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A Plancherel Formula for Representative Functions on Semisimple Lie Groups

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Abstract. A Plancherel formula is given for representative functions on a connected semisimple Lie group G. Since the matrix coefficients for the irreducible finite-dimensional representations are not necessarily square-integrable, an alternative to the Schur Orthogonality Relations is given using invariant differential operators. The corresponding operator analysis is summarized.
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1. Introduction

When G is a connected compact Lie group, understanding the square-integrable functions on G is sufficient for completely realizing the irreducible unitary representations of G, all of which are finite dimensional. This theory is summarized by the Schur Orthogonality Relations [8], the Peter-Weyl Theorem [7], and the Plancherel Formula. In particular, there exists a unitary equivalence of $G \times G$ representations

$$L^2(G) \cong \bigoplus_{\pi} V \otimes V^*.$$

where (π, V) ranges over a set of inequivalent irreducible representations of G. Each representation π contributes to $L^2(G)$ by passage to matrix coefficient functions, and one may choose a Hilbert basis of $L^2(G)$ consisting of such functions. Since

$$V \otimes V^* \cong Hom_{\mathbb{C}}(V, V)$$

as representations of $G \times G$, one may in turn express the L^2-norm using the Hilbert-Schmidt norm on each $Hom_{\mathbb{C}}(V, V)$.

When G is noncompact and semisimple, one may push forward an analogue to this analysis for $\mathcal{R}(G)$, the set of representative functions on G. Essentially, a representative function on G generates a finite-dimensional span under left and right translations by group elements. What follows parallels Chapter 14 of [2] and

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Section 1.5 of [5]. The analogues of the Schur Orthogonality Relations (Theorems 3.1 and 7.2) first appear in [3]; see also Chapter 3 of [4] for a similar proof in the finite case. Since the corresponding matrix coefficients are not square-integrable, the bi-invariant Hermitian form on $\mathcal{R}(G)$ is defined using invariant differential operators on G. In this work, the analysis is completed for $\mathcal{R}(G)$; that is, a convolution operator is defined (Section 4), the corresponding operator analysis is presented (Section 5), and the Plancherel Formula (Theorem 6.3) is given in this context. The Peter-Weyl Theorem holds by definition; a representative function is a finite sum of matrix coefficients for finite-dimensional representations.

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2. Representative Functions $\mathcal{R}(G)$

Suppose G is a real linear connected semisimple Lie group, and let K be a maximally compact subgroup of G. The condition rank G = rank K holds through Section 6. Section 7 addresses the general case.

We follow the convention that Lie algebras of groups are denoted by the corresponding Fraktur letter subscripted with a '0'; the subscript is dropped for the complexification. Fix a Cartan subgroup T in K with Lie algebra $\mathfrak{t}_0$; by the rank condition, T is a Cartan subgroup on G. Thus one may form the set of roots $\Delta(\mathfrak{g}, \mathfrak{t})$ and choose a positive system Δ^+.

The finite-dimensional representations of G are fully reducible, and the irreducible representations may be parameterized by the Theorem of the Highest Weight. Let (π, V) correspond to the irreducible representation with highest weight λ, and denote the dimension of V by d_λ or d_π, depending on whether λ is explicitly given.

With the equal rank assumption, each finite-dimensional representation of G admits an invariant Hermitian form that is non-degenerate. In this case, the dual representation π^* is equivalent to the complex-conjugate representation $\overline{\pi}$. When π is irreducible, this form is unique up to a nonzero real scalar. For any finite-dimensional representation π, a choice of Hermitian form $\langle \cdot, \cdot \rangle_\pi$ is fixed. When π is clear from context, the subscript is omitted.

Let f be a complex-valued function on G, and define left (resp. right) translation by the element g in G on f by

$$[L(g)f](x) = f(g^{-1}x) \qquad \text{(resp. } [R(g)f](x) = f(xg)\text{)}.$$

Definition 2.1. Let (π, V) be any finite-dimensional representation of G. For any u and v in V, the matrix coefficient function $\phi_{u,v} : G \to \mathbb{C}$ is defined by

$$\phi_{u,v}(g) = \langle \pi(g)u, v \rangle.$$

Definition 2.2 (cf. [1]). A function $f : G \to \mathbb{C}$ is called *representative* if it lies in the span of the matrix coefficients for a finite-dimensional representation of G. Equivalently, f is representative if its span under all left (equiv. right) translations by elements of G is finite dimensional. We denote the space of representative functions on G by $\mathcal{R}(G)$.

Definition 2.3. Let (π, V) be a finite-dimensional representation of G. The matrix coefficient map

$$\phi : V \otimes V^* \to \mathcal{R}(G)$$

is defined by linearly extending the map

$$u \otimes \langle \cdot, v \rangle \mapsto \phi_{u,v}.$$

Considering right and left translations simultaneously, one obtains a representation of $G \times G$ on $\mathcal{R}(G)$ by

$$[(R \otimes L)(g_1, g_2)f](x) = f(g_2^{-1} x g_1).$$

With this action, the constant functions occur in $\mathcal{R}(G)$ as the unique trivial constituent for $G \times G$. Furthermore, by full reducibility of finite-dimensional representations of $G \times G$, one has

$$\mathcal{R}(G) \cong \bigoplus_{\pi} V_\pi \otimes V_\pi^*,$$

where π ranges over a set of inequivalent, irreducible, finite-dimensional representations of G. Applying matrix coefficient maps to each constituent, $\bigoplus_\pi \pi \otimes \pi^*$ and $R \otimes L$ are equivalent as representations of $G \times G$.

3. An Invariant Form on $\mathcal{R}(G)$

To provide an analogue to the Plancherel Formula when rank G = rank K, an alternative to the traditional Schur Orthogonality Relations is recalled from [3]. Since matrix coefficient are defined using invariant forms, a non-degenerate bi-invariant Hermitian form may be defined naturally on $\mathcal{R}(G)$ as a substitute for invariant integration.

The main component in the definition of this form is the Casimir element Ω for $\mathfrak{g}$; for instance, see [5] or [6], Ch. 5.4. This element lies in the center of the universal enveloping algebra $U(\mathfrak{g})$, and it acts upon vectors for the irreducible representation π with highest weight λ by the real scalar

$$c_\pi = c_\lambda := |\lambda + \delta|^2 - |\delta|^2,$$

where δ denotes half the sum of the positive roots. This constant equals zero if and only if $\lambda = 0$; in this case, π is trivial.

Further define

$$D_r = \prod_{0<|\mu|\leq r} \left(1 - \frac{\Omega}{c_\mu}\right),$$

where μ ranges over all possible highest weights with length in the given range. One defines a bi-invariant Hermitian form on $\mathcal{R}(G)$ by

$$\langle\langle f, h \rangle\rangle := \lim_{n\to\infty} R(D_n)[f(y)\overline{h(y)}]|_{y=e}.$$

For a given finite-dimensional subrepresentation of $\mathcal{R}(G)$, there exists a positive integer n such that $R(D_n)$ annihilates all but the trivial constituent, and, once such an n is found, the result is unchanged for all larger n. Thus one may evaluate the limit at any y in G. Since the Casimir element is central, the differentiation is unchanged by inversion of y and by translations on either side of y by elements of G.

Theorem 3.1 (Schur Orthogonality Relations [3]). *Let (π, V) and (σ, U) be irreducible finite-dimensional representation of G with non-degenerate invariant Hermitian forms. Then*

1. *for u, v, u', v' in V,*

$$\langle\langle \phi_{u,v}, \phi_{u',v'} \rangle\rangle = \frac{1}{d_\pi} \langle u, u' \rangle \overline{\langle v, v' \rangle}, \qquad \textit{and}$$

2. *if π and σ are inequivalent, then $\langle\langle \phi_1, \phi_2 \rangle\rangle = 0$ for all matrix coefficients ϕ_1 (resp. ϕ_2) associated to π (resp. σ).*

Proof. Both sides of (1) express invariant sesquilinear forms for the irreducible $G \times G$ representation $\pi \otimes \pi^*$. Applying Schur's Lemma for $G \times G$, these forms differ by a scalar. To compute this scalar, let $\{u_i\}$ be a basis for V with corresponding dual basis $\{u_i'\}$; that is, $\langle u_i, u_j' \rangle = \delta_{ij}$ for all i and j, where δ denotes the Kronecker index. Then for fixed i,

$$\begin{aligned} \sum_j \phi_{u_i, u_j'}(g) \overline{\phi_{u_i', u_j}(g)} &= \sum_j \langle \pi(g) u_i, u_j' \rangle \langle u_j, \pi(g) u_i' \rangle \\ &= \sum_j \langle \langle \pi(g) u_i, u_j' \rangle u_j, \pi(g) u_i' \rangle \\ &= \langle \pi(g) u_i, \pi(g) u_i' \rangle \\ &= 1. \end{aligned}$$

For u, v, u', v' in V, the natural invariant Hermitian form on $V \otimes V^*$ is given by extending

$$\langle\langle u \otimes \langle \cdot, v \rangle, u' \otimes \langle \cdot, v' \rangle \rangle\rangle' = \langle u, u' \rangle \overline{\langle v, v' \rangle}.$$

Thus

$$\begin{aligned} \sum_j \langle\langle u_i \otimes \langle \cdot, u_j \rangle, u_i' \otimes \langle \cdot, u_j' \rangle \rangle\rangle' &= \sum_j \langle u_i, u_i' \rangle \overline{\langle u_j, u_j' \rangle} \\ &= d_\pi, \end{aligned}$$

and the result follows.

For (2), no trivial subrepresentation occurs in $\pi \otimes \sigma^*$, and the vanishing follows. ∎

It follows that

Proposition 3.2. *$\langle\langle \cdot, \cdot \rangle\rangle$ is non-degenerate on $\mathcal{R}(G)$.*

Proof. Suppose $\{u_i\}$ is any basis for the irreducible representation (π, V), and let $\{u_i'\}$ be the corresponding dual basis for V. The Schur Orthogonality Relations imply that

$$\langle\langle \phi_{u_i,u_j}, d_\pi \phi_{u_k',u_l'} \rangle\rangle = \delta_{ik}\delta_{jl},$$

so $\{\phi_{u_i,u_j'}\}$ forms a basis for the $\pi \otimes \pi^*$-subrepresentation in $\mathcal{R}(G)$. The proposition follows. ■

4. A Convolution for $\mathcal{R}(G)$

Recall that rank G = rank K. With the Schur Orthogonality Relations on $\mathcal{R}(G)$, an algebra structure with involution and convolution follows immediately in analogy with the traditional compact case.

Definition 4.1. For any function f in $\mathcal{R}(G)$, define involutions

$$(if)(g) = f(g^{-1})$$

and

$$f^*(g) = \overline{f(g^{-1})} = \overline{(if)(g)}.$$

The involution i intertwines $R \otimes L$ with $L \otimes R$ on $\mathcal{R}(G)$.

Proposition 4.2. *For all f and h in $\mathcal{R}(G)$,*

1. $\langle\langle if, ih \rangle\rangle = \langle\langle f, h \rangle\rangle$, *and*
2. $\langle\langle f^*, h^* \rangle\rangle = \langle\langle h, f \rangle\rangle$.

Proof. Since $i(f\overline{h}) = (if)\overline{(ih)}$ and $(f\overline{h})^* = f^*\overline{h^*}$, it is enough to consider the coefficient of the trivial constituent in each product. The proposition follows immediately.

Alternatively, we can compute on matrix coefficients for the irreducible representation (π, V). If u, v, w, x are in V,

$$\begin{aligned} \langle\langle i\phi_{u,v}, i\phi_{w,x} \rangle\rangle &= \langle\langle \overline{\phi_{v,u}}, \overline{\phi_{x,w}} \rangle\rangle \qquad \text{by invariance of } \langle\cdot,\cdot\rangle \\ &= d_\pi^{-1} \overline{\langle v, x \rangle} \langle u, w \rangle \\ &= \langle\langle \phi_{u,v}, \phi_{w,x} \rangle\rangle. \end{aligned}$$

Part (2) follows immediately. ■

Definition 4.3. For f and h in $\mathcal{R}(G)$, the convolution $f * h$ is defined by

$$\begin{aligned} [f * h](g) &:= \lim_{n\to\infty} R(D_n)[f(gy^{-1})h(y)]|_{y=e} = \overline{\langle\langle R(g^{-1})[f^*], h \rangle\rangle} \\ &= \lim_{n\to\infty} R(D_n)[f(y)h(y^{-1}g)]|_{y=e} = \langle\langle f, L(g)[h^*] \rangle\rangle. \end{aligned}$$

For fixed f and h, the limit converges for some finite n. The equality of limits follows from invariance under inversion and translations on y in the limit.

Proposition 4.4. *Let $f, h,$ and k be in $\mathcal{R}(G)$. Then one has*

1. $(f * h) * k = f * (h * k)$,
2. $(f * h)^* = h^* * f^*$, *and*
3. $\langle\langle f * h, k\rangle\rangle = \langle\langle f, k * h^*\rangle\rangle = \langle\langle h, f^* * k\rangle\rangle$.

Proof. To show (1), we apply inversion and translation to y :

$$\begin{aligned}
[(f * h) * k](g) &= \lim_{n\to\infty} R(D_n)[(f * h)(y)k(y^{-1}g)]|_{y=e} \\
&= \lim_{n\to\infty} R(D_n)[\lim_{m\to\infty} R(D_m)[f(z)h(z^{-1}y)]|_{z=e}k(y^{-1}g)]|_{y=e} \\
&= \lim_{m\to\infty} R(D_m)[\lim_{n\to\infty} R(D_n)[f(z)h(y)k(y^{-1}z^{-1}g)]|_{y=ze}]|_{z=e} \\
&= \lim_{m\to\infty} R(D_m)[f(z)(h * k)(z^{-1}g)]|_{z=e} \\
&= [f * (h * k)](g).
\end{aligned}$$

The arguments for (2) and (3) follow similarly.

Alternatively, let (π, V) and (σ, U) be irreducible representations of G, and suppose u, v are in V and u', v' are in U. Then

1. $\phi^*_{u,v} = \phi_{v,u}$,
2. if $\pi = \sigma$, then $d_\pi(\phi_{u,v} * \phi_{u',v'}) = \langle u, v'\rangle\phi_{u',v}$, and
3. if π and σ are inequivalent, then $\phi_{u,v} * \phi_{u',v'} = 0$.

The proposition follows by considering a matrix coefficients basis for $\mathcal{R}(G)$. ■

5. Projection Operators

Again recall that rank G = rank K. Let (π, V) be a finite-dimensional representation of G with non-degenerate invariant sesquilinear form $\langle\cdot,\cdot\rangle$. Again the decomposition of π into irreducible constituents follows the traditional compact case. Explicit projection operators are defined by applying convolution and adjoint to irreducible characters.

Definition 5.1. For u and v in V and f in $\mathcal{R}(G)$, define

$$\pi(f) : V \to V$$

by

$$\langle\pi(f)u, v\rangle = \langle\langle f, \overline{\phi_{u,v}}\rangle\rangle.$$

The adjoint $\pi(f)^*$ of $\pi(f)$ is defined with respect to $\langle\cdot,\cdot\rangle$; that is,

$$\langle\pi(f)u, v\rangle = \langle u, \pi(f)^*v\rangle.$$

Again by purely formal arguments, we have

Proposition 5.2. *For f and h in $\mathcal{R}(G)$,*

1. $\pi(f)^* = \pi(f^*)$,
2. $\pi(f * h) = \pi(f)\pi(h)$, *and*
3. $\pi(f * h)^* = \pi(h)^*\pi(f)^*$.

Following the compact case, irreducible characters are used to construct a complete set of orthogonal idempotents in $\mathcal{R}(G)$.

Definition 5.3. Denote the character of π by

$$\chi_\pi(g) = Trace[\pi(g)].$$

Equivalently, let $\{u_i\}$ be any basis for V, and let $\{u_i'\}$ be the corresponding dual basis. Then

$$\chi_\pi(g) = \sum \langle \pi(g)u_i, u_i' \rangle;$$

this sum is independent of the basis $\{u_i\}$ chosen.

Proposition 5.4. *Fix inequivalent irreducible representations (σ, U) and (τ, W) of G. Then*

1. $\langle\langle \chi_\sigma, \chi_\sigma \rangle\rangle = 1$,
2. $\langle\langle \chi_\sigma, \chi_\tau \rangle\rangle = 0$,
3. $\chi_\sigma^* = \chi_\sigma$,
4. *if u, v are in U, then* $d_\sigma \phi_{u,v} * \chi_\sigma = d_\sigma \chi_\sigma * \phi_{u,v} = \phi_{u,v}$,
5. $\chi_\sigma * \chi_\tau = 0$,
6. $d_\sigma \chi_\sigma * \chi_\sigma = \chi_\sigma$, *and*
7. *for any f in $\mathcal{R}(G)$,* $\chi_\sigma * f = f * \chi_\sigma$.

A projection operator onto the subrepresentation of σ-types in π is given by the following:

Definition 5.5. Let σ be an irreducible subrepresentation of π. Define the projection operator

$$E_\sigma : V \to V$$

by

$$E_\sigma(v) = d_\sigma \pi(\overline{\chi_\sigma})v.$$

Equivalently, for nontrivial σ,

$$E_\sigma(v) = \lim_{n\to\infty} \pi\left(\frac{\Omega D_n}{c_\sigma - \Omega}\right)v.$$

For fixed v, the limit converges for some finite n.

We verify the type property directly. Suppose $W \subseteq V$ and (π, W) is an irreducible subrepresentation of type σ. If v, w are in W, then

$$\begin{aligned}\langle E_\sigma v, w\rangle &= \langle d_\sigma \pi(\overline{\chi_\sigma})v, w\rangle \\ &= d_\sigma\langle\langle \overline{\chi_\sigma}, \overline{\phi_{v,w}}\rangle\rangle \\ &= d_\sigma\langle\langle \phi_{v,w}, \chi_\sigma\rangle\rangle = \langle v, w\rangle.\end{aligned}$$

On the other hand, if $\langle w, w'\rangle = 0$ for all w in W, then $\langle E_\sigma v, w'\rangle = 0$. The property follows by non-degeneracy of $\langle\cdot,\cdot\rangle$.

Proposition 5.6. *Suppose σ and τ are inequivalent irreducible subrepresentations of (π, V). Then*

1. $E_\sigma^* = E_\sigma$,
2. $E_\sigma E_\sigma = E_\sigma$,
3. $E_\tau E_\sigma = E_\sigma E_\tau = 0$,
4. *for all v in V, $v = \sum_\sigma E_\sigma v$, and*
5. $\langle v, w\rangle = \sum_\sigma \langle E_\sigma v, E_\sigma w\rangle$.

Now E_σ projects onto the subrepresentation of all σ-types in π. In particular, the multiplicity of σ in π equals $\text{Trace}(E_\sigma)/(\dim\sigma)$, and the trace of $\sigma(\chi_{\overline{\sigma}})$ equals 1.

6. Plancherel Formula

The corresponding analysis for $\mathcal{R}(G)$ follows below with the equal rank condition. Unraveling the various definitions yields a Plancherel Formula for $\mathcal{R}(G)$. Since

$$V \otimes V^* \cong Hom_{\mathbb{C}}(V, V),$$

the Schur Orthogonality Relations may be expressed in terms of the Hilbert-Schmidt norm on $Hom_{\mathbb{C}}(V, V)$.

Definition 6.1. For irreducible σ, define

$$P_\sigma : \mathcal{R}(G) \to \mathcal{R}(G)$$

by

$$P_\sigma(f) = d_\sigma f * \chi_\sigma = d_\sigma \chi_\sigma * f.$$

Equivalently, for nontrivial σ, one has

$$P_\sigma(f) = \lim_{n\to\infty} R\left(\frac{\Omega D_n}{c_\sigma - \Omega}\right) f.$$

For fixed f, the limit converges for some finite n.

Definition 6.1 follows by applying Definition 5.5 to the right action R on $\mathcal{R}(G)$. The following statements provide an analogue to Proposition 5.6:

Proposition 6.2. *Let σ and τ be inequivalent irreducible representations of G.*

1. *$P_\sigma^* = P_\sigma$ with respect to $\langle\langle \cdot, \cdot \rangle\rangle$,*
2. *$P_\sigma P_\sigma = P_\sigma$,*
3. *$P_\sigma P_\tau = P_\tau P_\sigma = 0$, and*
4. *for all f in $\mathcal{R}(G)$, $f = \sum_\sigma P_\sigma(f)$.*

Proof. Part (1) follows from Proposition 4.4 (3) and Proposition 5.4 (3). For part (2), let f be in $\mathcal{R}(G)$. Using associativity of convolution and Proposition 5.4 (6),

$$\begin{aligned} P_\sigma(P_\sigma f) &= P_\sigma(d_\sigma \chi_\sigma * f) \\ &= d_\sigma^2 \chi_\sigma * (\chi_\sigma * f) \\ &= d_\sigma \chi_\sigma * f = P_\sigma(f). \end{aligned}$$

Part (3) follows in a similar manner using Proposition 5.4 (5). Part (4) follows since the irreducible characters form a complete set of orthogonal idempotents in $\mathcal{R}(G)$. ■

The analogue to Proposition 5.6 (5) on $\mathcal{R}(G)$ is

Theorem 6.3 (Plancherel Formula)**.** *Suppose f and h are in $\mathcal{R}(G)$. Then*

$$\langle\langle f, h \rangle\rangle = \sum_\sigma d_\sigma \; Trace(\sigma(f)\sigma(h)^*),$$

where σ ranges over a set of inequivalent irreducible finite-dimensional representations of G. For fixed f and h, only finitely many σ are needed in the sum.

Proof. Since

$$\sigma(L(g)f) = \sigma(g)\sigma(f) \qquad \text{and} \qquad \sigma(R(g)f) = \sigma(f)\sigma(g^{-1}),$$

the sum on the right-hand side is a bi-invariant sesquilinear form on $\mathcal{R}(G)$, and thus both sides are equal up to a scalar on the irreducible constituents for $G \times G$. Setting $f = h = \chi_\sigma$, one has by Propositions 5.2 and 5.4 that

$$\begin{aligned} d_\sigma Trace(\overline{\sigma}(\chi_\sigma)\overline{\sigma}(\chi_\sigma)^*) &= d_\sigma Trace(\overline{\sigma}(\chi_\sigma * \chi_\sigma^*)) \\ &= Trace(\overline{\sigma}(\chi_\sigma)) = 1. \end{aligned}$$

■

7. The General Rank Case

The preceding results are recast to include the case where rank $G \neq$ rank K. Let Θ and θ be the associated Cartan involutions on G and $\mathfrak{g}_0$. Let T be any Cartan subgroup of K, and let $H = Z_G(T)$. Denote the associated Cartan decompositions by

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \qquad \text{and} \qquad \mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0.$$

Form the set of roots $\Delta(\mathfrak{g}, \mathfrak{h})$ and choose a θ-stable positive system Δ^+. We recall

Definition 7.1. Let (π, V) be a representation of G. The complex conjugate representation $(\overline{\pi}, \overline{V})$ is defined as follows: the space $\overline{V}$ equals V, but scalar multiplication is defined by $z \cdot v = \overline{z}v$ for z in $\mathbb{C}$ and v in V. For g in G and v in $\overline{V}$, $\overline{\pi}(g)v = \pi(g)v$.

Let (π, V) be an irreducible representation of G with highest weight λ. If λ vanishes on $\mathfrak{a}$, then there exists a non-degenerate invariant Hermitian form $\langle\cdot,\cdot\rangle$ on V and the analysis in the equal rank case applies. In this case, an equivalence between $\overline{\pi}$ and π^* may be defined by

$$\overline{V} \to V^*, \qquad v \mapsto \langle\cdot, v\rangle.$$

If λ is non-vanishing on $\mathfrak{a}$, then no such form exists, and one may appeal to the invariant sesquilinear pairing between V and $\overline{V^*}$: for u in V and v in $\overline{V^*}$,

$$\langle u, v\rangle := v(u).$$

For u in V and v in $\overline{V^*}$, we define the matrix coefficient

$$\phi_{u,v}(g) = v(\pi(g)u).$$

Then $\phi^*_{u,v}$ corresponds to a matrix coefficient for $\overline{\pi^*}$; the equivalence between $\overline{\pi}$ and $\overline{\pi}^{**}$ is implemented by sending u' in $\overline{V}$ to evaluation at u' on $\overline{V^*}$. Hence there exists an invariant sesquilinear pairing between the matrix coefficient spaces for π and $\overline{\pi}^*$:

$$\langle\langle \phi_{u,v}, \phi^*_{u',v'}\rangle\rangle' = v'(u)v(u').$$

On the other hand, $\langle\langle\cdot,\cdot\rangle\rangle$ may be defined on $\mathcal{R}(G)$ as before, and one has

Theorem 7.2 (Schur Orthogonality Relations)**.** *Suppose (π, V) is an irreducible, finite-dimensional representation of G.*

1. *For u, u' in V and v, v' in $\overline{V^*}$,*

$$\langle\langle \phi_{u,v}, \phi^*_{u',v'}\rangle\rangle = d_\pi^{-1} v'(u)v(u'), \text{ and}$$

2. *let (σ, U) be an irreducible representation of G such that σ is inequivalent to $\overline{\pi}^*$. Then $\langle\langle \phi_1, \phi_2\rangle\rangle = 0$ if ϕ_1 (resp. ϕ_2) is a matrix coefficient for π (resp. σ).*

Proof. First note that if V admits a non-degenerate invariant form, then the result follows from Theorem 3.1 and statement (1) in the proof of Proposition 4.4.

Otherwise we adapt the proof of Theorem 3.1. By Schur's Lemma, any invariant sesquilinear pairings between the matrix coefficent spaces for π and $\overline{\pi^*}$ are equal up to a scalar. Choose a basis $\{u_j\}$ for V with corresponding dual basis $\{u_j'\}$ for V^* (which equals $\overline{V^*}$ as a set of linear functionals). Now, for fixed i,

$$\begin{aligned}\sum_j \phi_{u_i,u_j'}(g)\ \overline{\phi^*_{u_j,u_i'}(g)} &= \sum_j u_j'(\pi(g)u_i)\ u_i'(\pi(g^{-1})u_j) \\ &= u_i'\Big(\sum_j u_j'(\pi(g)u_i)\ \pi(g^{-1})u_j\Big) \\ &= u_i'\Big(\sum_j (\pi^*(g^{-1})u_j')(u_i)\ \pi(g^{-1})u_j\Big) \\ &= u_i'(u_i) = 1.\end{aligned}$$

The last line follows since the dual basis of $\{\pi(g^{-1})u_j\}$ is $\{\pi^*(g^{-1})u_j'\}$.

On the other hand,

$$\begin{aligned}\sum_j \langle\langle \phi_{u_i,u_j'}\ \phi^*_{u_j,u_i'}\rangle\rangle' &= \sum_j u_i'(u_i)\ u_j'(u_j) \\ &= d_\pi.\end{aligned}$$

■

An immediate consequence of Theorem 7.2 is

Proposition 7.3. *$\langle\langle\cdot,\cdot\rangle\rangle$ is non-degenerate.*

The definitions of convolution and adjoint are intrinsic to $\mathcal{R}(G)$, so the changes occur in the corresponding formulas for matrix coefficients. Notably, for inequivalent, irreducible representations (π, V) and (σ, U), if u, u' are in V and v, v' are in $\overline{V^*}$,

1. $d_\pi(\phi_{u,v} * \phi_{u',v'}) = v'(u)\phi_{u',v}$, and
2. $\phi_1 * \phi_2 = 0$ if ϕ_1 (resp. ϕ_2) is a matrix coefficient associated to π (resp. σ).

The analysis is completed by noting the following changes to Sections 5 and 6. First we replace Proposition 5.2 (1) with

$$[\overline{\pi^*}(f)]^* = \pi(f^*);$$

here the adjoint is defined using the invariant pairing between V and $\overline{V^*}$ in Definition 5.1. Next one has

Definition 7.4. Let (π, V) be any finite-dimensional representation of G. Choose a basis $\{u_i\}$ for V with corresponding dual basis $\{u_i'\}$ for $\overline{V^*}$. Then the character of π is defined as

$$\chi_\pi(g) = Trace[\pi(g)] = \sum_i u_i'(\pi(g)u_i).$$

The sum is independent of the basis for V chosen.

The analogue to Proposition 5.4 is

Proposition 7.5. *Fix irreducible representations* (σ, U) *and* (τ, W) *of* G*. Then*

1. $\langle\langle \chi_\sigma, \chi_{\overline{\sigma^*}} \rangle\rangle = 1$,
2. $\langle\langle \chi_\sigma, \chi_\tau \rangle\rangle = 0$ *if* τ *and* $\overline{\sigma^*}$ *are inequivalent,*
3. $\chi_\sigma^* = \chi_{\overline{\sigma^*}}$,
4. *if* u *is in* U *and* v *is in* $\overline{U^*}$*, then* $d_\sigma \phi_{u,v} * \chi_\sigma = d_\sigma \chi_\sigma * \phi_{u,v} = \phi_{u,v}$,
5. $\chi_\sigma * \chi_\tau = 0$ *if* τ *and* σ *are inequivalent,*
6. $d_\sigma \chi_\sigma * \chi_\sigma = \chi_\sigma$, *and*
7. *for any* f *in* $\mathcal{R}(G)$, $\chi_\sigma * f = f * \chi_\sigma$.

For the remainder of the analysis, one redefines $E_\sigma = d_\sigma \pi(\overline{\chi}_{\overline{\sigma}^*}) = d_\sigma \pi(\chi_{\sigma^*})$, and, from that point forth, all definitions, propositions, and results follow with minor changes, including the Plancherel Formula. In particular, we replace $\sigma(h)^*$ with $[\overline{\sigma^*}(h)]^* = \sigma(h^*)$ in Theorem 6.3.

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