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VECTOR MEASURES ON TOPOLOGICAL SPACES

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Abstract. Let X be a completely regular Hausdorff space, E a quasi-complete locally convex space, $C(X)$ (resp. $C_b(X)$) the space of all (resp. all, bounded), scalar-valued continuous functions on X, and $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ be the classes of Borel and Baire subsets of X. We study the spaces $M_t(X,E)$, $M_\tau(X,E)$, $M_\sigma(X,E)$ of tight, τ-smooth, σ-smooth, E-valued Borel and Baire measures on X. Using strict topologies, we prove some measure representation theorems of linear operators between $C_b(X)$ and E and then prove some convergence theorems about integrable functions. Also, the Alexandrov's theorem is extended to the vector case and a representation theorem about the order-bounded, scalar-valued, linear maps from $C(X)$ is generalized to the vector-valued linear maps.

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1. Introduction and Notation

In this paper R stands for the set of real numbers, K denotes the field of real or complex numbers (we call them scalars) and X a completely regular Hausdorff space and E a quasi-complete locally convex space space over K with topology generated by an increasing family of semi-norms $\|.\|_p$, $p \in P$; E' denotes the topological dual of E. For a $p \in P$, $V_p = \{x \in E : \|x\|_p \leq 1\}$; polars are taken in the duality $\langle E, E' \rangle$. We denote by $C(X)$ the space of all K-valued continuous functions on X, and by $C_b(X)$ the space of all bounded elements of $C(X)$. The zero-sets of X are the elements of $\{f^{-1}(0) : f \in C_b(X)\}$; the positive-sets of X are sets of the form $X \setminus Z$ where Z is a zero-set. For locally convex spaces, the notation and results of [9] will be used. For a vector space F, F^* will denote its algebraic dual. N will denote the set of natural numbers. For topological measure theory the notation and results of ([10], [11], [5], [12]) will be used. All locally convex spaces are assumed to be Hausdorff and over K. The elements of the smallest σ-algebra, on X, relative to which all functions in $C_b(X)$ are measurable, are called Baire sets and the elements of the σ-algebra generated by open sets are called Borel sets. $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ are the classes of Borel and Baire subsets of X. $\tilde{X}$ will denote the Stone–Cech compactification of X and υX the real-compactification. $M_\sigma(X)$, $M_\tau(X)$, $M_t(X)$ denote the spaces of σ-additive, τ-smooth and tight Baire measures on X([12], [11]), respectively. The elements of $M_\sigma(X)$ are scalar-valued, countably additive measures on $\mathcal{B}_0(X)$. An element $\mu \in M_\sigma(X)$ is called τ-smooth if for any decreasing net $\{f_\alpha\} \subset$

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$C_b(X)$, $f_\alpha \downarrow 0$, we have $\mu(f_\alpha) \to 0$. Every τ-smooth measure has a unique extension to a Borel measure which is inner regular by closed subsets and outer regular by open subsets of X; an element $\mu \in M_\sigma(X)$ is called tight if for any uniformly bounded net $\{f_\alpha\} \subset C_b(X)$, $f_\alpha \to 0$, uniformly on the compact subsets of X, we have $\mu(f_\alpha) \to 0$. Every tight measure has a unique extension to a Borel measure which is inner regular by compact subsets and outer regular by open subsets of X ([12], [11]). Also, the so-called strict topologies β_z, $z = \sigma, \tau, t$ are defined on $C_b(X)$, with the result that $(C_b(X), \beta_z)' = M_z(X)$ (see [11]) (notation like β_1, β, β_0 is also used for these topologies in [10]). The topology β_t is the finest locally convex one on $C_b(X)$, agreeing with the topology of uniform convergence on the compact subsets of X, on the norm bounded subsets of $C_b(X)$. To define the topology β_σ, take a zero-set in $\tilde{X}$, $Z \subset \tilde{X} \setminus X$. The topology β_t on $C_b(\tilde{X} \setminus Z)$ is denoted by β_Z. Evidently, $C_b(\tilde{X} \setminus Z)$ can be identified with $C_b(X)$ (there is a natural one-to-one, onto, norm-preserving mapping) and so β_Z can be considered a locally convex topology on $C_b(X)$. The topology β_σ is defined as $\bigwedge\{\beta_Z : Z \text{ a zero-set in } \tilde{X},\ Z \subset \tilde{X} \setminus X\}$. Similarly, β_τ is defined as $\bigwedge\{\beta_C : C \text{ a compact set in } \tilde{X},\ C \subset \tilde{X} \setminus X\}$.

With norm topology on $C_b(X)$, the dual of $C_b(X)$ is denoted by $M(X)$; $M(X)$ can also be interpreted as the space of bounded finitely additive measures on the algebra generated by zero-sets of X, which are inner regular by zero-sets and outer regular by the positive-sets of X (Alexandrov Theorem [12], [11]).

For a function $f \in C_b(X)$, $\tilde{f}$ denotes its unique continuous extension to $\tilde{X}$. It can be easily verified that $\mathcal{B}(\tilde{X}) \cap X = \mathcal{B}(X)$ and $\mathcal{B}_0(\tilde{X}) \cap X = \mathcal{B}_0(X)$.

Now we come to vector-valued measures; the integrability of scalar-valued functions is taken in the sense of ([7]). If $\mathcal{A}$ is a σ-algebra of subsets of a set Y, $\mu : \mathcal{A} \to E$ a countably additive vector measure and $p \in P$, we denote the p-semi-variation of μ by $\bar{\mu}_p$, $\bar{\mu}_p(A) = \sup\{|g \circ \mu|(A) : g \in V_p^0\}$ (here V_p^0 is the polar of V_p in the duality $\langle E, E' \rangle$) [7]; we also consider the submeasure $\dot{\mu}_p : \mathcal{A} \to R^+$, $\dot{\mu}_p(A) = \sup\{\|\mu(B)\|_p : B \in \mathcal{A},\ B \subset A\}$ ([5], [3]). It is easy to verify that $\dot{\mu}_p$ is countably sub-additive [3] and $\dot{\mu}_p \le \bar{\mu}_p \le 4\dot{\mu}_p$. Also, there is a control measure for $\bar{\mu}_p$ to be denoted by λ_p; this control measure can be chosen in the closed convex hull of $\{|g \circ \mu| : g \in V_p^0\}$, with norm topology on measures ([7], p. 20, the proof of Theorem 1). This control measure also has the following properties: (i) $|f \circ \mu| \ll \lambda_p$ for every $f \in E'$ with $\|f\|_p \le 1$ (note that $\|f\|_p = \sup\{|f(x)| : x \in V_p\}$); (ii) if $\lambda_p(A) = 0$, then $\bar{\mu}_p(A) = 0$; (iii) $\lim_{\lambda_p(A) \to 0} \bar{\mu}_p(A) = 0$; (iv) $\lambda_p \le \bar{\mu}_p$. We also establish that if $f : Y \to K$ is a measurable function, $B \in \mathcal{A}$ and $|f| \le c$ on B, then $\| \int_B f d\mu \|_p \le c\bar{\mu}_p(B)$.

$L^1(\mu)$ denotes the space of μ-integrable functions ([7]). For any $f \in L^1(\mu)$, we take $\bar{\mu}_p(f) = \sup\{|g \circ \mu|(|f|) : g \in V_p^0\}$ ([7], Lemma 2, p. 23).

If $\mathcal{F}$ is an algebra of subsets of a set Y and $\mu : \mathcal{F} \to E$ a finitely additive measure, then μ is called exhaustive if for any disjoint sequence $\{A_n\} \subset \mathcal{F}$, we have $\mu(A_n) \to 0$; exhaustive measures are called strongly bounded measures in [2]; for quasi-complete E, a finitely additive μ is exhaustive if and only if $\mu(\mathcal{F})$

is relatively weakly compact in E (for Banach spaces, it is proved in [2] and can be easily extended to quasi-complete locally convex spaces).

If X is a compact Hausdorff space then there is a one-to-one correspondence between regular Borel E-valued measures μ and linear weakly compact operators $T : C(X) \to E$ such that $T(f) = \int f d\mu$, $\forall f \in C(X)$ ([8], Theorem 3.1, p. 163); regularity means that for any Borel $B \subset X$, $p \in P$, and $c > 0$, there exist a compact C and an open V, $C \subset B \subset V$ such that $\bar{\mu}_p(V \setminus C) < c$. In that case, for $p \in P$, the control measure λ_p is a positive regular Borel measure in X.

In this paper, by taking the strict topologies on $C_b(X)$ we get similar representation theorems for weakly compact and continuous linear maps from $C_b(X)$ into E. Some convergence type theorems having relevance to topology are also proved. With a norm topology on $C_b(X)$, the celebrated Alexandrov's theorem says that the dual of $C_b(X)$ is $M(X)$; we extend this result to weakly compact and continuous linear $\mu : C_b(X) \to E$. Another very well-known result in the scalar case is that a linear $\mu : C(X) \to R$, which maps order bounded subsets into bounded sets, comes from a countably additive $\mu \in M_\sigma(X)$, whose support is in υX; this result is also extended to linear maps $\mu : C(X) \to E$.

First we consider X to be a compact Hausdorff space and prove some properties of E-valued regular Borel measures on it; then we extend these properties to completely regular Hausdorff spaces.

2. Representation Theorems

Theorem 1. *Let X be a compact Hausdorff space and μ an E-valued regular Borel measure on X.*

(i) *Suppose $\{f_\alpha\}$ is an increasing net of non-negative, lower semi-continuous functions in $L^1(\mu)$, converging to $f \in L^1(\mu)$, pointwise on X. Then* $\lim \bar{\mu}_p(f - f_\alpha) = 0$*; in particular* $\lim \int f_\alpha d\mu = \int f d\mu$.

(ii) *Given a $p \in P$, there exists the largest open set $U_p \subset X$ such that $\bar{\mu}_p(U_p) = 0$; this $X \setminus U_p$ is called the support of $\bar{\mu}_p$ and has the property that for any $f \in C_b(X)$, $f \geq 0$, and f not identically 0 on $X \setminus U_p$, one has $\bar{\mu}_p(f) > 0$.*

Proof. (i) Fix a $p \in P$ and let λ_p be the corresponding control measure. Since λ_p is in the norm-closed, absolutely convex hull of $\{|g \circ \mu| : g \in E',\ \|g\|_p \leq 1\}$, it follows that f is λ_p-integrable. As λ_p is a regular Borel measure, $\lim \int f_\alpha d\lambda_p = \int f d\lambda_p$. This means there are an increasing sequence $\{f_{\alpha(n)}\}$ and a Borel $B \subset X$ such that $\lambda_p(X \setminus B) = 0$ and $f_{\alpha(n)} \to f$ pointwise on B. Using the fact $f_{\alpha(n)} \leq f$, $\forall n$, by ([7], Theorem 1, p. 20), $\bar{\mu}_p(f - f_{\alpha(n)}) \to 0$. This proves this result.

(ii) Fix a $p \in P$ and let $\mathcal{V} = \{U \subset X : U$ open and $\bar{\mu}_p(U) = 0\}$. By the subadditivity of $\bar{\mu}_p$, for any finite collection $\{U_i\ (1 \leq i \leq n)\} \subset \mathcal{V}$, $\bar{\mu}_p(\cup U_i) = 0$. From (i) $\bar{\mu}_p(\cup\{U : U \in \mathcal{V}\}) = 0$. The other statement is easy to prove. □

Now assume that X is a completely regular Hausdorff space and $\mathcal{B}(X)$ and $\mathcal{B}_0(X)$ be the classes of Borel and Baire subsets of X ([11]). If it is not necessary to specify the space X, we will also denote them by $\mathcal{B}$ and $\mathcal{B}_0$. Let $M_\sigma(X, E) =$

$\{(\mu : \mathcal{B}_0 \to E) : g \circ \mu \in M_\sigma(X),\ \forall g \in E'\}$. This implies that every $\mu \in M_\sigma(X, E)$ is countably additive in the original topology of E.

Theorem 2. *Suppose X is a completely regular Hausdorff space and $\mu \in M_\sigma(X, E)$ is a countably additive Baire measure. Then*

(i) *μ is inner regular by zero-sets and outer regular by positive sets;*

(ii) *the linear mapping $\mu : (C_b(X), \beta_\sigma) \to E$ is continuous and bounded sets are mapped into relatively weakly compact sets.*

Conversely, if a linear mapping $\mu : (C_b(X), \beta_\sigma) \to E$ is continuous and maps bounded sets into relatively weakly compact sets, then there exists a unique countably additive Baire measure $\nu : \mathcal{B}_0 \to E$ such that $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$.

Proof. (i) Note $\mathcal{B}_0(\tilde{X}) \cap X = \mathcal{B}_0(X)$. Define $\tilde{\mu} : \mathcal{B}_0(\tilde{X}) \to E$, $\tilde{\mu}(B) = \mu(B \cap X)$. This means $\tilde{\mu}(B) = 0$ when $B \cap X = \varnothing$. Take a $p \in P$, $c > 0$ and a Baire set $B \subset X$. Select a Baire $\tilde{B} \subset \tilde{X}$ such that $\tilde{B} \cap X = B$. Since a Baire measure on a compact Hausdorff space is regular ([5]), there exists a zero-set Z and a positive set V in $\tilde{X}$ such that $Z \subset \tilde{B} \subset V$ and $\bar{\tilde{\mu}}_p(V \setminus Z) \le c$. From this it follows that $\bar{\mu}_p(V \cap X \setminus Z \cap X) \le c$. This proves the regularity of μ.

(ii) Since the range of a countably additive E-valued measure is a relatively weakly compact subset of E, the unit ball of $C_b(X)$ is mapped into a relatively weakly compact subset of E under the mapping $\mu : (C_b(X), \beta_\sigma) \to E$. Also β_σ-bounded sets are norm-bounded ([11]) and so the bounded sets are mapped into a relatively weakly compact subset of E.

Now for every $g \in E'$, $g \circ \mu \in M_\sigma(X)$ and so, with weak topology on E, the mapping $\mu : (C_b(X), \beta_\sigma) \to E$ is continuous. Since β_σ is Mackey ([11]), the mapping is also continuous with the original topology on E ([9], 7.4, p. 149).

Conversely, suppose that $\mu : (C_b(X), \beta_\sigma) \to E$ is a linear and continuous mapping and the bounded sets are mapped into a relatively weakly compact subset of E. With sup-norm topology on $C(\tilde{X})$, the mapping $\tilde{\mu} : C(\tilde{X}) \to E$, $\tilde{\mu}(f) = \mu(f_{|X})$, $\forall f \in C(\tilde{X})$, is linear and weakly compact and so $\tilde{\mu}$ can be considered a regular Baire measure on $\tilde{X}$. If $Z \subset \tilde{X} \setminus X$ is a zero-set, there exists a sequence $\{f_n\} \subset C(\tilde{X})$ such that $f_n \downarrow \chi_Z$. This means, in $(C_b(X), \beta_\sigma)$, $f_{n|X} \to 0$. Thus for every zero-set $Z \subset \tilde{X} \setminus X$, $\tilde{\mu}(Z) = 0$, and so, for every $p \in P, \bar{\tilde{\mu}}_p(B) = 0$, for all Baire sets $B \subset \tilde{X} \setminus X$. For any Baire set $A \subset X$, define $\nu(A) = \tilde{\mu}(B)$, B being any Baire subset of $\tilde{X}$, with $B \cap X = A$. It is a routine verification that ν is well-defined, is countably additive and for the integration of any $f \in C_b(X)$, $\int f d\nu = \int f d\mu$. Also if there is another Baire measure ν_1, on X, such that $\int f d\nu = \int f d\nu_1$ for every $f \in C_b(X)$, then we have $\nu(Z) = \nu_1(Z)$ for every zero-set $Z \subset X$; by regularity, this will imply $\nu = \nu_1$. So the uniqueness is established. □

A Baire measure $\mu : \mathcal{B}_0 \to E$ is called τ-smooth if for every $g \in E'$, $g \circ \mu \in M_\tau(X)$. The set of all E-valued τ-smooth measures is denoted by $M_\tau(X, E)$.

Theorem 3. *Suppose X is a completely regular Hausdorff space and $\mu : \mathcal{B}_0 \to E$ is a τ-smooth measure. Then*

(i) *μ can be extended to a Borel measure which is inner regular by closed sets and outer regular by open sets (we call this extension a regular Borel Measure);*

(ii) *the linear mapping $\mu : (C_b(X), \beta_\tau) \to E$ is continuous and bounded sets are mapped into relatively weakly compact sets;*

(iii) *considering μ a Borel measure, suppose $\{f_\alpha\}$ is an increasing net of non-negative, lower semi-continuous functions in $L^1(\mu)$, converging pointwise to an $f \in L^1(\mu)$. Then $\lim \bar{\mu}_p(f - f_\alpha) = 0$; in particular $\lim \int f_\alpha d\mu = \int f d\mu$.*

(iv) *Given a $p \in P$, there exists the largest open set $U_p \subset X$ such that $\bar{\mu}_p(U_p) = 0$; this $X \setminus U_p$ is called the support of $\bar{\mu}_p$ and has the property that for any $f \in C_b(X)$, $f \geq 0$, and f not identically 0 on $X \setminus U_p$, one has $\bar{\mu}_p(f) > 0$.*

(v) *The Borel regular extension of μ, satisfying the condition that, for an increasing net $\{V_\alpha\}$ of open subsets of X with $\cup V_\alpha = V$, we have $\lim \mu(V_\alpha) = \mu(V)$, is unique.*

Conversely, if a linear mapping $\mu : (C_b(X), \beta_\tau) \to E$ is continuous and maps bounded sets into relatively weakly compact sets, then there exists a unique τ-smooth measure $\nu : \mathcal{B}_0 \to E$ such that $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$.

Proof. (i). We have $\mathcal{B}(\tilde{X}) \cap X = \mathcal{B}(X)$. As $C_b(X) \subset L^1(\mu)$, we get a linear continuous $\tilde{\mu} : C(\tilde{X}) \to E$, $\tilde{\mu}(f) = \mu(f_{|X})$, $\forall f \in C(\tilde{X})$. Thus $\tilde{\mu}$ can be considered as a regular Borel measure on $\tilde{X}$. Take a closed set $C \subset \tilde{X} \setminus X$; there exists a net $\{f_\alpha\} \subset C(\tilde{X})$ such that $f_\alpha \downarrow \chi_C$. This means, in $(C_b(X), \beta_\tau)$, $f_{\alpha|X} \to 0$. Thus for every closed set $C \subset \tilde{X} \setminus X$, $\tilde{\mu}(C) = 0$, and so, by regularity, for every $p \in P$, $\bar{\tilde{\mu}}_p(B) = 0$, for all Borel sets $B \subset \tilde{X} \setminus X$. For any Borel set $A \subset X$, define $\nu(A) = \tilde{\mu}(B)$, B being any Borel subset of $\tilde{X}$, with $B \cap X = A$. It is a routine verification that ν is well-defined, is countably additive and for the integration of any $f \in C_b(X)$ we have $\int f d\nu = \int f d\mu$. Also by the regularity of $\tilde{\mu}$ it can be easily verified that μ is inner regular by closed sets and outer regular by open sets.

(ii) To prove the continuity of $\mu : (C_b(X), \beta_\tau) \to E$, we get $\tilde{\mu} : C(\tilde{X}) \to E$ as done above. Fix a $p \in P$, put M $= \bar{\tilde{\mu}}_p(\tilde{X})$, and fix an $n \in N$. Take a compact $C \subset \tilde{X} \setminus X$. Now the topology β_C is identical with the topology β_t on $C_b(\tilde{X} \setminus C)$, if we identify $C_b(X)$ with $C_b(\tilde{X} \setminus C)$ ([11]). Thus it is enough to prove that $\tilde{\mu} : (C_b(\tilde{X} \setminus C), \beta_t) \to E$ is continuous. We will use the fact that β_t is the finest locally convex topology agreeing with the compact-open topology on norm-bounded sets. Take a compact $K \subset \tilde{X} \setminus C$ such that $\bar{\tilde{\mu}}_p((\tilde{X} \setminus C) \setminus K) \leq \frac{1}{3n}$. Since $\tilde{\mu}_p(C) = 0$, we have $\bar{\tilde{\mu}}_p(\tilde{X} \setminus K) \leq \frac{1}{3n}$. Take an $f \in C_b(X)$, $|f| \leq n$, $|\tilde{f}| \leq \frac{1}{2nM}$ on K. Now $\int \tilde{f} d\tilde{\mu} = \int_K \tilde{f} d\tilde{\mu} + \int_{\tilde{X} \setminus K} \tilde{f} d\tilde{\mu}$. Taking the $\|.\|_p$-norm on both sides, we get $\|\mu(f)\|_p \leq \frac{1}{2nM} M + \frac{1}{3n} n \leq 1$. This proves the continuity of μ.

(iii) Since $g \circ \mu \in M_\tau(X)$, $\forall g \in E'$, we get that the control measure $\lambda_p \in M_\tau(X)$. As in Theorem 1, this means $\lim \int f_\alpha d\lambda_p = \int f d\lambda_p$. So we get an increasing sequence $f_{\alpha(n)}$ and a Borel $B \subset X$ such that $\lambda_p(X \setminus B) = 0$ and $f_{\alpha(n)} \to f$ pointwise on B. Using the fact $f_{\alpha(n)} \leq f$, $\forall n$, by ([7], Theorem 1, p. 20), $\bar{\mu}_p(f - f_{\alpha(n)}) \to 0$. This proves the result.

(iv) The proof is identical to the one given in Theorem 1 (ii).

(v) Suppose ν_1 and ν_2 are two regular Borel extensions of ν, satisfying the given condition. Fix an open set $V \subset X$ and take an increasing net $\{U_\alpha\}$ of positive-sets in X such that $U_\alpha \uparrow V$. By (iii) $\nu_1(V) = \nu_2(V)$ and so, by regularity, $\nu_1 = \nu_2$.

Conversely, suppose that $\mu : (C_b(X), \beta_\tau) \to E$ is a linear and continuous mapping and the bounded sets are mapped into relatively weakly compact subset of E. Proceeding as in Theorem 2, we get a unique countably additive Baire measure ν on X such that $\int f d\nu = \mu(f)$, for every $f \in C_b(X)$. Now for every $g \in E'$, $g \circ \mu : (C_b(X), \beta_\tau) \to K$ is a linear and continuous and $g \circ \mu \in M_\tau(X)$. This means ν is τ-smooth. □

A countably additive Baire measure $\mu : \mathcal{B}_0 \to E$ is called tight if for every $g \in E'$, $g \circ \mu \in M_t(X)$. The set of all E-valued tight measures will be denoted by $M_t(X, E)$. It is a trivial verification that a tight measure $\mu : \mathcal{B}_0 \to E$ is also τ-smooth.

Theorem 4. *Suppose X is a completely regular Hausdorff space and $\mu : \mathcal{B}_0 \to E$ is a tight measure. Then*

(i) *μ can be extended to a Borel measure which is inner regular by compact sets and outer regular by open sets;*

(ii) *the linear mapping $\mu : (C_b(X), \beta_t) \to E$ is continuous and bounded sets are mapped into relatively weakly compact sets;*

(iii) *considering μ a Borel measure, suppose $\{f_\alpha\}$ is an increasing net of non-negative, lower semi-continuous functions in $L^1(\mu)$, converging pointwise to an $f \in L^1(\mu)$, pointwise on X. Then $\lim \bar{\mu}_p(f - f_\alpha) = 0$, $\forall p \in P$; in particular $\lim \int f_\alpha d\mu = \int f d\mu$;*

(iv) *the regular Borel extension of μ, satisfying condition* (i)*, is unique.*

Conversely, if a linear mapping $\mu : (C_b(X), \beta_t) \to E$ is continuous and maps bounded sets into relatively weakly compact sets, then there exists a unique tight measure $\nu : \mathcal{B}_0 \to E$ such that $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$.

Proof. (i). Since the measure is τ-smooth, using Theorem 3, it can be uniquely extended to a Borel measure, satisfying condition (iii) of Theorem 3. Now considering this a Borel measure and using the fact for every $g \in E'$, $g \circ \mu \in M_t(X)$, we get that μ is regular in the weak topology on E. By ([8], Theorem 1.6, p. 159), μ is inner regular by compact subsets of X; it is a simple verification that this implies that μ is outer regular by open subsets of X.

(ii) To prove the continuity of $\mu : (C_b(X), \beta_t) \to E$, we will use the fact that β_t is the finest locally convex topology agreeing with the compact-open topology on norm-bounded sets. Fix a $p \in P$, an $n \in N$ and a $c > 0$. Take an $M > 0$ such that $\bar{\mu}_p(X) \leq M$. Take a compact $C \subset X$ such that $\bar{\mu}_p(X \setminus C) \leq \frac{1}{2n}$. Now for any $f \in C_b(X)$, $\|f\| \leq n$ and $|f| \leq \frac{1}{2M}$ on C we have $\mu(f) = \int_C f d\mu + \int_{X \setminus C} f d\mu$. Taking the $\|\cdot\|_p$-norm on both sides, we get $\|\mu(f)\|_p \leq \|\int_C f d\mu\|_p + \|\int_{X \setminus C}\|_p \leq \frac{1}{2M} M + n\frac{1}{2n} < 1$.

Since β_t is the finest locally convex topology, agreeing with the compact-open topology on bounded sets, we prove that μ is continuous. Also, since

μ is countably additive, the bounded sets are mapped into relatively weakly compact subsets of E.

(iii) Since the measure μ is τ-smooth, this follows from (iii) of Theorem 3.

(iv) Let μ_i (i=1, 2) be two Borel extensions of μ, satisfying (i). Take an open V and a compact C in X. There is a zero-set Z in X, $C \subset Z \subset V$. Since $\mu_1 = \mu_2$ on zero-sets, we get $\mu_1(V) = \mu_2(V)$. By regularity, $\mu_1 = \mu_2$.

Conversely, suppose that $\mu : (C_b(X), \beta_t) \to E$ is a linear and continuous mapping and the bounded sets are mapped into relatively weakly compact subsets of E. Proceeding as in Theorem 2, we get a unique countably additive Baire measure ν on X such that $\int f d\nu = \mu(f)$ for every $f \in C_b(X)$. Now for every $g \in E'$, $g \circ \mu : (C_b(X), \beta_t) \to K$ is a linear and continuous and so $g \circ \mu \in M_t(X)$. This means ν is tight. □

3. Alexandrov's Theorem

In this section, we extend the celebrated Alexandrov representation theorem to the vector-valued measures. In the scalar case, in a simple form, this theorem says:

Suppose X is a completely regular Hausdorff space, $\mathcal{F}$ the algebra generated by zero-sets and $\mu : C_b(X) \to K$ a continuous linear mapping. Then there exists a unique, finitely additive measure $\nu : \mathcal{F} \to R$ such that

(i) ν is inner regular by zero-sets and outer regular by positive-sets;

(ii) $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$. ([12], Theorem 6, p. 163; [11],). Note $C_b(X)$ is contained in the uniform closure of $\mathcal{F}$-simple functions on X in the space of all bounded functions on X and so each $f \in C_b(X)$ is ν-integrable.

We state and prove the following extension. Our proof is obtained by the regularity properties of the corresponding regular Borel measure on $\tilde{X}$ and is very different from that given in [11]. We start with a lemma.

Lemma 5. *If Z_1 and Z_2 are zero-sets in X, then $\overline{Z_1 \cap Z_2} = \overline{Z_1} \cap \overline{Z_2}$ (for a subset $A \subset X$, $\overline{A}$ denotes the closure of A in $\tilde{X}$). Hence if $Z_1 \cap Z_2 = \varnothing$, then $\overline{Z_1} \cap \overline{Z_2} = \varnothing$.*

Proof. Suppose this is not true. Take a point $a \in \overline{Z_1} \cap \overline{Z_2} \setminus \overline{Z_1 \cap Z_2}$ (note $Z_1 \cap Z_2$ can be empty). Take an $f \in C_b(X)$, $0 \leq f \leq 1$, such that $\tilde{f}(a) = 1$ and $f = 0$ on $Z_1 \cap Z_2$. For $i = 1, 2$, take $h_i \in C_b(X)$ such that $0 \leq h_i \leq 1$ and $Z_i = h_i^{-1}(0)$. Define $f_i(x) = f(x)\frac{h_i(x)}{h_1(x)+h_2(x)}$, for $x \notin Z_1 \cap Z_2$, and 0 otherwise. These functions are continuous and $f = f_1 + f_2$. Thus $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$. Since $f_i = 0$ on Z_i, $\tilde{f}_i = 0$ on $\overline{Z_i}$ and so $\tilde{f}_1 + \tilde{f}_2 = 0$ on $\overline{Z_1} \cap \overline{Z_2}$. This means $\tilde{f}(a) = 0$, a contradiction. □

Now we come to the main theorem.

Theorem 6. *Suppose X is a completely regular Hausdorff space and $\mu : C_b(X) \to E$ a weakly compact linear mapping. Then there exists a unique finitely additive, exhaustive measure $\nu : \mathcal{F} \to E$ such that*

(i) ν is inner regular by zero-sets and outer regular by positive-sets;

(ii) $\int f d\nu = \mu(f)$, $\forall f \in C_b(X)$.

Proof. Considering $\tilde{\mu} : C(\tilde{X}) \to E$, we get an E-valued regular Borel measure $\tilde{\mu} : \mathcal{B}(\tilde{X}) \to E$. If A is a subset of X or $\tilde{X}$, $\overline{A}$ will denote the closure of A in $\tilde{X}$. We prove this theorem in several steps.

I. Let $\overline{\mathcal{Z}} = \{\overline{A} : A \text{ a zero-set in } X\}$. Then for every $Q \in \overline{\mathcal{Z}}$ and $c > 0$, there exists $W \in \overline{\mathcal{Z}}$ such that $W \subset \tilde{X} \setminus Q$ and $\bar{\tilde{\mu}}_p((\tilde{X} \setminus Q) \setminus W) < c$.

Proof. Using the inner regularity of $\bar{\tilde{\mu}}_p$ and Urysohn's lemma, we can take a positive set $V \subset \tilde{X} \setminus Q$ having the property that $\bar{\tilde{\mu}}_p((\tilde{X} \setminus Q) \setminus V) < \frac{c}{2}$. Take a $g \in C(\tilde{X})$, $0 \le g \le 1$, such that $V = g^{-1}(0, 1]$. Put $V_n = \{x \in \tilde{X} : g(x) > \frac{1}{n}\}$ and $Z_n = \{x \in \tilde{X} : g(x) \ge \frac{1}{n}\}$. Now, using the fact that X is dense in $\tilde{X}$, we have $V_n \subset \overline{(V_n \cap X)} \subset \overline{(Z_n \cap X)} \subset Z_n \subset V_{n+1}$ By choosing n sufficiently large we can assume $\bar{\tilde{\mu}}_p(V \setminus V_n) < \frac{c}{2}$. Taking $W = \overline{(Z_{n+1} \cap X)}$, we get the result.

II. Let $\mathcal{A}$ be the algebra, in $\tilde{X}$, generated by $\overline{\mathcal{Z}}$ and denote by $\mathcal{A}_0$ the elements of $\mathcal{A}$ which have the property that these elements and their complements are inner regular by the elements of $\overline{\mathcal{Z}}$. Then $\mathcal{A}_0 = \mathcal{A}$.

Proof. We use I to prove. By I, $\mathcal{A}_0 \supset \overline{\mathcal{Z}}$. By definition, $\mathcal{A}_0$ is closed under complements. Also, using Lemma 5, it is a routine verification that if A and B are in $\mathcal{A}_0$, then $A \cup B$ and $A \cap B$ are also in $\mathcal{A}_0$. This proves the result.

III. Let $\mathcal{F}$ be the algebra, in X, generated by zero-sets in X. Then it is a simple verification that $\mathcal{A} \cap X \supset \mathcal{F}$. Also if $A \in \mathcal{A}$ and $A \cap X = \varnothing$, then $\bar{\tilde{\mu}}_p(A) = 0$. To prove this, take any $\overline{Z} \in \overline{\mathcal{Z}}$, Z being a zero-set in X, such that $\overline{Z} \subset A$. This means Z is empty and so $\bar{\tilde{\mu}}_p(A) = 0$. Now we can define a $\nu : \mathcal{F} \to E$, $\nu(B) = \tilde{\mu}(A)$, A being any element in $\mathcal{A}$ with $B = A \cap X$; it is a trivial verification that ν is well-defined, is finitely additive and it is inner regular by zero-sets in X and outer regular by positive-sets in X. We also have $\nu(Z) = \tilde{\mu}(\overline{Z})$ for any zero-set $Z \subset X$. Since $\nu(\mathcal{F})$ is relatively weakly compact in E, ν is exhaustive ($\equiv$strongly additive) ([2], Corollary 3, p. 28; this is proved for Banach space E, but easily extends to the quasi-complete locally convex space E). Also, for any $B \in \mathcal{F}$, $\bar{\nu}_p(B) \le \bar{\tilde{\mu}}_p(A)$, where A is any element in $\mathcal{A}$ such that $B = A \cap X$.

IV. For any $f \in C_b(X)$, $\mu(f) = \int f d\nu$.

Proof. Assume $\bar{\tilde{\mu}}_p(X) \le 1$. Fix a $c > 0$ and take an $f \in C_b(X)$, $0 \le f \le 1$. Then there is a non-negative, $\mathcal{F}$-simple function $\sum_{i=1}^n a_i \chi_{B_i}$ such that B_i's are mutually disjoint, their union is X and $|f - \sum_{i=1}^n a_i \chi_{B_i}| < c$ on X. Take mutually disjoint $\{A_i\} \subset \mathcal{A}$ such that $B_i = A_i \cap X$ for every i. Also take mutually disjoint zero-sets $\{Z_i\} \subset X$ such that $\bar{\tilde{\mu}}_p(A_i \setminus \overline{Z_i}) < \frac{c}{n}$, for each i. Now

$$\begin{aligned} \| \int f d\nu - \sum a_i \nu(Z_i) \|_p &\le \| \int f d\nu - \sum a_i \nu(B_i) \|_p + \| \sum a_i \nu(B_i \setminus Z_i) \|_p \\ &\le c + \| \sum a_i \tilde{\mu}(A_i \setminus \overline{Z_i}) \|_p \le c + n \frac{c}{n} = 2c. \end{aligned}$$

Also, $|f - \sum_{i=1}^n a_i \chi_{B_i}| \le c$ implies that $|\tilde{f} - \sum_{i=1}^n a_i \chi_{\overline{(Z_i)}}| \le c$ on $\cup(\overline{Z_i})$ (note $\overline{Z_i}$ are also mutually disjoint by Lemma 5). So $\| \int \tilde{f} d\tilde{\mu} - \sum a_i \nu(Z_i) \|_p = \| \int \tilde{f} d\tilde{\mu} -$

$\sum a_i\tilde{\mu}(\overline{Z_i})\|_p \leq c + \| \sum a_i\tilde{\mu}(A_i \setminus \overline{Z_i})\|_p \leq c + n.\frac{c}{n} = 2c$. This prove that $\mu(f) = \nu(f)$.

V. Uniqueness.

Proof. Let $\nu : \mathcal{F} \to E$ be a finitely additive regular (inner regular by zero-sets in X and outer regular by positive-sets in X) measure, having a relatively weakly compact range, such that $\int f d\nu = 0$, $\forall f \in C_b(X)$. This means ν is exhaustive and so $\bar{\nu}_p(X) < \infty$, $\forall p \in P$. If $\nu \neq 0$, then there is $p \in P$, a zero-set $Z \subset X$, and a $c > 0$ such that $\|\nu(Z)\|_p = 2c$. Take a a positive-set $U \supset Z$ such that $\bar{\nu}_p(U \setminus Z) < c$. Then take an $f \in C_b(X)$, $0 \leq f \leq 1$, $f(Z) = \{1\}$, $f(X \setminus U) = \{0\}$. We get $0 = \int f d\nu = \int_Z f d\nu + \int_{U\setminus Z} f d\nu$. This means $\nu(Z) = -\int_{U\setminus Z} f d\nu$ and so $2c \leq 1.\bar{\nu}_p(U \setminus Z) < c$. This contradiction proves the uniqueness. □

We denote by $M(X, E)$ the set of all exhaustive, finitely additive $\nu : \mathcal{F} \to E$ which are inner regular by zero-sets in X and outer regular by positive-sets in X; they are the collection of all weakly compact, continuous linear maps $\nu : C_b(X) \to E$.

4. Representation Theorem for $C(X)$ with a Completely Regular X

In this section we assume that $K = R$. A subset $B \subset C(X)$ will be called order-bounded if there are elements f and g in $C(X)$ such that $f \leq b \leq g$, $\forall b \in B$. It is well-known that a linear map $\mu : C(X) \to R$, which maps order-bounded sets into bounded sets, gives a unique $\nu \in M_\sigma(X)$ such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \int f d\nu$ ([12], Theorem 23; [4]).

We will extend this fact to the vector case.

Theorem 7. *Let $\mu : C(X) \to E$ be a linear map such that order-bounded subsets are mapped into relatively weakly compact subsets of E. Then*

(i) *There is a unique $\nu \in M_\sigma(X, E)$ such that $C(X) \subset L^1(\nu)$ and $\mu(f) = \int f d\nu$;*

(ii) *for every $p \in P$ there is compact $C \subset \upsilon X$ (the real-compactification of X), depending on p, such that $\bar{\tilde{\nu}}_p(\tilde{X} \setminus C) = 0$ ([4]).*

Proof. (i) We will use the fact that, when $E = R$, the result is known. First restrict μ to $C_b(X)$; this means μ is a weakly compact linear operator and $\forall h \in E'$, $h \circ \mu \in M_\sigma(X)$ (here we are using the fact that, for $E = R$, the result is known). So there exists a $\nu \in M_\sigma(X, E)$ such that $\mu(f) = \int f d\nu$, $\forall f \in C_b(X)$ and $C(X) \subset L^1(|h \circ \nu|)$ for every $h \in E'$ and $h \circ \mu(f) = \int f d(h \circ \nu)$, for every $f \in C(X)$.

Now we will prove that $C(X) \subset L^1(\nu)$. Let S be the closed unit ball of $C_b(X)$. Fix an $f \in C(X)$, $f \geq 0$, and $A \in \mathcal{B}_0$. Take a net $\{g_\alpha\} \subset S$ such that $\int |g_\alpha - \chi_A| d|\lambda| \to 0$, for every $\lambda \in M_\sigma(X)$. Now $\{g_\alpha f\}$ is order-bounded in $C(X)$ and so $\{\mu(g_\alpha f)\}$ in relatively weakly compact in E. By taking subsets, if necessary, assume $\mu(g_\alpha f) \to x \in E$ weakly. Fix an $h \in E'$. We have $h \circ \mu(g_\alpha f) \to h(x)$ and so $\int (g_\alpha f) d(h \circ \nu) \to h(x)$. Now since f is integrable with respect to $|h \circ \nu|$, we get $\int (g_\alpha f) d(h \circ \nu) \to \int_A f d(h \circ \nu)$. From this we get that $\int_A f d(h \circ \nu) = h(x)$, $\forall h \in E'$. This implies that $f \in L^1(\nu)$. Now, from $\int f d(h \circ \nu) = h \circ \mu(f)$, $\forall h \in E'$, it follows that $\int f d\nu = \mu(f)$. We denote ν by μ.

(ii). Fix a $p \in P$. By the mapping $x \to \{f(x)\}_{f \in C(X)}$, X can be imbedded in $R^{C(X)}$, with product topology. Denoting $[-\infty, \infty]$ by $\bar{R}$, we get that X is embedded in the compact Hausdorff space $\bar{R}^{C(X)}$ (with product topology). The closure of X, in $R^{C(X)}$, is the real-compactification of X and will be denoted by υX; the closure of X, in $\bar{R}^{C(X)}$, is the Stone–Cech compactification and will be denote by $\tilde{X}$. Every $f \in C(X)$ extends continuously to υX (it will be real-valued; just the component-wise values); it also extends continuously to $\tilde{X}$ (can have values $\pm\infty$; just the component-wise values).

We will complete the proof (ii) in several steps:

I. For an $f \in C(X)$, there is a $c \geq 0$ such that if $U = \{x \in X : |f(x)| > c\}$ then $\bar{\mu}_p(U) = 0$.

Proof. The result will be proved if we prove under the assumption that $f \geq 0$. Suppose $\bar{\mu}_p(W_n) > 0$, $\forall n \in N$, where $W_n = \{x \in X : |f(x) > n\}$. Then there are sequences $\{a_n\}$ and $\{b_n\}$ of positive real numbers such that, for every n, $a_n < b_n < a_{n+1}$, $\lim a_n \to \infty$ and $\bar{\mu}_p(U_n) = c_n > 0$, where $U_n = f^{-1}(a_n, b_n)$. Take a sequence $\{h_n\} \subset E'$ such that $|h_n(V_p)| \leq 1$ and $|h_n \circ \mu|(U_n) > c_n$, $\forall n$. Choose $\{g_n\} \subset C_b(X)$, $0 \leq g_n \leq \chi_{U_n}$ such that $|h_n \circ \mu(g_n)| > c_n$, $\forall n$. Let $f_0 = \sum \frac{n}{c_n} g_n f$. Then $f_0 \in C(X)$ and $|h_n \circ \mu|(f_0) \geq \frac{n}{c_n} |h_n \circ \mu(g_n)| \geq n$, $\forall n$. Since $f_0 \in L^1(\nu)$, this is a contradiction. The smallest such c (which will exist because of countable additivity) will be denoted by c_f.

II. Let $\tilde{\mu}$ be the regular Borel measure on $\tilde{X}$ associated with $\mu \in M_\sigma(X, E)$. For an $f \in C(X)$, let $A_f = \{x \in X : |f(x)| \leq c_f\}$ (c_f is defined in I). Then $\bar{\tilde{\mu}}_p(\tilde{X} \setminus \overline{A_f}) = 0$.

Proof. Suppose this is not true. Then there is an $h \in E'$ with $|h(V_p| \leq 1$, and a $g \in C(\tilde{X})$ such that $|g| \leq \chi_{\tilde{X} \setminus \overline{A_f}}$ and $|(h \circ \tilde{\mu})(g)| > 0$. This means $|g|_{|X} \leq \chi_{X \setminus A_f}$ and $|(h \circ \mu)(g)| > 0$ which is a contradiction by I.

III. For an $f \in C(X)$ let $\bar{f}$ be its extension to υX. Let $A_{\bar{f}} = \{x \in \upsilon X : |\bar{f}(x)| \leq c_f\}$. Then $\bar{\tilde{\mu}}_p(\tilde{X} \setminus \overline{A_{\bar{f}}}) = 0$. Consequently, $\bar{\tilde{\mu}}_p(\tilde{X} \setminus \cap\{\overline{A_{\bar{f}}} : f \in C(X)\}) = 0$.

Proof. Since $\overline{A_{\bar{f}}} \supset \overline{A_f}$, the result follows by II.

IV. $\cap\{\overline{A_{\bar{f}}} : f \in C(X)\} = \cap\{A_{\bar{f}} : f \in C(X)\}$.

Proof. To prove this, take a $y \in \cap\{\overline{A_{\bar{f}}} : f \in C(X)\}$. Fix an $f \in C(X)$. Suppose $y \notin A_{\bar{f}}$; then $y \notin \upsilon X$. Take a $g \in C(X)$ such that $\bar{g}(y) = \infty$. This means $y \notin \cap\overline{A_{\bar{g}}}$.This contradiction proves the result.

V. If $C = \cap\{A_{\bar{f}} : f \in C(X)$, then C is a compact subset of υX and $\bar{\tilde{\mu}}_p(\tilde{X} \setminus C) = 0$.

Proof. It follows from IV that C is compact in υX. Now, from III, $\bar{\tilde{\mu}}_p(\tilde{X} \setminus C) = 0$. This proves the result. □

In the following corollary we take E to be an order complete locally convex vector lattice such that if a bounded net $\{x_\alpha\}$ order converges to x then $x_\alpha \to x$ in E; these assumptions imply that E is an ideal in E'' and order intervals in E are $\sigma(E, E')$-compact ([1], Theorem 11.13, p. 170). By ([9], 7.5, Corollary 1), if E is an order complete vector lattice whose order is regular and of minimal type, then E with order topology ([9], Sec. 6, p. 230) has the above property (examples of these spaces are given in [9], p. 240).

Corollary 8. *Let E be an order complete locally convex vector lattice with the property if a bounded net $\{x_\alpha\}$ order converges to x then $x_\alpha \to x$ in E. Let $\mu : C(X) \to E$ be a positive linear map. Then*

(i) *There is a unique $\nu \in M_\sigma(X, E)$ such that $C(X) \subset L^1(\nu)$, $\nu \geq 0$ (this means $f \in C(X)$, $f \geq 0$ implies $\nu(f) \geq 0$) and $\mu(f) = \int f d\nu$, $\forall f \in C(X)$;*

(ii) *for every $p \in P$ there is compact $C \subset \upsilon X$ (the real-compactification of X), depending on p, such that $\bar{\tilde{\nu}}_p(\tilde{X} \setminus C) = 0$.*

Proof. The assumptions on μ and E imply that order bounded sets are mapped into relatively $\sigma(E, E')$-compact subsets E. The result follows from Theorem 7. □

Let $M_c(X, E) = \{\mu \in M(X, E) : \text{supp}(\bar{\tilde{\mu}}_p) \subset \upsilon X,\ \forall p \in P\}$. It is easy to see that $M_c(X, E) \subset M_\sigma(X, E)$: Take a $\mu \in M_c(X, E)$ and a bounded sequence $\{f_n\} \subset C_b(X)$, $f_n \to 0$, pointwise to 0 in X; this means $f_n \to 0$, pointwise on υX (well-known result). Now $\mu(f_n) = \tilde{\mu}\ (\tilde{f_n}) \to 0$ implies that $\mu \in M_\sigma(X, E)$.

The following corollary is somewhat converse to Theorem 7; it says that measures in $M_c(X, E)$ map order-bounded subsets of $C(X)$ into relatively weakly compact subsets of E.

Corollary 9. *Let $\mu \in M_c(X, E)$. Then $C(X) \subset L^1(\mu)$ and in the linear map $\mu : C(X) \to E$, order-bounded subsets are mapped into relatively weakly compact subsets of E.*

Proof. Take an $f \in C(X)$, $f \geq 0$. Fix a $p \in P$ and let $C = \text{supp}(\bar{\tilde{\mu}}_p$. Put $M = \sup \tilde{f}(C)$. $U = \{x \in \tilde{X} : \tilde{f}(x) > M\}$ is an open Baire set in $\tilde{X}$ and is disjoint from C so that $\tilde{\mu}_p(U) = 0$. From this it easily follows that $\bar{\mu}_p(U \cap X) = 0$ (note $U \cap X$ is a Baire set in X). Also, $f \leq M$ *a.e.*$[\lambda_p]$. Since the constant functions are in $L^1(\mu)$, by ([7], Theorem 2, p. 30), $f \in L^1(\mu)$.

Putting $h = \inf(f, M)$, we have $f = h$ *a.e.*$[\lambda_p]$. Let K be an absolutely convex, weakly compact subset of E such that $\mu(S) \subset K$ (S being the closed unit ball of $C_b(X)$). This means $\mu(h) \in MK$. Since $f = h$ *a.e.*$[\lambda_p]$, we have $\mu(f) \in MK$. This proves that order-bounded sets are mapped into relatively weakly compact sets. □

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