{q}_\Phi$. Then Q can be identified with the quotient of the set of invertible upper triangular matrices by a closed normal subgroup. Let $s,t\in\Bbbk^*$. Assume that $\mathfrak{q}_\Phi^{f_{\boldsymbol{\lambda}_t}}$ and $\mathfrak{q}_\Phi^{f_{\boldsymbol{\lambda}_s}}$ are Q-conjugate, then there exist $\lambda\in\Bbbk^*$ and an invertible triangular matrix P such that $PZ_sP^{-1}-\lambda Z_t\in\mathfrak{i}$. By the definition of $\mathfrak{i}$, for any element $L\in\mathfrak{i}$ and for any upper triangular matrix R, we have $LR\in\mathfrak{i}$. It follows that $PZ_s-\lambda Z_tP\in\mathfrak{i}$. By a direct computation, we obtain that $t=s$.

Recall that for $f,g\in\mathfrak{q}_\Phi^*$, if f and g are Q-conjugate, then $\mathfrak{q}_\Phi^f$ and $\mathfrak{q}_\Phi^g$ are also Q-conjugate. We define

$$\begin{array}{rccl}\Psi & : & Q\times\Bbbk^* & \rightarrow & \mathfrak{q}_\Phi^* \\ & & (x,t) & \mapsto & x.f_{\boldsymbol{\lambda}_t}\end{array}$$

By the above consideration, we deduce that

$$\Psi^{-1}(f_{\boldsymbol{\lambda}_t})=\{(x,s)\in Q\times\Bbbk^*;x.f_{\boldsymbol{\lambda}_s}=f_{\boldsymbol{\lambda}_t}\}=\mathrm{Stab}_Q(f_{\boldsymbol{\lambda}_t}).$$

Since $\dim\mathrm{Im}\Psi=\dim(Q\times\Bbbk^*)-\dim\Psi^{-1}(f_{\boldsymbol{\lambda}_t})$, we have $\dim\mathrm{Im}\Psi=\dim Q$. Therefore $Q.\Bbbk^*$ contains a non-empty open subset of $\mathfrak{q}_\Phi^*$ which does not contain any stable linear form.

Since the set of stable linear forms of $\mathfrak{q}_\Phi$ is an open subset of $\mathfrak{q}_\Phi^*$, the result follows immediately. ∎

6. Remarks on the exactness of the upper bounds

Assume that $\mathfrak{g}$ is of type C_7. Using the numbering of simple roots in [12], set

$$\begin{aligned}
&\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,\\
&\beta_2 = \alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7,\\
&\beta_3 = 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7,\\
&\Phi = \{\alpha \in \Delta^+; \alpha \geqslant \beta_i, \text{ for some } i \text{ such that } 1 \leqslant i \leqslant 3\}.
\end{aligned}$$

We check by hand that the minimal length of an H-sequence associated to $\Delta^+ \setminus \Phi$ is 8. For example, the following H-sequence $\mathbf{h} = (\theta_1, \dots, \theta_s)$ is of length 8:

$$\begin{aligned}
&\theta_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, && \theta_5 = \alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7,\\
&\theta_2 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7, && \theta_6 = 2\alpha_5 + 2\alpha_6 + \alpha_7,\\
&\theta_3 = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7, && \theta_7 = \alpha_5 + \alpha_6,\\
&\theta_4 = \alpha_3 + \alpha_4 + \alpha_5, && \theta_8 = \alpha_5.
\end{aligned}$$

and $d(\mathbf{h}) = 7$.

By Theorem 3.2, we have

$$\chi(\mathfrak{q}_\Phi) \leqslant \ell + \ell(\mathbf{h}) - 2d(\mathbf{h}) = 1, \text{ and } \chi(\mathfrak{m}_\Phi) \leqslant \ell(\mathbf{h}) = 8.$$

So $\chi(\mathfrak{q}_\Phi) = 1$ by Proposition 3.3. But by considering an arbitrary linear form, we found that $\chi(\mathfrak{m}_\Phi) \leqslant 6$. Thus the upper bound for the index of $\mathfrak{m}_\Phi$ is not always exact when $\mathfrak{g}$ is not of type A.

We did some computations using GAP4 on arbitrary linear forms when $\mathfrak{g}$ is of rank less than or equal to 6, and we have not found an example where the upper bound for $\chi(\mathfrak{q}_\Phi)$ is not exact. This leads us to formulate the following conjecture:

Conjecture 6.1. Let $\Phi \subset \Delta^+$ such that $\mathfrak{g}^\Phi$ is an ad-nilpotent ideal of $\mathfrak{g}$. There exists an H-sequence $\mathbf{h}$ of $\Delta^+ \setminus \Phi$ such that

$$\chi(\mathfrak{q}_\Phi) = \ell + \ell(\mathbf{h}) - 2d(\mathbf{h}).$$

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