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On a Lie Group Characterization of Quasi-local Symmetries of Nonlinear Evolution Equations

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Abstract. We develop an efficient algebraic approach to classifying nonlinear evolution equations in one spatial dimension that admit non-local transformation groups (quasi-local symmetries), i.e., groups involving integrals of the dependent variable. It applies to evolution equations invariant under Lie point symmetries leaving the temporal variable invariant. We construct inequivalent realizations of two- and three-dimensional Lie algebras leading to evolution equations admitting quasi-local symmetries. Finally, we generalize the approach in question for the case of an arbitrary system of evolution equations in two independent variables.

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1. Introduction

In the present paper we study symmetries of nonlinear evolution equations in one spatial dimension

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n), \quad n \geq 2. \tag{1}$$

Here $u = u(t, x)$ is a real-valued function of two real variables t, x, $u_i = \partial^i u/\partial x^i$, $i = 1,\ 2, \dots, n$, and F is an arbitrary smooth real-valued function.

There is an extensive body of literature devoted to analysis and applications of local (Lie and higher Lie) symmetries to evolution equations of the form (1) (see, e.g., [9] and the references therein). However, much less is known about nonlocal symmetries of nonlinear equations (1). Unlike the case of local symmetries there is still no regular method for constructing nonlocal symmetries of nonlinear partial differential equations.

In what follows we develop group-theoretical approach to classifying nonlocal symmetries of nonlinear evolution equations (1).

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The most general Lie point transformation group leaving differential equation (1) invariant has the form (see, e.g., [20, 21])

$$t' = T(t, \vec{\theta}), \quad x' = X(t, x, u, \vec{\theta}), \quad u = U(t, x, u, \vec{\theta}). \qquad (2)$$

Here T, X, U are smooth real-valued functions satisfying the non-singularity condition $\frac{D(T,X,U)}{D(t,x,u)} \not\equiv 0$ in some open domain of $\mathbb{R}^3$ and $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_r) \in \mathbb{R}^r$ is the vector of group parameters.

If a transformation of the space of variables t, x, u changes the specific form of (1) leaving invariant its differential structure, then we arrive at the concept of equivalence group. More precisely, if the (locally) invertible change of variables,

$$t \to t' = T(t, x, u), \quad x \to x' = X(t, x, u), \quad u \to u' = U(t, x, u)$$

maps Eq.(1) into a possibly different n-th order evolution equation

$$u'_{t'} = G(t', x', u', u'_1, u'_2, \ldots, u'_n),$$

then this change of variables is called an equivalence transformation. The set of all possible equivalence transformations forms a finite- or infinite-dimensional group, $\mathcal{E}$, which is called the equivalence group of equation (1). Clearly, if we require that $G \equiv F$ then the equivalence group boils down to the symmetry group of Eq.(1). Consequently, Lie point symmetry group of a given equation is a subgroup of its equivalence group. Further details can be found in [17].

Let partial differential equation (1) be invariant under Lie point transformation group (2). What would happen to this Lie symmetry if we perform a transformation from the equivalence group of the equation under study? Evidently, transformation group (2) after being rewritten in the 'new' variables t', x', u' becomes Lie point symmetry of the transformed equation.

Now suppose that we allow for a more general equivalence transformation group

$$t \to t' = T(t, x, u, \vec{v}), \quad x \to x' = X(t, x, u, \vec{v}), \quad u \to u' = U(t, x, u, \vec{v}),$$

where $\vec{v} = (u_1, u_2, \ldots, u_p, \partial^{-1}u, \partial^{-2}u, \ldots \partial^{-s}u)$ with $\partial^{-1}u = \int u(t,x)dx$ and $\partial^{-k-1} = \partial^{-1}\partial^{-k}$. Saying it another way, we allow for an equivalence transformation to include derivatives and integrals of the dependent variable u. If such a transformation still preserves the differential structure of evolution equation (1), what would happen to Lie point symmetries of the latter? The answer is, 'it depends'. In some cases, Lie point symmetry transforms into another Lie point symmetry. However, it could happen that some Lie point symmetries would 'disappear' after performing equivalence transformation, meaning that they cannot be found within the framework of the infinitesimal Lie method. The reason is that the transformation rule for the variables t, x, u might contain derivatives and integrals of u, which are beyond reach for the standard Lie method. One needs to apply the generalized Lie [10]-[4] or non-Lie [7] approaches to be able to handle those symmetries.

Akhatov, Gazizov and Ibragimov were the first to introduce the concept of quasi-local symmetry [1], which is a special case of nonlocal symmetry. Independently, the notion of quasi-local symmetry has been suggested in [14]. We employ the term 'quasi-local symmetry' in a more narrow sense meaning Lie point symmetries which after nonlocal equivalence transformation of an equation under study turn into non-Lie symmetries. Note that a similar concept was studied by Kaptsov in [11].

In the present paper we suggest a simple regular method for deriving quasi-local symmetries (QLS) of evolution equations. Note that the basic idea of the method has been suggested in our paper [2] and some non-trivial examples of second-order evolution equations with QLS are given in [21].

What is more, we demonstrate that the method developed readily applies to arbitrary systems of evolution equations with two independent variables.

2. Method description

The most general Lie point transformation group admitted by evolution equation (1) is of the form (2). The infinitesimal operator, Q, of this group reads as [20]

$$Q = \tau(t)\partial + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u. \tag{3}$$

Provided $\tau \equiv 0$, there is a transformation,

$$t \to \bar{t} = t, \quad x \to \bar{x} = X(t,x,u), \quad u \to \bar{u} = U(t,x,u), \tag{4}$$

that reduces Q to the canonical form ∂_u (we drop the bars). Evolution equation (1) now becomes

$$u_t = f(t,x,u_1,u_2,\ldots,u_n). \tag{5}$$

Note that the right-hand side of Eq.(5) does not depend explicitly on u.

Differentiating (5) with respect to x yields

$$u_{tx} = \frac{\partial f}{\partial x} + \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} u_{i+1}.$$

Making the change of variables

$$\bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u_x \tag{6}$$

and dropping the bars we finally get

$$u_t = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} + \sum_{i=2}^{n} \frac{\partial f}{\partial u_{i-1}} u_i, \tag{7}$$

where $f = f(t,x,u,u_1,\ldots,u_{n-1})$.

Thus the nonlocal transformation (6) preserves the differential structure of the class of evolution equations (5).

Let the differential equation (5) admit r-parameter Lie point transformation group (2) with $\vec{\theta} = (\theta_1,\ldots,\theta_r)$ and $r \geq 2$. To obtain the symmetry group of Eq.(7) we need to transform (2) according to (6). To this end we compute the first prolongation of formulas (2) and derive the transformation rule for the first derivative of u

$$\frac{\partial u'}{\partial x'} = \frac{U_u u_x + U_x}{X_u u_x + X_x},$$

so that symmetry group (2) reads

$$t' = T(t, \vec{\theta}), \quad x' = X(t, x, v, \vec{\theta}), \quad u' = \frac{U_v u + U_x}{X_v u + X_x} \tag{8}$$

with $v = \partial^{-1} u$ and $U = (t, x, v, \vec{\theta})$. Consequently, if the right-hand sides of (8) depend explicitly on the nonlocal variable v, then the transformation group (8) is a quasi-local symmetry of Eq.(7).

Evidently, the transformations (8) include the variable v if and only if

$$X_v \neq 0 \text{ or } \frac{\partial}{\partial v}\left(\frac{U_v u + U_x}{X_v u + X_x}\right) \neq 0$$

or, equivalently (since all the functions involved are real-valued),

$$\begin{aligned}(X_v)^2 + (U_{vv}X_v - U_v X_{vv})^2 + (U_{xv}X_x - U_x X_{xv})^2 \\ +(U_{vv}X_x + U_{xv}X_v - U_x X_{vv} - U_v X_{xv})^2 \neq 0.\end{aligned} \tag{9}$$

If $X_v \neq 0$, then the above inequality holds true. If X_v does vanish identically, then (9) reduces to $U_{xv}^2 + U_{vv}^2 \neq 0$. It is straightforward to express the above constraints in terms of the coefficients of the corresponding infinitesimal operator of group (2). As a result, we get the following assertion.

Theorem 2.1. *Equation (5) can be reduced to the evolution equation (7) having QLS if it admits Lie point symmetry, whose infinitesimal generator satisfies one of the inequalities*

$$\frac{\partial \xi}{\partial u} \neq 0, \tag{10}$$

$$\frac{\partial \xi}{\partial u} = 0, \quad \left(\frac{\partial^2 \eta}{\partial u \partial x}\right)^2 + \left(\frac{\partial^2 \eta}{\partial u \partial u}\right)^2 \neq 0. \tag{11}$$

Now we can formulate the procedure for constructing evolution equations of the form (1) admitting QLS.

1. We compute the maximal Lie point symmetry group $\mathcal{S}$ of differential equation (1).
2. We classify inequivalent one-parameter subgroups of $\mathcal{S}$ and select subgroups $\mathcal{S}_1, \ldots \mathcal{S}_p$ whose infinitesimal operators are of the form $Q = \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$.
3. For each subgroup, $\mathcal{S}_i$, we construct a change of variables (4) reducing the corresponding infinitesimal operator Q to the canonical operator $\partial_{\bar{u}}$, which leads to an evolution equation of the form (5).

4. Since the invariance group, $\bar{\mathcal{S}}$, admitted by (5) is isomorphic to $\mathcal{S}$, we can utilize the results of subgroup classification of $\mathcal{S}$. For each of the one-parameter subgroups of $\bar{\mathcal{S}}$ we check whether its infinitesimal generators satisfies one of conditions (10), (11) of Theorem 2.1. This yields the list of evolution equations that can be reduced to those having QLS.
5. Performing a nonlocal change of variables (6) we obtain the evolution equations (7) admitting quasi-local symmetries (8).

In what follows, we classify the realizations of two- and three-parameter Lie point transformation groups leading to evolution equations (1) that admit QLS.

Hereafter we assume that evolution equation (1) admits a Lie symmetry $Q = \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ and therefore can be reduced to the form (5). Differential equation (5) is guaranteed to admit at least the one-parameter symmetry group generated by the operator ∂_u. What we are going to do is to describe all realizations of two- and three-dimensional Lie algebras, which

- are invariance algebras of equations of the form (5), and,
- have coefficients satisfying one of the inequalities (10), (11) from Theorem 2.1.

With these realizations in hand, the problem of describing equations having QLS reduces to a straightforward application of the infinitesimal Lie method [17, 15, 8], which boils down to integrating characteristic system for calculating differential invariants of the corresponding Lie algebras of first-order differential operators.

Let us recall that the most general symmetry generator admitted by (5) is of the form (3), while the most general equivalence group admitted by Eq.(1) reads as (see, e.g., [12])

$$\bar{t} = T(t), \quad \bar{x} = X(t,x,u), \quad \bar{u} = U(t,x,u), \tag{12}$$

where T, X, U are arbitrary smooth real-valued functions.

We can always choose basis operators of the invariance algebra of Eq.(5) so that

$$e_0 = \partial_u, \quad e_i = \tau_i(t)\partial_t + \xi_i(t,x,u)\partial_x + \eta_i(t,x,u)\partial_u,$$

where $i = 1, \ldots, r$. Note that by the Magadeev theorem [13] the maximal possible value for r is $n+4$, n being the order of evolution equation (5), provided (5) is not locally equivalent to a linear equation. In particular, for the second-order evolution equation we have $r \leq 6$. By the definition of Lie algebra there are constant $r \times r$ matrix C and constant r-component vector $\vec{c}$ such that

$$[e_0,\, e_i] = \sum_{j=1}^{r} C_{ij} e_j + c_i e_0, \quad i = 1, \ldots, r. \tag{13}$$

Here $[Q_1,\, Q_2] \equiv Q_1 Q_2 - Q_2 Q_1$.

The system of equations (13) is the starting point of our classification algorithm. First of all, let us note that by re-arranging the basis of the Lie algebra,

$e_\mu \to \sum_{\nu=0}^{r} a_{\mu\nu} e_\nu,\ \mu = 0, 1, \ldots, r$, we can always reduce the constant matrix C to the canonical form. Consequently, without any loss of generality we may assume that the matrix C is in the canonical form.

Computing commutators in the left-hand sides of (13) and equating the coefficients of linearly-independent operators $\partial_t, \partial_x, \partial_u$ we get the following system of partial differential equations:

$$\frac{\partial \vec{\xi}}{\partial u} = C\vec{\xi}, \quad \frac{\partial \vec{\eta}}{\partial u} = C\vec{\eta} + \vec{c}, \quad C\vec{\tau} = 0, \tag{14}$$

where $\vec{\xi} = (\xi_1, \ldots, \xi_r)$, $\vec{\eta} = (\eta_1, \ldots, \eta_r)$, $\vec{\tau} = (\tau_1, \ldots, \tau_r)$. After integrating differential equations (14) we need to ensure that the operators $e_1, \ldots, e_r$ do form a basis of Lie algebra and satisfy the additional set of commutation relations

$$[e_i,\, e_j] = \sum_{k=1}^{r} c_{ij}^k e_k, \quad k = 1, \ldots, r.$$

Next, we simplify the form of operators $e_1, \ldots, e_r$ using the suitable equivalence transformations from the group $\mathcal{E}$. As the final step, we verify that at least one of the coefficients of one the operators $e_1, \ldots, e_r$ satisfy either (10) or (11).

3. Realizations of two-dimensional QLS algebras

For the case $r = 1$, system (14) reduces to a pair of non-coupled differential equations

$$\xi_u = \lambda\xi, \quad \eta_u = \lambda\eta + c, \quad \lambda\tau = 0, \tag{15}$$

where λ, c are constants.

While integrating (15) we need to differentiate between the two cases $\lambda \neq 0$ and $\lambda = 0$.

Case 1. $\lambda \neq 0$. The general solution of (15) has the form

$$\tau(t) = 0, \quad \xi(t,x,u) = W_1(t,x)\exp(\lambda u),$$
$$\eta(t,x,u) = W_2(t,x)\exp(\lambda u) - c\lambda^{-1}.$$

where W_1, W_2 are arbitrary smooth functions. So that the two-dimensional Lie algebra $\langle e_0, e_1\rangle$ reads as

$$e_0 = \partial_u, \quad e_1 = W_1(t,x)\exp(\lambda u)\partial_x + \left(W_2(t,x)\exp(\lambda u) - c\lambda^{-1}\right)\partial_u.$$

As $\lambda \neq 0$ we can always re-scale u, i.e., make a transformation $u \to ku$, $k = \mathrm{const}$, in order to get $\lambda = 1$. Next taking as new e_1 the linear combination $c\lambda^{-1}e_0 + e_1$ we get rid of the term in e_1 which is proportional to c, namely,

$$e_0 = \partial_u, \quad e_1 = W_1(t,x)\exp(u)\partial_x + W_2(t,x)\exp(u)\partial_u.$$

It is not difficult to verify that the most general subgroup of the equivalence group $\mathcal{E}$ not altering the form of equation (5) and operator ∂_u is given by the formulas

$$\bar{t} = T(t), \quad \bar{x} = X(t,x), \quad \bar{u} = u + U(t,x), \tag{16}$$

where T, X, U are arbitrary smooth functions.

Since the functions W_1 and W_2 do not vanish simultaneously, we have three possible subcases, (1) $W_1 \neq 0, W_2 \neq 0$, (2) $W_1 \neq 0, W_2 = 0$, and $W_1 = 0, W_2 \neq 0$.

Case 1.1. $W_1 \neq 0, W_2 \neq 0$. Applying (16) with $T = t$ we reduce the operator e_1 to the form

$$e_1 = \epsilon_1 \exp(u)\partial_x + \epsilon_2 \exp(u)\partial_u,$$

where $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$. Combining equivalence transformation $t \to t,\ x \to -x,\ u \to u$ and re-scaling $e_1 \to -e_1$ we get the final form of the basis elements e_0, e_1

$$e_0 = \partial_u, \quad e_1 = \exp(u)(\partial_x + \partial_u).$$

Since $\xi_u = \exp(u) \neq 0$, the condition (10) of Theorem 2.1 holds true and the evolution equation invariant under the above algebra is equivalent to a quasi-linear evolution equation that admits QLS.

Case 1.2. $W_1 \neq 0, W_2 \neq 0$. Applying transformation (15) with $T(t) = t, U(t, x) = 0, k = 1$ we reduce the operator e_1 to the form $e_1 = \exp(u)\partial_x$. Again, the coefficient $\xi = \exp(u)$ obeys condition (10) of Theorem 2.1 and, therefore, it gives rise to the two-dimensional Lie algebra

$$e_0 = \partial_u, \quad e_1 = \exp(u)\partial_x$$

that leads to an evolution equation admitting QLS.

Case 1.3. $W_1 = 0, W_2 \neq 0$. Applying transformation (15) with $T(t) = t, X(t, x) = x$ we reduce the operator e_1 to the form $e_1 = \pm \exp(u)\partial_u$. Re-scaling, if necessary, the operator e_1 to $-e_1$ we can make sure that e_1 reads as $\exp(u)\partial_u$ and finally get

$$e_0 = \partial_u, \quad e_2 = \exp(u)\partial_u.$$

Since the coefficient $\eta = \exp(u)$ obeys condition (11) of Theorem 2.1, an evolution equation invariant under the above algebra is equivalent to a partial differential equation of the form (7) which has QLS.

Case 2. $\lambda = 0$. System (15) is readily integrated to yield

$$T(t) = W_0(t), \quad \xi = W_1(t, x), \quad \eta = W_2(t, x) + cu.$$

Here W_0, W_1, W_2 are arbitrary smooth functions. Checking the conditions of Theorem 2.1 we see that neither of them can be satisfied by the coefficients of the operator e_1. Consequently, this case yields no equations admitting QLS.

Below we give the full list of $\mathcal{E}$-inequivalent realizations of two-dimensional Lie algebras spanned by the operators $e_0 = \partial_u$, $e_1 = T(t)\partial_t + \xi(t, x, u)\partial_x + \eta(t, x, u)\partial_u$ satisfying the conditions of Theorem 2.1

$$\begin{aligned} A_2^1 &: \langle \partial_u,\ \exp(u)(\partial_x + \partial_u) \rangle, \\ A_2^2 &: \langle \partial_u,\ \exp(u)\partial_x \rangle, \\ A_2^3 &: \langle \partial_u,\ \exp(u)\partial_u \rangle. \end{aligned}$$

Evolution equations (5) invariant under the above algebras are reduced to differential equations that admit QLS. The corresponding QLS are obtained by re-writing the transformation groups generated by the operators e_1 in terms of new (nonlocal) variables t, x, u and $\partial^{-1}u$.

Consider, for example, the algebra $A_2^2 = \langle \partial_u,\ \exp(u)\partial_x \rangle$. Applying the standard infinitesimal Lie algorithm [17] we obtain the determining equations for the function f

$$-u_x f - f_x + u_x^2 f_{u_x} + (u_x^3 + 3u_x u_{xx}) f_{u_{xx}} = 0.$$

The general solution of the above equation reads as

$$f(t, x, u_x, u_{xx}) = u_x \tilde{f}(\omega_0, \omega_1, \omega_2),$$

where $\tilde{f}$ is an arbitrary smooth function and $\omega_0 = t$, $\omega_1 = (xu_x - 1)u_x^{-1}$, $\omega_2 = (u_{xx} + u_x^2)u_x^{-3}$ are absolute invariants of the transformation group generated by the operators ∂_u and $\exp(u)\partial_x$. Consequently, the evolution equation invariant under the algebra A_2^2 is of the form

$$u_t = u_x \tilde{f}(\omega_0, \omega_1, \omega_2).$$

Differentiating the above equation with respect to x and replacing u_x with u according to (6) we arrive at the evolution equation

$$u_t = u_x \tilde{f} + \frac{u_x + u^2}{u^2} \tilde{f}_{\omega_1} + \frac{u u_{xx} - 3(u^2 + 1)u_x}{u^4} \tilde{f}_{\omega_2}$$

with $\omega_0 = t$, $\omega_1 = x - u^{-1}$, $\omega_2 = (u_x + u^2)u^{-3}$. This differential equation admits the following quasi-local symmetry group

$$t' = t,\ x' = x + \theta \exp(v), \quad u' = \frac{u}{1 + \theta u \exp(v)}.$$

where θ is a group parameter and $v = \partial^{-1}u$.

4. Realizations of three-dimensional QLS algebras

Consider system of partial differential equations (14) with $r = 2$. The constant 2×2 matrix C has been reduced to the canonical real Jordan form. There are three inequivalent cases that need to be considered separately, namely, when eigenvalues

1. are complex conjugate,
2. are real, and the matrix C is diagonal,
3. are real and the matrix C is the 2×2 canonical Jordan box

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

We consider in detail the class of realizations of three-dimensional Lie algebras obtained for the case when C has two complex eigenvalues λ_1, λ_2. For the remaining two classes we present the final results only.

4.1. Case of diagonal canonical form with complex eigenvalues. As the characteristic equation of the real matrix C is real, the eigenvalues have to satisfy the additional constraint $\lambda_1^* = \lambda_2$, where the star stands for complex conjugation. Consequently, if we define $\nu = (\lambda_1+\lambda_2)/2$ and $a = (\lambda_1-\lambda_2)/(2i)$, then the general solution of (14) can be represented in the form

$$\begin{aligned}
&\tau_1 = 0, \quad \tau_2 = 0,\\
&\xi_1 = (W_1(t,x)\cos(au) + W_2(t,x)\sin(au))\exp(\nu u),\\
&\xi_2 = (W_2(t,x)\cos(au) - W_1(t,x)\sin(au))\exp(\nu u),\\
&\eta_1 = (W_3(t,x)\cos(au) + W_4(t,x)\sin(au))\exp(\nu u) + b_1,\\
&\eta_2 = (W_4(t,x)\cos(au) - W_3(t,x)\sin(au))\exp(\nu u) + b_2.
\end{aligned} \tag{17}$$

Here W_1, W_2, W_3, W_4 are arbitrary smooth real-valued functions, b_1, b_2 are real constants. Hence, the most general form of the basis operators e_1, e_2 is

$$\begin{aligned}
e_1 &= (W_1(t,x)\cos(au) + W_2(t,x)\sin(au))\exp(\nu u)\partial_x\\
&\quad +\Big((W_3(t,x)\cos(au) + W_4(t,x)\sin(au))\exp(\nu u) + b_1\Big)\partial_u,\\
e_2 &= (W_2(t,x)\cos(au) - W_1(t,x)\sin(au))\exp(\nu u)\partial_x\\
&\quad +\Big((W_4(t,x)\cos(au) - W_3(t,x)\sin(au))\exp(\nu u) + b_2\Big)\partial_u.
\end{aligned}$$

Note that $a \neq 0$, otherwise, λ_1, λ_2 are not complex. By re-scaling the variable u we can make a equal to 1. Next, applying to the operators e_1, e_2 an equivalence transformation (15) with $T(t) = t, X(t,x) = x$ we can get rid of the function W_2. With these remarks the operators e_1, e_2 take the form

$$\begin{aligned}
e_1 &= \Big((W_3(t,x)\cos u + W_4(t,x)\sin u)\exp(\nu u) + b_1\Big)\partial_u\\
&\quad +W_1(t,x)\cos u\exp(\nu u)\partial_x,\\
e_2 &= \Big((W_4(t,x)\cos u - W_3(t,x)\sin u)\exp(\nu u) + b_2\Big)\partial_u\\
&\quad -W_1(t,x)\sin u\exp(\nu u)\partial_x.
\end{aligned} \tag{18}$$

In what follows we need to distinguish between the cases, $\nu \neq 0$ and $\nu = 0$.
Case 1. $\nu \neq 0$. Performing, if necessary, transformation (15) with $T(t) = t, U(t,x) = u$ we can always make non-vanishing identically function W equal to 1. Next, taking as e_1 and e_2 the linear combinations $e_1 - b_1e_0$ and $e_2 - b_2e_0$ we can get rid of parameters b_1, b_2.

Now we need to ensure that the operators e_0, e_1, e_2 do form a realization of a Lie algebra. To this end we have to verify that the relation

$$[e_1,\ e_2] = \alpha e_1 + \beta e_2 + \gamma e_0 \tag{19}$$

with some real α, β, γ holds true. Calculating the commutators and equating the coefficients of linearly-independent operators $\partial_t, \partial_x, \partial_u$ we get the system of differential equations for W_3, W_4. Its general solution is given by the formulas

$$W_3 = \nu(\nu^2+1)^{-1}(x+p(t))^{-1}, \quad W_4 = -(\nu^2+1)^{-1}(x+p(t))^{-1}.$$

Here $p(t)$ is an arbitrary smooth function. Making the equivalence transformation (15) with $T(t)=t, X(t,x)=p(t), U(t,x)=0$ we eliminate the function $p(t)$ and arrive at the following realization of the three-dimensional Lie algebra

$$\begin{aligned} e_0 &= \partial_u, \\ e_1 &= \exp(\nu u)\cos u\partial_x + (\nu^2+1)^{-1}(\nu\cos u - \sin u)x^{-1}\exp(\nu u)\partial_u, \\ e_2 &= -\exp(\nu u)\sin u\partial_x - (\nu^2+1)^{-1}(\cos u - \nu\sin u)x^{-1}\exp(\nu u)\partial_u, \end{aligned}$$

Evidently, the coefficients of e_1, e_2 satisfy condition (10) of Theorem 2.1 and, consequently, evolution equation invariant under the symmetry algebra e_0, e_1, e_2 can be reduced to the one having QLS.

Case 2. $\nu = 0$. Operators (18) take the form

$$\begin{aligned} e_1 &= \Big((W_3(t,x)\cos u + W_4(t,x)\sin u) + b_1\Big)\partial_u + W_1(t,x)\cos u\partial_x, \\ e_2 &= \Big((W_4(t,x)\cos u - W_3(t,x)\sin u) + b_2\Big)\partial_u - W_1(t,x)\sin u\partial_x. \end{aligned}$$

Replacing e_1 and e_2 with the linear combinations $e_1 - b_1e_0$ and $e_2 - b_2e_0$ eliminates the parameters b_1, b_2. So that we can assume that $b_1 = 0, b_2 = 0$ without any loss of generality.

Case 2.1. $W_1 \neq 0$. Utilizing equivalence transformation (15) with $T = t, U = 0$ we can make W_1 equal to 1. After simple algebra we prove that for the operators e_1, e_2 to satisfy the remaining commutation relation (19) the functions W_3, W_4 have to take one of the following forms:

$$\begin{aligned} &W_3 = 0, \quad W_4 = \mu\tan(\mu x), \\ &W_3 = 0, \quad W_4 = -\mu\tanh(\mu x), \\ &W_3 = 0, \quad W_4 = x^{-1}, \end{aligned}$$

where μ is an arbitrary real parameter. Inserting the above expressions into the corresponding formulas for e_1, e_2 we finally get

$$\begin{aligned} e_0 &= \partial_u, \\ e_1 &= \cos u\partial_x + \mu\tan(\mu x)\sin u\partial_u, \\ e_2 &= -\sin u\partial_x + \mu\tan(\mu x)\cos u\partial_u; \\ e_0 &= \partial_u, \\ e_1 &= \cos u\partial_x - \mu\tanh(\mu x)\sin u\partial_u, \\ e_2 &= -\sin u\partial_x - \mu\tanh(\mu x)\cos u\partial_u; \\ e_0 &= \partial_u, \\ e_1 &= \cos u\partial_x - x^{-1}\sin u\partial_u, \\ e_2 &= -\sin u\partial_x - x^{-1}\cos u\partial_u. \end{aligned}$$

Case 2.2. $W_1 = 0$. In this case using transformation (15) with $T = t, X = x$ we can eliminate W_4. Inserting the corresponding expressions for e_1, e_2 into (19) and solving the obtained equations within the equivalence relation $\mathcal{E}$ yields

$$e_0 = \partial_u, \quad e_1 = \cos u\partial_u, \quad e_2 = \sin u\partial_u.$$

Note that all realizations of three-dimensional Lie algebras obtained under Cases 2.1, 2.2 satisfy the conditions of Theorem 2.1. Consequently, evolution equations invariant with respect to the above algebras can be transformed into equations admitting QLS.

Summing up we present the full list of realizations of three-dimensional Lie algebras, obtained for the case when 2×2 matrix C in (14) has two complex eigenvalues.

$$\begin{aligned} A_3^1 \ : \ & \langle \partial_u,\ \exp(\mu u)\cos u\partial_x + (\mu^2+1)^{-1}(\mu\cos u - \sin u)x^{-1}\exp(\mu u)\partial_u, \\ & -\exp(\mu u)\sin u\partial_x - (\mu^2+1)^{-1}(\cos u - \mu\sin u)x^{-1}\exp(\mu u)\partial_u\rangle, \end{aligned}$$

$$\begin{aligned} A_3^2 \ &: \ \langle \partial_u,\ \cos u\partial_x + \mu\tan(\mu x)\sin u\partial_u, -\sin u\partial_x + \mu\tan(\mu x)\cos u\partial_u\rangle, \\ A_3^3 \ &: \ \langle \partial_u,\ \cos u\partial_x - \mu\tanh(\mu x)\sin u\partial_u,\ -\sin u\partial_x - \mu\tanh(\mu x)\cos u\partial_u\rangle, \\ A_3^4 \ &: \ \langle \partial_u,\ \cos u\partial_x - x^{-1}\sin u\partial_u,\ -\sin u\partial_x - x^{-1}\cos u\partial_u\rangle, \\ A_3^5 \ &: \ \langle \partial_u,\ \cos u\partial_u,\ \sin u\partial_u\rangle. \end{aligned}$$

Here μ is an arbitrary real constant.

Evolution equations (5) invariant under the algebras $A_3^1, \ldots, A_3^5$ can be reduced to equations admitting QLS.

4.2. Case of diagonal canonical form with real eigenvalues. Without loss of generality we may assume that the matrix C from (14) has been reduced to the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Since λ_1, λ_2 do not vanish simultaneously we may assume that $\lambda_1 \neq 0$. Re-scaling u we make λ_1 equal to 1. With this choice of C, system (14) takes the form

$$\begin{aligned} & T_1 = 0, \quad \lambda_2 T_2 = 0, \\ & \xi_{1u} = \xi_1, \quad \xi_{2u} = \lambda_2\xi_2, \\ & \eta_{1u} = \eta_1 + c_1, \quad \eta_{2u} = \lambda_2\eta_2 + c_2. \end{aligned}$$

Integrating the above system within the equivalence relation $\mathcal{E}$, inserting the result into the remaining commutation relation (19) and solving the equations obtained we arrive at the following realizations of three-dimensional Lie algebras:

$$
\begin{aligned}
A_3^6 \ &: \ \langle \partial_u,\ \exp(u)\partial_x,\ (x^2+\sigma(t))\exp(-u)\partial_x + 2x\exp(-u)\partial_u\rangle,\\
A_3^7 \ &: \ \langle \partial_u,\ \exp(u)\partial_x + \epsilon\exp(u)\partial_u,\ (\sigma(t)\exp(x) \pm \exp(2x) + \mu)\exp(-u)\partial_x\\
&\qquad +(\pm\exp(2x) - \mu)\partial_u\rangle,\\
A_3^8 \ &: \ \langle \partial_u,\ \exp(u)\partial_x + \exp(u)\partial_u,\ (\sigma(t)\exp(-\mu x) \pm \exp((1-\mu)x))\\
&\qquad \times\exp(\mu u)\partial_x \pm \exp((1-\mu)x))\exp(\mu u)\partial_u\rangle,\\
A_3^9 \ &: \ \langle \partial_u,\ \exp(u)\partial_x,\ \sigma(t)x\exp(\mu u)\partial_x + \sigma(t)\exp(\mu u)\partial_u\rangle,\\
A_3^{10} \ &: \ \langle \partial_u,\ \exp(u)\partial_x,\ \sigma(t)\exp(\mu u)\partial_u\rangle,\\
A_3^{11} \ &: \ \langle \partial_u,\ \exp(u)(\partial_x + \partial_u),\ (\sigma(t) + \exp(x))\partial_x + \exp(x)\partial_u + \epsilon\partial_t\rangle,\\
A_3^{12} \ &: \ \langle \partial_u,\ \exp(u)(\partial_x + \partial_u),\ \sigma(t)\partial_x + \epsilon\partial_t\rangle,\\
A_3^{13} \ &: \ \langle \partial_u,\ \exp(u)\partial_x,\ \sigma(t)(x\partial_x + \partial_u) + \epsilon\partial_t\rangle,\\
A_3^{14} \ &: \ \langle \partial_u,\ \exp(u)\partial_x,\ \sigma(t)(\partial_x + \partial_u) + \epsilon\partial_t\rangle,\\
A_3^{15} \ &: \ \langle \partial_u,\ \exp(u)\partial_u,\ \sigma(t)\partial_u + \epsilon\partial_t\rangle,\\
A_3^{16} \ &: \ \langle \partial_u,\ \exp(u)\partial_u,\ \partial_x + \sigma(t)\partial_u\rangle,\\
A_3^{17} \ &: \ \langle \partial_u,\ \exp(u)(\partial_x + \partial_u),\ \exp(u)(\sigma_1(t)\exp(-x) + \sigma_2(t))\partial_x\\
&\qquad +\sigma_1(t)\exp(u)\partial_u\rangle,\\
A_3^{18} \ &: \ \langle \partial_u,\ \exp(u)\partial_x,\ \exp(u)(\sigma_1(t)x + \sigma_2(t))\partial_x + \sigma_1(t)\exp(u)\partial_u\rangle,\\
A_3^{19} \ &: \ \langle \partial_u,\ \exp(u)\partial_u,\ x\exp(u)\partial_u\rangle,\\
A_3^{20} \ &: \ \langle \partial_u,\ \exp(u)\partial_u,\ t\exp(u)\partial_u\rangle.
\end{aligned}
$$

Here $\sigma(t), \sigma_1(t), \sigma_2(t)$ are arbitrary real-valued smooth functions, μ is an arbitrary real parameter, and $\epsilon = 0, 1$.

By construction coefficients of algebras $A_3^6 - A_3^{20}$ satisfy the conditions of Theorem 2.1. Consequently, evolution equations (5) invariant under these algebras can be reduced to ones having QLS.

4.3. Case of non-diagonal canonical form. For the case under consideration, the matrix C from (14) is of the form $\begin{pmatrix} \theta & 1 \\ 0 & \theta \end{pmatrix}$. System of partial differential equations (13) now reads as

$$
\begin{aligned}
&\theta T_1 + T_2 = 0, \quad \theta T_2 = 0,\\
&\xi_{1u} = \theta\xi_1 + \xi_2, \quad \xi_{2u} = \theta\xi_2,\\
&\eta_{1u} = \theta\eta_1 + \eta_2 + c_1, \quad \eta_{2u} = \theta\eta_2 + c_2.
\end{aligned}
$$

Integrating the above system within the equivalence relation $\mathcal{E}$, inserting the result into (19) and solving the relations obtained we obtain eleven realizations of three-dimensional Lie algebras:

$$\begin{aligned}
A_3^{21} &: \langle \partial_u,\ u\exp(u)\partial_x + x^{-1}(u+1)\exp(u)\partial_u,\ \exp(u)\partial_x + x^{-1}\exp(u)\partial_u\rangle,\\
A_3^{22} &: \langle \partial_u,\ u\partial_x + (\mu u^2 + \sigma(t)\exp(4\mu x))\partial_u,\ \partial_x + 2\mu\partial_u\rangle,\\
A_3^{23} &: \langle \partial_u,\ u\partial_x + (\mu x + \nu u + \sigma_1(t))\partial_u + \sigma_2(t)\partial_t,\ \partial_x\rangle,\\
A_3^{24} &: \langle \partial_u,\ \partial_x + \mu u\tan(\mu x)\partial_u,\ \tan(\mu x)\partial_u\rangle,\\
A_3^{25} &: \langle \partial_u,\ \partial_x - \mu u\tanh(\mu x)\partial_u,\ \tanh(\mu x)\partial_u\rangle,\\
A_3^{26} &: \langle \partial_u,\ \partial_x - x^{-1}u\partial_u,\ x^{-1}\partial_u\rangle,\\
A_3^{27} &: \langle \partial_u,\ (u^2 + x)\partial_u,\ u\partial_u\rangle,\\
A_3^{28} &: \langle \partial_u,\ (u^2 + t)\partial_u,\ u\partial_u\rangle,\\
A_3^{29} &: \langle \partial_u,\ u(\nu + 2\mu\tan(\mu t + \alpha x))\partial_u + 2\partial_t,\ (\nu + 2\mu\tan(\mu t + \alpha x))\partial_u\rangle,\\
A_3^{30} &: \langle \partial_u,\ u(\nu + 2\mu\tanh(-\mu t + \alpha x))\partial_u + 2\partial_t,\ (\nu + 2\mu\\
&\qquad \times\tanh(-\mu t + \alpha x))\partial_u\rangle,\\
A_3^{31} &: \langle \partial_u,\ (\mu\nu x - \mu t - 2)(t - \nu x)^{-1}u\partial_u + 2t\partial_t,\ (\mu\nu x - \mu t - 2)\\
&\qquad \times(t - \nu x)^{-1}u\partial_u\rangle.
\end{aligned}$$

Here $\sigma(t), \sigma_1(t), \sigma_2(t)$ are arbitrary real-valued smooth functions, α, μ, ν are arbitrary real parameters.

Evolution equations (5) admitting symmetry algebras $A_3^{21}, \ldots, A_3^{31}$ can be reduced to equations having QLS.

5. Some generalizations

The technique developed in the previous sections naturally expands to cover general systems of evolution equations

$$\vec{u}_t = \vec{F}(t, x, \vec{u}, \vec{u}_1, \ldots, \vec{u}_n), \quad n \geq 2. \tag{20}$$

Here $\vec{F} = (F^1, \ldots, F^m)$ is an arbitrary m-component real-valued smooth function, $\vec{u} = (u^1, \ldots, u^m) \in \mathbb{R}^m$, and $\vec{u}_i = (\partial^i \vec{u})/(\partial x^i)$, $i = 1, \ldots, n$.

Suppose that system of evolution equations (20) admits m-parameter Abelian symmetry group which leaves the temporal variable, t, invariant. The infinitesimal generators of this group have to be of the form

$$Q_i = \xi_i(t, x, \vec{u})\partial_x + \sum_{j=1}^{m} \eta_{ij}(t, x, \vec{u})\partial_{u^j}, \quad i = 1, \ldots, n \tag{21}$$

and what is more, the rank of the matrix composed of the coefficients of differential operators $\partial_x, \partial_{u^1}, \ldots, \partial_{u^m}$ equals to m. Given these conditions, there exists a change of variables

$$\bar{t} = t, \quad \bar{x} = X(t, x, \vec{u}), \quad \vec{\bar{u}} = \vec{U}(t, x, \vec{u}) \tag{22}$$

that reduces operators (21) to canonical ones $\bar{Q}_i = \partial_{\bar{u}^i}$, $i = 1, \ldots, m$ (see, e.g., [17]). Consequently system of evolution equations (20) takes the form

$$\vec{u}_t = \vec{f}(t, x, \vec{u}_1, \ldots, \vec{u}_n), \quad n \geq 2. \tag{23}$$

Note that we dropped the bars.

Now we can apply the same trick we utilized for the case of a single evolution equation. Namely, we differentiate (23) with respect to x and map $\vec{u} \to \vec{u}_x$ thus getting

$$\vec{u}_t = \frac{\partial \vec{f}}{\partial x} + \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\partial \vec{f}}{\partial u^j_{i-1}} u^j_i \tag{24}$$

with $u^j_0 \equiv u^j$.

If system of evolution equations (23) admits the Lie point transformation group

$$t' = T(t, \theta), \quad x' = X(t, x, \vec{u}, \theta), \quad \vec{u}' = \vec{U}(t, x, \vec{u}, \theta), \tag{25}$$

where $\theta \in \mathbb{R}$ is a group parameter, then the transformed system of equations (25) admits the group

$$\begin{aligned} t' &= T(t, \theta), \\ x' &= X(t, x, \vec{v}, \theta), \\ \vec{u}' &= \frac{\vec{U}_x + \sum_{i=1}^{m} \vec{U}_{v^i} u^i}{X_x + \sum_{i=1}^{m} X_{v^i} u^i} \end{aligned} \tag{26}$$

with $v^i = \partial^{-1} u^i \equiv \int u^i dx$ and $\vec{U} = \vec{U}(t, x, \vec{v}, \theta)$. Consequently, provided either of relations

$$\frac{\partial X}{\partial v^j} \neq 0, \quad \frac{\partial}{\partial v^j} \left(\frac{\vec{U}_x + \sum_{i=1}^{m} \vec{U}_{v^i} u^i}{X_x + \sum_{i=1}^{m} X_{v^i} u^i} \right) \neq 0 \tag{27}$$

holds for some j, $1 \leq j \leq m$, system of evolution equations admits QLS (26).

Set of relations (27) is equivalent to the following system of inequalities

$$\sum_{i=1}^{m} X^2_{v^i} \neq 0,$$

or

$$\sum_{i=1}^{m} X^2_{v^i} = 0, \quad \sum_{i,j=1}^{m} (U^j_{xv^i})^2 + \sum_{i,j,k=1}^{m} (U^k_{v^i v^j})^2 \neq 0.$$

Note that all the functions involved are real-valued, so that vanishing of the sum of squares requires for every summand to vanish individually. Rewriting the obtained relations in terms of coefficients of the corresponding infinitesimal operators we arrive at the following assertion.

Theorem 5.1. *System of evolution equations (23) can be reduced to a system having QLS if it admits Lie point symmetry whose infinitesimal operator* $Q = \tau(t)\partial_t + \xi(t, x, \vec{u})\partial_x + \sum_{i=1}^{m} \eta_i(t, x, \vec{u})\partial_{u^i}$ *satisfies one of the inequalities*

$$\sum_{i=1}^{m} \xi^2_{v^i} \neq 0, \tag{28}$$

$$\sum_{i=1}^{m} \xi^2_{v^i} = 0, \quad \sum_{i,j=1}^{m} (\eta^j_{xv^i})^2 + \sum_{i,j,k=1}^{m} (\eta^k_{v^i v^j})^2 \neq 0. \tag{29}$$

Summing up we formulate the procedure for classification of systems of evolution equations (20) admitting QLS.

1. We compute the maximal Lie point symmetry group $\mathcal{S}$ of system of partial differential equations (20).
2. We classify inequivalent m-parameter Abelian subgroups $\mathcal{S}_1, \dots \mathcal{S}_p$ of the group $\mathcal{S}$ and select subgroups whose infinitesimal operators are of the form (21).
3. For each subgroup $\mathcal{S}_i$ we construct change of variables (22) reducing commuting infinitesimal operators, Q_i, to the canonical forms $\partial_{\bar{u}^i}$, which leads to system of evolution equations (23).
4. Since the invariance group, $\bar{\mathcal{S}}$, admitted by (23) is isomorphic to $\mathcal{S}$, we can utilize the results of subgroup classification of $\mathcal{S}$. For each of the m-parameter Abelian subgroups of $\bar{\mathcal{S}}$ we check whether their infinitesimal generators satisfy one of conditions (28), (29) of Theorem 5.1. This yields the list of systems of evolution equations that can be reduced to those having QLS.
5. Performing the nonlocal change of variables $u^i \to u^i_x$, $i = 1, \dots, m$ we obtain systems of evolution equations (24) admitting quasi-local symmetries (26).

We intend to devote a special publication to application of this algorithm to Schrödinger-type systems of partial differential equations. Here we present an example of Galilei-invariant nonlinear Schrödinger equation, which leads to the equation possessing QLS.

Consider the nonlinear Schrödinger equation

$$\begin{aligned} i\psi_t = \psi_{xx} + 2(x+i\alpha)^{-1}\psi_x - (i/2)(x+i\alpha) \\ +F(2i\alpha(x+i\alpha)\psi_x - (x-i\alpha)(\psi-\psi^*)), \end{aligned} \tag{30}$$

where $\psi = \psi_{RE}(t,x) + i\psi_{IM}(t,x)$, $\psi^* = \psi_{RE}(t,x) - i\psi_{IM}(t,x)$, $\alpha \neq 0$ is an arbitrary real constant and F is an arbitrary complex-valued function. According to [22], Eq.(30) admits the Lie algebra of the Galilei group having the following basis operators:

$$\begin{aligned} e_1 &= \partial_t, \\ e_2 &= \partial_\psi + \partial_{\psi^*}, \\ e_3 &= (x+i\alpha)^{-1}\partial_\psi + (x-i\alpha)^{-1}\partial_{\psi^*}, \\ e_4 &= \partial_x - (t + (x+i\alpha)^{-1}\psi)\partial_\psi - (t + (x-i\alpha)^{-1}\psi^*)\partial_{\psi^*}. \end{aligned}$$

Operators e_2, e_3 commute and the rank of the matrix of coefficients of operators $\partial_t, \partial_x, \partial_\psi, \partial_{\psi^*}$ is equal to 2. Consequently, there is a change of variables that reduces e_2, e_3 to canonical forms ∂_u, ∂_v. Indeed, making the change of variables

$$u(t,x) = (1/2)(\psi+\psi^*), \quad v(t,x) = (2i\alpha)^{-1}(x^2+\alpha^2)(\psi-\psi^*) \tag{31}$$

transforms e_1, e_2 to become $e_1 = \partial_u$, $e_2 = \partial_v$. So we can apply the above approach to Eq.(30) transformed according to (31). As the coefficients of the transformed operator e_3 satisfy (29), it leads to QLS of the system of evolution equations of the form (24).

6. Concluding remarks

In the present paper we develop the efficient approach to constructing evolution type partial differential equations which admit quasi-local symmetries. It is important to emphasize that the approach in question can be applied iteratively. Namely, if the transformed equation possesses Lie point symmetries which satisfy conditions of Theorem 2.1, then it again can be transformed to a new evolution equation admitting QLS and so on. What is more, the equation obtained as the Nth iteration of the algorithm admits QLS which involves nonlocal variables $\partial^{-1}u, \partial^{-2}u, \ldots, \partial^{-N}u$.

It is of great interest to explore quasi-local symmetries of nonlinear multi-component evolution equations. The most natural objects are the nonlinear Schrödinger-type equations and systems of nonlinear reaction-diffusion equations.

There is a different approach to analyzing nonlocal symmetries for some special differential equations based on the notion of potential symmetries introduced by Bluman [5, 3]. Recently, we have established that for the case of arbitrary evolution equation in one spatial variable any potential symmetry is quasi-local. More precisely, for any potential symmetry of an equation of the form (1) there is a (non-point) transformation reducing it to a group of contact transformations leaving invariant the properly transformed equation (1) [18]. A similar assertion holds for the case of system of evolution equations in one spatial variable [19]. A detailed discussion of alternative approaches to analysis of nonlocal symmetries of partial differential equations can be found in [6, 16].

A study of the above mentioned problems is in progress now and will be reported in our future publications.

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