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ASYMPTOTIC BEHAVIOR OF RELATIVELY NONEXPANSIVE OPERATORS IN BANACH SPACES

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Abstract. Let K be a closed convex subset of a Banach space X and let F be a nonempty closed convex subset of K. We consider complete metric spaces of self-mappings of K which fix all the points of F and are relatively nonexpansive with respect to a given convex function f on X. We prove (under certain assumptions on f) that the iterates of a generic mapping in these spaces converge strongly to a retraction onto F.

0. Introduction

In this paper we consider the problem of whether and under what conditions, relatively nonexpansive operators T defined on, and with values in, a nonempty, closed convex subset K of a Banach space $(X, ||\cdot||)$ have the

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property that the sequences $\{T^k x\}_{k=1}^{\infty}$ converge strongly to fixed points of T, whenever $x \in K$.

We say that an operator $T : K \to K$ is *relatively nonexpansive* with respect to the convex function $f : X \to \mathbb{R}^1 \cup \{\infty\}$ if K is a subset of the algebraic interior $\mathcal{D}^0$ of the domain of f,

$$\mathcal{D} := \mathrm{dom}(f) = \{x \in X : f(x) < \infty\},$$

the function f is lower semicontinuous on K and there exists a point $z \in K$ such that, for any $x \in K$, we have

$$D_f(z, Tx) \leq D_f(z, x), \tag{0.1}$$

where $D_f : X \times \mathcal{D}^0 \to [0, \infty)$ stands for the Bregman distance given by

$$D_f(y, x) = f(y) - f(x) + f^0(x, x - y), \tag{0.2}$$

and $f^0(x, d)$ denotes the right-hand derivative of f at x in the direction d. In this case, the point z is called a *pole* of T with respect to f.

The problem described above occurs in mathematics in various forms. For example, if the operator T is such that, for some $z \in K$,

$$||z - Tx|| \leq ||z - x||, \tag{0.3}$$

for all $x \in K$, then it is relatively nonexpansive with respect to any of the functions $f(x) = ||x - z||^r$ with $r > 1$. Clearly, the nonexpansive operators on bounded closed convex subsets of uniformly convex Banach spaces fall in this class and there is a rich literature dedicated to the possible convergence of the sequences $\{T^k x\}_{k=1}^{\infty}$ generated by such operators (see, for instance, [1], [15] and the references therein). In general, relatively nonexpansive operators with respect to arbitrary convex functions f may not be quasi-nonexpansive in the sense of (0.3). Examples of such operators can be found in [5]. The asymptotic behavior of operators which are relatively nonexpansive with respect to some function f without necessarily being nonexpansive in the classical sense of the term is of special interest in the convergence analysis of feasibility, optimization and equilibrium methods for solving problems of image processing, rational resource allocation, and optimal control. The most typical examples in this regard are the Bregman projections and the Yosida type operators which are the cornerstones of the common fixed point and optimization algorithms discussed in [5] (see also the references therein). These operators satisfy a stronger condition than (0.1), namely, they are *strongly nonexpansive* with respect to f in the sense that, for some $z \in K$, we have

$$D_f(z, Tx) + D_f(Tx, x) \leq D_f(z, x) \tag{0.4}$$

for all $x \in K$.

The asymptotic behavior of related classes of operators was also studied in [2, 4, 11, 17]. It is known that, in general, sequences $\{T^k x\}_{k=1}^{\infty}$ generated by operators T which are relatively nonexpansive with respect to a convex function f may converge weakly, but not necessarily strongly. On the other hand, experiments with many iterative procedures based on computing sequences $\{T^k x\}_{k=1}^{\infty}$ generated by relatively nonexpansive operators T show that, in practice, these procedures do seem to converge strongly. The aim of this paper is to show that, under quite mild conditions, strong convergence of the sequences $\{T^k x\}_{k=1}^{\infty}$ generated by relatively nonexpansive operators is the rule and that weak, but not strong, convergence is the exception. To this end, we consider the set $\mathcal{M} = \mathcal{M}(f, K, F)$ of all operators $T : K \to K$ which are relatively nonexpansive with respect to the same convex function $f : X \to \mathbb{R}^1 \cup \{\infty\}$ and which have a nonempty closed convex set F of common poles. We assume that the function f satisfies the following conditions:

A(i) For any nonempty bounded set $E \subset K$ and any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\text{if } x \in E,\ z \in F \text{ and } D_f(z, x) \le \delta, \text{ then } ||z - x|| \le \varepsilon. \tag{0.5}$$

A(ii) There exists $\theta \in F$ such that the restriction to K of the function $g(\cdot) := D_f(\theta, \cdot)$ has the following property: For any subset $E \subset K$, $g(E)$ is bounded if and only if E is bounded.

A(iii) For any $z \in F$, the function $D_f(z, \cdot) :\ K \to \mathbb{R}^1$ is convex and lower semicontinuous.

A(iv) For any $x \in K$, there exists a vector $Px \in F$ such that

$$D_f(Px, x) \le D_f(z, x) \text{ for all } z \in F. \tag{0.6}$$

In practical situations one also uses the following stronger version of A(i):

For any nonempty bounded set $E \subset K$, $\inf\{\nu_f(x, t) :\ x \in E\}$ is positive for all $t > 0$, where

$$\nu_f(x, t) = \inf\{D_f(y, x) :\ y \in X \text{ and } ||y - x|| = t\}.$$

In [5] this condition is termed *sequential compatibility* of the function f with the relative topology of the set K. We will show (see Lemma 1.1 below) that sequential compatibility implies A(i). In its turn, condition A(i) implies that all $z \in F$ are common fixed points of the operators in $\mathcal{M}$. Condition A(ii) guarantees that any operator $T \in \mathcal{M}$ is bounded on bounded subsets of K (a feature which is essential in our proofs) because, for any bounded set $E \subset K$, we have

$$D_f(\theta, Tx) \le D_f(\theta, x),$$

where, according to condition A(ii), the function $D_f(\theta, \cdot)$ is bounded on E, and therefore so is the set $\{Tx :\ \ x \in E\}$. Condition A(ii), even taken in conjunction with A(i), is satisfied by many useful functions and, among them, by many functions which are sequentially compatible with the relative topology of K. In contrast, condition A(iii) is quite restrictive. However, it does hold for many functions f which are of interest in current applications (see the examples below). The vector Px satisfying (0.6) was termed the *Bregman projection* with respect to f of x onto F in [10]. Condition A(iv) is automatically satisfied when X is reflexive and f is totally convex on K (in particular, when f is sequentially compatible with the relative topology of K) as follows from [5, Proposition 2.1.5(i)]. In this case, if f is differentiable on the algebraic interior of its domain, then, for each $x \in K$, there exists a unique vector Px in F which satisfies (0.6) and the operator P satisfies condition (0.4) (cf. [5, Prop. 2.1.5(ii)]). We now mention four typical situations in which all the conditions A(i)-A(iv) are satisfied simultaneously.

(i) (cf. [5]) X is a Hilbert space, K and F are nonempty closed convex subsets of X such that $F \subset K$ and $f(x) = ||x||^2$;

(ii) (cf. [3]) $F \subset K \subset \mathbb{R}^n_{++}$ and f is the negentropy;

(iii) (cf. [6]) X is a Lebesgue space L^p or l^p, $1 < p \leq 2$, $f(x) = ||x||^p$ and K consists of either nonnegative or nonpositive functions;

(iv) (cf. [7]) X is smooth and uniformly convex, F is a singleton $\{z\}$, and $f(x) = ||x - z||^r$ with $r > 1$.

We provide the set $\mathcal{M} = \mathcal{M}(f, K, F)$ with the uniformity determined by the following base:

$$E(N, \varepsilon) = \{(T_1, T_2) \in \mathcal{M} \times \mathcal{M} : \ ||T_1 x - T_2 x|| \leq \varepsilon$$

$$\text{for all } x \in K \text{ satisfying } ||x|| \leq N\},$$

where $N, \varepsilon > 0$. Clearly this uniform space is metrizable and complete. We equip the space $\mathcal{M}$ with the topology induced by this uniformity. Let $\mathcal{M}_c$ be the set of all operators in $\mathcal{M}$ which are continuous on K. This is a closed subset of $\mathcal{M}$ and we endow it with the relative topology. The subset of $\mathcal{M}_c$ consisting of those operators which are uniformly continuous on bounded subsets of K is denoted by $\mathcal{M}_u$. Again, this set is closed in $\mathcal{M}$ and we endow it with the relative topology. We show (see Theorems 2.1, 3.1 and 3.2) below that the sequence of powers of a generic mapping T in $\mathcal{M}_u$, $\mathcal{M}_c$ and $\mathcal{M}$ respectively, converges in the uniform topology to a relatively nonexpansive operator which belongs to the same space and is a retraction onto F. Consequently, the sequences $\{T^k x\}_{k=1}^{\infty}$ generated by a generic mapping T are strongly convergent to points in F, i.e., to fixed points of T. The basic mathematical tools we employ are the methods of generic analysis which have already been proved useful in the theory of dynamical systems ([8], [13], [14], [16] and [18]) as well as in the calculus of variations (see [19] and [20]).

In particular, we have shown in [8] that the iterates of a generic operator in certain other spaces of relatively nonexpansive operators converge strongly to its unique fixed point. As we have just noted above, in the different situation considered here, the iterates of a generic operator converge to a retraction onto its fixed point set F.

Our paper is organized as follows. In Section 1 we prove two preliminary lemmas regarding the convex function f and the Bregman projection P. In Section 2 we state our generic result (Theorem 2.1) for the space $\mathcal{M}_u$. This result is proved in Section 5. Our generic results for the spaces $\mathcal{M}$ (Theorems 3.1 and 3.2) and $\mathcal{M}_c$ (Theorems 7.1–7.3) are stated in Section 3 and 7 respectively. These results are proved in Sections 6 and 8. Section 4 is devoted to two auxiliary assertions.

We emphasize that in contrast with many individual convergence theorems, all of our results hold in a general Banach space.

1. Preliminaries

This section is devoted to two lemmas. The first one shows that sequential compatibility implies condition A(i) while the second shows that the retraction, the existence of which is stipulated in condition A(iv), belongs to $\mathcal{M}$.

Lemma 1.1. *If the convex function f is sequentially compatibile with the relative topology of K, then it satisfies condition* A(i).

Proof. Let the convex function f be sequentially compatible with the relative topology of K. For any nonempty set $E \subset K$ and any $t \geq 0$, set

$$\nu_f(E,t) = \inf\{D_f(y,x) : \ x \in E, \ y \in X \text{ and } ||y-x|| = t\}.$$

Since f is assumed to be sequentially compatible with the relative topology of K, $\nu_f(E,t) > 0$ for any nonempty bounded set $E \subset K$ and any $t > 0$, and the function $\nu_f(x,\cdot)$ is strictly increasing (see [5, Proposition 1.2.2]).

Assume now that we are given a nonempty bounded subset M of K and an $\varepsilon > 0$. Let $\delta = \nu_f(M,\varepsilon)$. If $x \in M$, $y \in F$ and $D_f(y,x) \leq \delta$, then

$$\nu_f(x, ||y-x||) \leq D_f(y,x) \leq \delta \leq \nu_f(x,\varepsilon).$$

Since the function $\nu_f(x,\cdot)$ is strictly increasing we conclude that $||y-x|| \leq \varepsilon$. Lemma 1.1 is proved. □

Note that the functions in the examples (i)–(iv) listed in the Introduction are all sequentially compatible with the relative topology of any closed convex subset of their respective domains.

Lemma 1.2. *Let an operator* $P: K \to F$ *be as guaranteed in condition* A(iv). *Then for any* $x \in K$ *and for any* $z \in F$, *we have*

$$D_f(z, Px) \le D_f(z, x). \tag{1.1}$$

Proof. Fix $x \in K$ and $z \in F$. Denote $\hat{x} = Px$ and let

$$u(\alpha) = \hat{x} + \alpha(z - \hat{x}) \tag{1.2}$$

for any $\alpha \in [0, 1]$. Observe that $D_f(\cdot, x)$ and f are convex and, therefore, the following limits exist, and for all $y \in K$ and $d \in X$,

$$\begin{aligned}
&[D_f(\cdot, x)]^0(y, d) = \lim_{t\to 0^+} [D_f(y + td, x) - D_f(y, x)]/t \\
&= \lim_{t\to 0^+} [f(y + td) - f(x) + f^0(x, x - y - td) - (f(y) - f(x) \\
&+ f^0(x, x - y))]/t \\
&= \lim_{t\to 0^+} [f(y + td) - f(y)]/t + \lim_{t\to 0^+} [f^0(x, x - y - td) - f^0(x, x - y)]/t \\
&= f^0(y, d) + \lim_{t\to 0^+} [f^0(x, x - y - td) - f^0(x, x - y)]/t.
\end{aligned}$$

The function $f^0(x, \cdot)$ is subadditive and positively homogeneous because f is convex. Consequently, we have

$$f^0(x, x - y) \le f^0(x, x - y - td) + tf^0(x, d).$$

Combining this inequality and the previous formula we get

$$[D_f(\cdot, x)]^0(y, d) \ge f^0(y, d) - f^0(x, d). \tag{1.3}$$

Now since $\hat{x} = Px$, we have by (0.6) and (1.3) that for any $\alpha \in (0, 1]$,

$$\begin{aligned}
0 &\ge D_f(\hat{x}, x) - D_f(u(\alpha), x) \ge [D_f(\cdot, x)]^0(u(\alpha), \hat{x} - u(\alpha)) \\
&= [D_f(\cdot, x)]^0(u(\alpha), -\alpha(z - \hat{x})) = \alpha[D_f(\cdot, x)]^0(u(\alpha), \hat{x} - z)) \\
&\ge \alpha[f^0(u(\alpha), \hat{x} - z) - f^0(x, \hat{x} - z)].
\end{aligned}$$

Hence, for any $\alpha \in (0, 1]$ we get

$$f^0(x, \hat{x} - z) \ge f^0(u(\alpha), \hat{x} - z). \tag{1.4}$$

Note that by A(iii) the function $\phi(x) = f^0(x, x-z)$, $x \in K$, is lower semicontinuous. Hence the function $\phi(u(\alpha))$, $\alpha \in [0, 1]$ is also lower semicontinuous. Since

$$\phi(u(\alpha)) = f^0(u(\alpha), u(\alpha) - z) = (1 - \alpha)f^0(u(\alpha), \hat{x} - z),\ \alpha \in [0, 1),$$

the function $\alpha \mapsto f^0(u(\alpha), \hat{x} - z)$, $\alpha \in [0, 1)$, is lower semicontinuous too.

Taking the $\liminf_{\alpha\to 0^+}$ of both sides of the inequality (1.4), we see that

$$f^0(x, \hat{x} - z) \ge f^0(\hat{x}, \hat{x} - z).$$

This, in turn, implies that

$$f(z) - f(\hat{x}) + f^0(x, \hat{x} - z) \geq D_f(z, \hat{x}).$$

Since $f^0(x, \cdot)$ is sublinear, it follows that

$$f(z) - f(\hat{x}) + f^0(x, \hat{x} - x) + f^0(x, x - z) \geq D_f(z, \hat{x}).$$

Hence

$$D_f(z, x) + [f(z) - f(\hat{x}) - f(z) + f(x) + f^0(x, \hat{x} - x)] \geq D_f(z, \hat{x}). \tag{1.5}$$

Note that the quantity between square brackets is exactly

$$-[f(\hat{x}) - f(x) - f^0(x, \hat{x} - x)] \leq 0$$

because f is convex. This inequality and (1.5) imply (1.1). The proof of Lemma 1.2 is complete. □

In the sequel we will use the following notation.

For each $x \in K$ and each nonempty $G \subset K$, set

$$\rho_f(x, G) = \inf\{D_f(z, x) : \ z \in G\}. \tag{1.6}$$

2. Convergence of powers for a class of uniformly continuous operators

In this section we assume that the operator P, the existence of which is stipulated in condition A(iv), belongs to $\mathcal{M}_u$, and that the following condition is satisfied:

$$\begin{gathered}\text{For each bounded set } K_0 \subset K \text{ and each } \varepsilon > 0, \text{ there is } \delta > 0 \\ \text{such that if } x \in K_0,\ z \in F \text{ and } ||z - x|| \leq \delta, \text{ then } D_f(z, x) \leq \varepsilon.\end{gathered} \tag{2.1}$$

Remark. Note that condition (2.1) holds if the function f is Lipschitzian on each bounded subset of K.

Theorem 2.1. *There exists a set $\mathcal{F} \subset \mathcal{M}_u$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_u$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

(i) *There exists $P_B \in \mathcal{M}_u$ such that $P_B(K) = F$ and $B^n x \to P_B x$ as $n \to \infty$, uniformly on bounded subsets of K;*

(ii) *For each* $\varepsilon > 0$ *and each bounded set* $C \subset K$*, there exist a neighborhood* U *of* B *in* $\mathcal{M}_u$ *and an integer* $N \geq 1$ *such that for each* $S \in U$*, each* $x \in C$ *and each integer* $n \geq N$*,*

$$||S^n x - P_B x|| \leq \varepsilon.$$

This theorem will be established in Section 5.

3. Convergence of powers for a class of operators with a uniformly continuous Bregman distance

In this section we assume that the function $D_f(\cdot, \cdot) : F \times K \to \mathbb{R}^1$ is uniformly continuous on bounded subsets of $F \times K$ and state two theorems the proofs of which will be given in Section 6.

Theorem 3.1. *There exists a set* $\mathcal{F} \subset \mathcal{M}$ *which is a countable intersection of open everywhere dense subsets of* $\mathcal{M}$ *such that for each* $B \in \mathcal{F}$ *the following assertions hold:*

1. *There exists* $P_B \in \mathcal{M}$ *such that* $P_B(K) = F$ *and* $B^n x \to P_B x$ *as* $n \to \infty$*, uniformly on bounded subsets of* K*; if* $B \in \mathcal{M}_c$*, then* $P_B \in \mathcal{M}_c$*.*
2. *For each* $\varepsilon > 0$ *and each nonempty bounded set* $C \subset K$*, there exists a neighborhood* U *of* B *in* $\mathcal{M}$ *and a natural number* N *such that for each* $S \in U$ *and each* $x \in C$*, there is* $z(S, x) \in F$ *such that* $||S^n x - z(S, x)|| \leq \varepsilon$ *for all integers* $n \geq N$*.*

Moreover, if $P \in \mathcal{M}_c$*, then there exists a set* $\mathcal{F}_c \subset \mathcal{F} \cap \mathcal{M}_c$ *which is a countable intersection of open everywhere dense subsets of* $\mathcal{M}_c$*.*

Theorem 3.2. *Let the set* $\mathcal{F} \subset \mathcal{M}$ *be as guaranteed in Theorem* 3.1*,* $B \in \mathcal{F} \cap \mathcal{M}_c$*,* $P_B z = \lim_{n\to\infty} B^n z$*,* $z \in K$*, and let* $x \in K$*,* $\varepsilon > 0$*. Then there exists a neighborhood* U *of* B *in* $\mathcal{M}$*, a number* $\delta > 0$ *and a natural number* N *such that for each* $y \in K$ *satisfying* $||x - y|| \leq \delta$*, each* $S \in U$ *and each integer* $n \geq N$*,* $||S^n y - P_B x|| \leq \varepsilon$*.*

4. Auxiliary results

In this section we prove two lemmas which will be used in the proofs of our theorems. We use the convention that $S^0 x = x$ for each $x \in K$ and each $S \in \mathcal{M}$.

For each $\gamma \in (0, 1)$ and each $T \in \mathcal{M}$ define a mapping $T_\gamma : K \to K$ by

$$T_\gamma x = \gamma P x + (1 - \gamma) T x, \; x \in K, \tag{4.1}$$

where P is the operator the existence of which is stipulated in condition A(iv).

Lemma 4.1. *Let $T \in \mathcal{M}$ and $\gamma \in (0,1)$. Then $T_\gamma \in \mathcal{M}$. If $T, P \in \mathcal{M}_u$ (respectively, $T, P \in \mathcal{M}_c$), then $T_\gamma \in \mathcal{M}_u$ (respectively, $T_\gamma \in \mathcal{M}_c$).*

Proof. Clearly $T_\gamma \in \mathcal{M}$ and $T_\gamma x = x$ for all $x \in F$. By (4.1), A(iii), (0.1), A(iv) and Lemma 1.2, for each $z \in F$ and each $x \in K$,

$$\begin{aligned} D_f(z, T_\gamma x) = D_f(z, \gamma Px + (1-\gamma)Tx) \le \gamma D_f(z, Px) \\ + (1-\gamma) D_f(z, Tx) \le D_f(z, x). \end{aligned}$$

Thus $T_\gamma \in \mathcal{M}$. Clearly, $T_\gamma \in \mathcal{M}_u$ if $T, P \in \mathcal{M}_u$ and $T_\gamma \in \mathcal{M}_c$ if $T, P \in \mathcal{M}_c$. Lemma 4.1 is proved. □

It is obvious that for each $T \in \mathcal{M}$,

$$T_\gamma \to T \text{ as } \gamma \to 0^+ \text{ in } \mathcal{M}. \tag{4.2}$$

Lemma 4.2. *Let $T \in \mathcal{M}$, $\gamma \in (0,1)$ and let $x \in K$. Then*

$$\rho_f(T_\gamma x, F) \le (1-\gamma^2)\rho_f(x, F). \tag{4.3}$$

Proof. Let $\varepsilon > 0$. There exists $y \in F$ such that (see (1.6))

$$D_f(y, x) \le \rho_f(x, F) + \varepsilon. \tag{4.4}$$

It follows from (4.1), A(iv), Lemma 1.2, A(iii) and (0.1) that

$$\begin{aligned} \rho_f(T_\gamma x, F) =& \rho_f(\gamma Px + (1-\gamma)Tx, F) \\ \le& D_f(\gamma Px + (1-\gamma)y, (1-\gamma)Tx + \gamma Px) \\ \le& \gamma D_f(Px, \gamma Px + (1-\gamma)Tx) \\ &+ (1-\gamma) D_f(y, \gamma Px + (1-\gamma)Tx) \\ \le& \gamma^2 D_f(Px, Px) + \gamma(1-\gamma) D_f(Px, Tx) \\ &+ (1-\gamma)\gamma D_f(y, Px) + (1-\gamma)^2 D_f(y, Tx) \\ \le& \gamma(1-\gamma) D_f(Px, x) + (1-\gamma)\gamma D_f(y, Px) \\ &+ (1-\gamma)^2 D_f(y, Tx). \end{aligned} \tag{4.5}$$

It follows from (4.5), A(iv), Lemma 1.2 and (4.4) that

$$\begin{aligned} \rho_f(T_\gamma x, F) &\le \gamma(1-\gamma)\rho_f(x, F) + (1-\gamma)\gamma D_f(y, x) + (1-\gamma)^2 D_f(y, x) \\ &\le \varepsilon + (1-\gamma^2)\rho_f(x, F). \end{aligned}$$

Since ε is an arbitrary positive number we conclude that (4.3) holds. This completes the proof of Lemma 4.2. □

5. Proof of Theorem 2.1

Before we prove Theorem 2.1 we quote [8, Proposition 5.1] and prove one more lemma.

Proposition 5.1. *Let K_0 be a bounded subset of K, $T \in \mathcal{M}_u$, $\varepsilon > 0$, and let $n \geq 1$ be an integer. Then there exists a neighborhood U of T in $\mathcal{M}_u$ such that for each $S \in U$ and each $x \in K_0$, the inequality $||T^n x - S^n x|| \leq \varepsilon$ holds.*

Lemma 5.1. *Let $T \in \mathcal{M}_u$, $\gamma \in (0,1)$, $\varepsilon > 0$ and let K_0 be a nonempty bounded subset of K. Then there exist a neighborhood U of T_γ in $\mathcal{M}_u$ and a natural number N such that for each $x \in K_0$ there exists $Qx \in F$ such that for each integer $n \geq N$ and each $S \in U$,*

$$||S^n x - Qx|| \leq \varepsilon.$$

Proof. Set

$$K_1 = \bigcup \{S^i(K_0) : \ S \in \mathcal{M}, \ i \geq 0\}. \tag{5.1}$$

Assumption A(ii) and (0.1) imply that the set K_1 is bounded. Evidently,

$$S(K_1) \subset K_1 \text{ for all } S \in \mathcal{M}^{(F)}. \tag{5.2}$$

By A(i) there exists $\varepsilon_0 \in (0, \varepsilon)$ such that

$$\text{if } x \in K_1, \ z \in F \text{ and } D_f(z,x) \leq \varepsilon_0, \text{ then } ||z - x|| \leq 4^{-1}\varepsilon. \tag{5.3}$$

By (2.1) there is $\varepsilon_1 \in (0, 2^{-1}\varepsilon_0)$ such that

$$\text{if } x \in K_1, \ z \in F \text{ and } ||x - z|| \leq 2\varepsilon_1, \text{ then } D_f(z,x) \leq 2^{-1}\varepsilon_0. \tag{5.4}$$

By A(i) there is $\varepsilon_2 \in (0, 2^{-1}\varepsilon_1)$ such that

$$\text{if } x \in K_1, \ z \in F \text{ and } D_f(z,x) \leq 2\varepsilon_2, \text{ then } ||x - z|| \leq 2^{-1}\varepsilon_1. \tag{5.5}$$

Set

$$c_0 = \sup\{\rho_f(x,F) : \ x \in K_1\}. \tag{5.6}$$

By A(ii), $c_0 < \infty$. Choose a natural number $N \geq 4$ such that

$$(1 - \gamma^2)^N (c_0 + 1) \leq 2^{-1}\varepsilon_2. \tag{5.7}$$

It follows from Lemma 4.2, (5.6) and (5.7) that for each $x \in K_1$,

$$\rho_f(T_\gamma^N x, F) \leq (1 - \gamma^2)^N \rho_f(x,F) \leq (1 - \gamma^2)^N c_0 < 2^{-1}\varepsilon_2.$$

Thus for each $x \in K_1$ there is $Qx \in F$ such that $D_f(Qx, T_\gamma^N x) \le 2^{-1}\varepsilon_2$. When combined with (5.2) and (5.5), the last inequality implies that

$$||T_\gamma^N x - Qx|| \le 2^{-1}\varepsilon_1 \text{ for all } x \in K_1. \tag{5.8}$$

By Proposition 5.1, there exists a neighborhood U of T_γ in $\mathcal{M}_u$ such that for each $x \in K_1$ and each $S \in U$,

$$||S^N x - T_\gamma^N x|| \le 4^{-1}\varepsilon_1. \tag{5.9}$$

Assume that $x \in K_0$ and $S \in U$. Evidently, $\{S^i x\}_{i=0}^\infty \subset K_1$. By (5.8) and (5.9), $||S^N x - Qx|| \le 3 \cdot 4^{-1}\varepsilon_1$. It follows from this inequality and (5.4) that $D_f(Qx, S^N x) \le 2^{-1}\varepsilon_0$. Since $S \in \mathcal{M}_u$, it follows from the last inequality that $D_f(Qx, S^n x) \le 2^{-1}\varepsilon_0$ for all integers $n \ge N$. Combined with (5.3) this implies that $||Qx - S^n x|| \le \varepsilon$ for all integers $n \ge N$. Lemma 5.1 is proved. □

Proof of Theorem 2.1. By (4.2), the set $\{T_\gamma : \ T \in \mathcal{M}_u, \ \gamma \in (0,1)\}$ is an everywhere dense subset of $\mathcal{M}_u$. For each natural number i set

$$K_i = \{x \in K : \ ||x - \theta|| \le i\}. \tag{5.10}$$

By Lemma 5.1, for each $T \in \mathcal{M}_u$, each $\gamma \in (0,1)$ and each integer $i \ge 1$, there exist an open neighborhood $\mathcal{U}(T,\gamma,i)$ of T_γ in $\mathcal{M}_u$ and a natural number $N(T,\gamma,i)$ such that the following property holds:

P(i) For each $x \in K_{2^i}$, there is $Qx \in F$ such that

$$||S^n x - Qx|| \le 2^{-i} \text{ for all integers } n \ge N(T,\gamma,i) \text{ and all } S \in \mathcal{U}(T,\gamma,i).$$

Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \{\mathcal{U}(T,\gamma,q) : \ T \in \mathcal{M}_u, \ \gamma \in (0,1)\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_u$.

Let $B \in \mathcal{F}$, $\varepsilon > 0$ and let C be a bounded subset of K. There exists an integer $q \ge 1$ such that

$$C \subset K_{2^q} \text{ and } 2^{-q} < 4^{-1}\varepsilon. \tag{5.11}$$

There also exist $T \in \mathcal{M}_u$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(T,\gamma,q). \tag{5.12}$$

It now follows from Property P(i), (5.11) and (5.12) that the following property also holds:

P(ii) For each $x \in C$ there is $Qx \in F$ such that

$$||S^n x - Qx|| \le 4^{-1}\varepsilon$$

for each integer $n \ge N(T,\gamma,q)$ and each $S \in \mathcal{U}(T,\gamma,q)$.

Property P(ii) and (5.12) imply that for each $x \in C$ and each integer $n \geq N(T, \gamma, q)$,

$$||B^n x - Qx|| \leq 4^{-1}\varepsilon. \tag{5.13}$$

Since ε is an arbitrary positive number and C is an arbitrary bounded subset of K, we conclude that for each $x \in K$, $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore for each $x \in K$ there exists

$$P_B x = \lim_{n\to\infty} B^n x. \tag{5.14}$$

By (5.13) and (5.14), for each $x \in C$,

$$||P_B x - Qx|| \leq 4^{-1}\varepsilon. \tag{5.15}$$

Once again, since ε is an arbitrary positive number and C is an arbitrary bounded subset of K, we conclude that

$$P_B(K) = F. \tag{5.16}$$

It now follows from property P(ii) and (5.15) that for each $x \in C$, each $S \in \mathcal{U}(T, \gamma, q)$ and each integer $n \geq N(T, \gamma, q)$,

$$||S^n x - P_B x|| \leq 2^{-1}\varepsilon.$$

This completes the proof of Theorem 2.1. □

6. Proofs of Theorems 3.1 and 3.2

We begin with four lemmas.

Lemma 6.1. *Let K_0 be a nonempty bounded subset of K and β a positive number. Then the set $\{(z, y) \in F \times K_0 : D_f(z, y) \leq \beta\}$ is bounded.*

Proof. If this claim were not true, then there would exist a sequence $\{(z_i, x_i)\}_{i=1}^{\infty} \subset F \times K_0$ such that

$$D_f(z_i, x_i) \leq \beta,\ i = 1, 2, \ldots,\ \text{and } ||z_i|| \to \infty \text{ as } i \to \infty. \tag{6.1}$$

By (0.1), $D_f(z_i, Px_i) \leq \beta$, $i = 1, 2, \ldots$. Clearly, the sequence $\{Px_i\}_{i=1}^{\infty}$ is bounded. We may assume that $||z_i - Px_i|| \geq 16$, $i = 1, 2, \ldots$. For each integer $i \geq 1$ there exists $\alpha_i > 0$ such that

$$||[(1 - \alpha_i)Px_i + \alpha_i z_i] - Px_i|| = 1. \tag{6.2}$$

Clearly $\alpha_i \to 0$ as $i \to \infty$. It is easy to see that for each integer $i \geq 1$,

$$D_f((1 - \alpha_i)Px_i + \alpha_i z_i, Px_i) \leq \alpha_i D_f(z_i, Px_i) \leq \alpha_i \beta \to 0 \text{ as } i \to \infty.$$

Combined with A(i) this implies that $||Px_i - [(1 - \alpha_i)Px_i + \alpha_i z_i]|| \to 0$ as $i \to \infty$. Since this contradicts (6.2), Lemma 6.1 follows. □

Lemma 6.2. *Let $T \in \mathcal{M}$, $\gamma, \varepsilon \in (0,1)$ and let K_0 be a nonempty bounded subset of K. Then there exists a neighborhood U of T_γ in $\mathcal{M}$ such that for each $S \in U$ and each $x \in K_0$ satisfying $\rho_f(x,F) > \varepsilon$, the following inequality holds:*

$$\rho_f(Sx, F) \le \rho_f(x,F) - \varepsilon\gamma^2/4. \tag{6.3}$$

Proof. Set

$$K_1 = \bigcup\{S^i(K_0) : \ S \in \mathcal{M}, \ i \ge 0\}. \tag{6.4}$$

Assumption A(ii) and (0.1) imply that the set K_1 is bounded. Evidently, $S(K_1) \subset K_1$ for all $S \in \mathcal{M}$. By A(ii) there exists $c_0 > 0$ such that

$$4 + \sup\{D_f(\theta, x) : \ x \in K_1\} < c_0. \tag{6.5}$$

By Lemma 6.1 there exists a number $c_1 > 0$ such that

$$\text{if } (z,x) \in F \times K_1 \text{ and } D_f(z,x) \le c_0 + 2, \text{ then } ||z|| \le c_1. \tag{6.6}$$

We may assume without loss of generality that

$$c_1 > \sup\{||Px|| : \ x \in K_1\}. \tag{6.7}$$

Since $D_f(\cdot,\cdot)$ is uniformly continuous on bounded subsets of $F \times K$, there exists a number $\delta \in (0, 2^{-1})$ such that for each pair of points,

$$(z, x_1), \ (z, x_2) \in \{\xi \in F : \ ||\xi|| \le c_1\} \times K_1$$

satisfying $||x_1 - x_2|| \le \delta$, the following inequality holds:

$$|D_f(z,x_1) - D_f(z,x_2)| \le 4^{-1}\varepsilon\gamma^2. \tag{6.8}$$

Set

$$U = \{S \in \mathcal{M} : \ ||Sx - T_\gamma x|| \le \delta \text{ for all } x \in K_1\}. \tag{6.9}$$

Clearly U is a neighborhood of T_γ in $\mathcal{M}$.

Assume that

$$S \in U, \ x \in K_0 \text{ and } \rho_f(x,F) > \varepsilon. \tag{6.10}$$

We will show that (6.3) is valid. By Lemma 4.2,

$$\rho_f(T_\gamma x, F) \le (1-\gamma^2)\rho_f(x,F). \tag{6.11}$$

Let

$$\Delta \in (0, 4^{-1}\gamma^2\varepsilon). \tag{6.12}$$

There is $z \in F$ such that

$$D_f(z, T_\gamma x) \le (1-\gamma^2)\rho_f(x,F) + \Delta. \tag{6.13}$$

By (6.13), (6.12), (6.5) and (6.6),

$$D_f(z, T_\gamma x) \le c_0 \text{ and } ||z|| \le c_1. \tag{6.14}$$

By (6.9) and (6.10),

$$||T_\gamma x - Sx|| \le \delta. \tag{6.15}$$

By (6.14) and (6.4),

$$(z, T_\gamma x),\ (z, Sx) \in \{\xi \in F:\ ||\xi|| \le c_1\} \times K_1. \tag{6.16}$$

By (6.16), (6.15) and the definition of δ (see (6.8)),

$$|D_f(z, T_\gamma x) - D_f(z, Sx)| \le 4^{-1}\varepsilon\gamma^2.$$

Combined with (6.13) and (6.12) this implies that

$$\rho_f(Sx, F) \le D_f(z, Sx) \le 4^{-1}\varepsilon\gamma^2 + D_f(z, T_\gamma x) \le$$
$$4^{-1}\varepsilon\gamma^2 + (1-\gamma^2)\rho_f(x, F) + \Delta$$
$$\le (1-\gamma^2)\rho_f(x, F) + 2^{-1}\varepsilon\gamma^2.$$

Thus

$$\rho_f(Sx, F) \le (1-\gamma^2)\rho_f(x, F) + 2^{-1}\varepsilon\gamma^2.$$

The inequality (6.3) follows from this inequality and (6.10). Lemma 6.2 is proved. □

Lemma 6.3. *Let $T \in \mathcal{M}$, $\gamma, \varepsilon \in (0,1)$ and let K_0 be a nonempty bounded subset of K. Then there exist a neighborhood U of T_γ in $\mathcal{M}$ and a natural number N such that for each $S \in U$ and each $x \in K_0$,*

$$\rho_f(S^N x, F) \le \varepsilon. \tag{6.17}$$

Proof. Define the set K_1 by (6.4). Assumption A(ii) and (0.1) imply that the set K_1 is bounded. Clearly $S(K_1) \subset K_1$ for all $S \in \mathcal{M}_u$. By A(ii) there is a positive number c_0 such that (6.5) is valid. By Lemma 6.2 there exists a neighborhood U of T_γ in $\mathcal{M}$ such that for each $S \in U$ and each $x \in K_1$ satisfying $\rho_f(x, F) > \varepsilon$, the following inequality holds:

$$\rho_f(Sx, F) \le \rho_f(x, F) - \varepsilon\gamma^2/4. \tag{6.18}$$

Choose a natural number N for which

$$8^{-1}\varepsilon\gamma^2 N > c_0 + 1. \tag{6.19}$$

Assume that $S \in U$ and $x \in K_0$. We will show that inequality (6.17) is valid. If it were not, then we would have $\rho(S^i x, F) > \varepsilon$ for all $i = 0, \ldots, N$. Combined with the definition of U (see (6.18)), these inequalities imply that for all $i = 0, \ldots, N-1$,

$$\rho_f(S^{i+1} x, F) \le \rho_f(S^i x, F) - \varepsilon\gamma^2/4.$$

Therefore

$$\rho_f(S^N x, F) \leq \rho_f(x, F) - \varepsilon\gamma^2 N/4.$$

By this inequality, (6.5) and (6.19),

$$0 \leq \rho_f(S^n x, F) \leq c_0 - 4^{-1}\varepsilon\gamma^2 N \leq -1.$$

This contradiction proves (6.17) and Lemma 6.3 follows. □

Lemma 6.4. *Let $T \in \mathcal{M}$, $\gamma, \varepsilon \in (0,1)$ and let K_0 be a nonempty bounded subset of K. Then there exist a neighborhood U of T_γ in $\mathcal{M}$ and a natural number N such that for each $S \in U$ and each $x \in K_0$, there is $z(S,x) \in F$ such that*

$$||S^i x - z(S,x)|| \leq \varepsilon \text{ for all integers } i \geq N. \tag{6.20}$$

Proof. Define K_1 by (6.4). Assumption A(ii) and (0.1) imply that K_1 is bounded. By Assumption A(i) there exists $\delta \in (0,1)$ such that

$$\text{if } x \in K_1,\ z \in F \text{ and } D_f(z,x) \leq \delta, \text{ then } ||x - z|| \leq 2^{-1}\varepsilon. \tag{6.21}$$

By Lemma 6.3, there exists a neighborhood U of T_γ in $\mathcal{M}$ and a natural number N such that

$$\rho_f(S^N x, F) \leq \delta/2 \text{ for each } S \in U \text{ and } x \in K_1.$$

This implies that for each $x \in K_0$ and each $S \in U$ there is $z(S,x) \in F$ for which $D_f(z(S,x), S^N x) < \delta$. Combined with (6.21) this implies that for each $x \in K_0$, each $S \in U$, and each integer $i \geq N$,

$$D_f(z(S,x), S^i x) < \delta \text{ and } ||S^i x - z(S,x)|| \leq 2^{-1}\varepsilon.$$

Lemma 6.4 is proved. □

Proof of Theorem 3.1. By (4.2) the set $\{T_\gamma :\ T \in \mathcal{M},\ \gamma \in (0,1)\}$ is an everywhere dense subset of $\mathcal{M}$ and if $P \in \mathcal{M}_c$, then $\{T_\gamma :\ T \in \mathcal{M}_c,\ \gamma \in (0,1)\}$ is an everywhere dense subset of $\mathcal{M}_c$. For each natural number i set

$$K_i = \{x \in K :\ ||x - \theta|| \leq i\}. \tag{6.22}$$

By Lemma 6.4, for each $T \in \mathcal{M}$, each $\gamma \in (0,1)$, and each integer $i \geq 1$, there exist an open neighborhood $\mathcal{U}(T,\gamma,i)$ of T_γ in $\mathcal{M}$ and a natural number $N(T,\gamma,i)$ such that the following property holds:

P(iii) For each $x \in K_{2^i}$ and each $S \in \mathcal{U}(T,\gamma,i)$, there is $z(S,x) \in F$ such that

$$||S^n x - z(S,x)|| \leq 2^{-i} \text{ for all integers } n \geq N(T,\gamma,i).$$

Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \{\mathcal{U}(T, \gamma, q) :\ T \in \mathcal{M},\ \gamma \in (0,1)\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}$. If $P \in \mathcal{M}_c$, then we define

$$\mathcal{F}_c = [\bigcap_{q=1}^{\infty} \bigcup \{\mathcal{U}(T, \gamma, q) :\ T \in \mathcal{M}_c,\ \gamma \in (0,1)\}] \cap \mathcal{M}_c.$$

In this case $\mathcal{F}_c \subset \mathcal{F}$ and $\mathcal{F}_c$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c$.

Let $B \in \mathcal{F}$, $\varepsilon > 0$, and let C be a bounded subset of K. There exists an integer $q \geq 1$ such that

$$C \subset K_{2^q} \text{ and } 2^{-q} < 4^{-1}\varepsilon. \tag{6.23}$$

There also exist $T \in \mathcal{M}$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(T, \gamma, q). \tag{6.24}$$

Note that if $P \in \mathcal{M}_c$ and $B \in \mathcal{F}_c$, then $T \in \mathcal{M}_c$.

It follows from Property P(iii), (6.23) and (6.24) that the following property also holds:

P(iv) For each $S \in \mathcal{U}(T, \gamma, q)$ and each $x \in C$, there is $z(S, x) \in F$ such that $||S^n x - z(S, x)|| \leq 4^{-1}\varepsilon$ for each integer $n \geq N(T, \gamma, q)$.

The relation (6.24) and property P(iv) imply that for each $x \in C$ and each integer $n \geq N(T, \gamma, q)$,

$$||B^n x - z(B, x)|| \leq 4^{-1}\varepsilon. \tag{6.25}$$

Since ε is an arbitrary positive number and C is an arbitrary bounded subset of K, we conclude that for each $x \in K$, $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence. Therefore for each $x \in K$ there exists

$$P_B x = \lim_{n \to \infty} B^n x.$$

Now (6.25) implies that for each $x \in C$,

$$||P_B x - z(B, x)|| \leq 4^{-1}\varepsilon. \tag{6.26}$$

Once again, since ε is an arbitrary positive number and C is an arbitrary bounded subset of K, we conclude that

$$P_B(K) = F.$$

It follows from (6.25) and (6.26) that for each $x \in C$ and each integer $n \geq N(T, \gamma, q)$,

$$||B^n x - P_B x|| \leq 2^{-1}\varepsilon.$$

This implies that $P_B \in \mathcal{M}$ and if $B \in \mathcal{M}_c$, then $P_B \in \mathcal{M}_c$. Theorem 3.1 is established. $\square$

We will use the next lemma in the proof of Theorem 3.2.

Lemma 6.5. *Let $B \in \mathcal{M}_c$, $x \in K$, $\varepsilon \in (0,1)$ and let $N \geq 1$ be an integer. Then there exist a neighborhood U of B in $\mathcal{M}$ and a number $\delta > 0$ such that for each $S \in U$ and each $y \in K$ satisfying $||y - x|| \leq \delta$, the following inequality holds:*

$$||S^n y - B^n x|| \leq \varepsilon.$$

This lemma is proved by induction on n.

Proof of Theorem 3.2. By Theorem 3.1, there exist a natural number N and a neighborhood U_0 of B in $\mathcal{M}$ such that

$$||P_B y - B^n y|| \leq 8^{-1}\varepsilon \text{ for each } y \in K \text{ satisfying } ||y - x|| \leq 1 \text{ and each } n \geq N; \tag{6.27}$$

and for each $S \in U_0$ and each $y \in K$ satisfying $||y - x|| \leq 1$, there is $z(S, y) \in F$ such that

$$||S^n y - z(S, y)|| \leq 8^{-1}\varepsilon \text{ for all integers } n \geq N. \tag{6.28}$$

By Lemma 6.5, there exist a number $\delta \in (0,1)$ and a neighborhood U of B in $\mathcal{M}$ such that $U \subset U_0$ and

$$||S^N y - B^N x|| \leq 8^{-1}\varepsilon \text{ for each } S \in U \text{ and each } y \in K \text{ for which } ||y - x|| \leq \delta. \tag{6.29}$$

Assume that

$$y \in K,\ ||x - y|| \leq \delta \text{ and } S \in U. \tag{6.30}$$

By (6.30), (6.29) and (6.27),

$$||S^N y - B^N x|| \leq 8^{-1}\varepsilon,\ ||S^N y - z(S, y)|| \leq 8^{-1}\varepsilon \text{ and } ||P_B x - B^N x|| \leq 8^{-1}\varepsilon.$$

These inequalities imply that

$$||z(S, y) - P_B x|| \leq 3 \cdot 8^{-1}\varepsilon.$$

Combined with (6.28) the last inequality implies that

$$||S^n y - P_B x|| \leq 2^{-1}\varepsilon \text{ for all integers } n \geq N.$$

This completes the proof of Theorem 3.2. □

7. Convergence of powers for a class of continuous operators

In this section we assume that $P \in \mathcal{M}_c$ and that the function

$$D_f(z, \cdot) : K \to \mathbb{R}^1 \text{ is continuous for all } z \in F. \tag{7.1}$$

Theorem 7.1. *Let $x \in K$. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c$ such that for each $B \in \mathcal{F}$ the following assertions hold:*

1. *There exists $\lim_{n\to\infty} B^n x \in F$.*
2. *For each $\varepsilon > 0$ there exist a neighborhood U of B in $\mathcal{M}_c$, a natural number N and a number $\delta > 0$ such that for each $S \in U$, each $y \in K$ satisfying $||y - x|| \le \delta$ and each integer $n \ge N$, $||S^n y - \lim_{i\to\infty} B^i x|| \le \varepsilon$.*

We equip the space $K \times \mathcal{M}_c$ with the product topology.

Theorem 7.2. *There exists a set $\mathcal{F} \subset K \times \mathcal{M}_c$ which is a countable intersection of open everywhere dense subsets of $K \times \mathcal{M}_c$ such that for each $(z, B) \in \mathcal{F}$ the following assertions hold:*

1. *There exists $\lim_{n\to\infty} B^n z \in F$.*
2. *For each $\varepsilon > 0$ there exist a neighborhood U of (z, B) in $K \times \mathcal{M}_c$ and a natural number N such that for each $(y, S) \in U$ and each integer $n \ge N$,*

$$||S^n y - \lim_{i\to\infty} B^i z|| \le \varepsilon.$$

Theorem 7.3. *Assume that the set K_0 is a nonempty separable closed subset of K. Then there exists a set $\mathcal{F} \subset \mathcal{M}_c$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c$ such that for each $T \in \mathcal{F}$ there exists a set $\mathcal{K}_T \subset K_0$ which is a countable intersection of open everywhere dense subsets of K_0 with the relative topology such that the following assertions hold:*

1. *For each $x \in \mathcal{K}_T$ there exists $\lim_{n\to\infty} T^n x \in F$.*
2. *For each $x \in \mathcal{K}_T$ and each $\varepsilon > 0$, there exist an integer $N \ge 1$ and a neighborhood U of (x, T) in $\mathcal{K} \times \mathcal{M}_c$ such that for each $(y, S) \in U$ and each integer $i \ge N$, $||S^i y - \lim_{n\to\infty} T^n x|| \le \varepsilon$.*

8. Proofs of Theorems 7.1–7.3

We precede the proofs of Theorems 7.1 and 7.2 by the following lemma.

Lemma 8.1. *Let $T \in \mathcal{M}_c$, $\gamma, \varepsilon \in (0,1)$ and let $x \in K$. Then there exist a neighborhood U of T_γ in $\mathcal{M}_c$, a natural number N, a point $\widehat{z} \in F$ and a number $\delta > 0$ such that for each $S \in U$, each $y \in K$ satisfying $||y - x|| \leq \delta$ and each integer $n \geq N$,*

$$||S^n y - \widehat{z}|| \leq \varepsilon. \tag{8.1}$$

Proof. Define

$$K_1 = \bigcup \{S^i(\{y \in K : \; ||y - x|| \leq 1\}) : \; S \in \mathcal{M}, \; i = 0, 1, \dots \}. \tag{8.2}$$

By A(ii) and (0.1), the set K_1 is bounded. By A(i) there is $\varepsilon_0 \in (0, \varepsilon/2)$ such that

$$\text{if } z \in F, \; y \in K_1 \text{ and } D_f(z, y) \leq 2\varepsilon_0, \text{ then } ||z - y|| \leq \varepsilon/2. \tag{8.3}$$

Choose a natural number N for which

$$(1 - \gamma^2)^N (\rho_f(x, F) + 1) < \varepsilon_0/8. \tag{8.4}$$

By Lemma 4.2 this implies that

$$\rho_f(T_\gamma^N x, F) \leq (1 - \gamma^2)^N \rho_f(x, F) < \varepsilon_0/8.$$

Therefore there exists $\widehat{z} \in F$ for which

$$D_f(\widehat{z}, T_\gamma^N x) < \varepsilon_0/8. \tag{8.5}$$

Since the function $D_f(\widehat{z}, \cdot) : K \to \mathbb{R}^1$ is continuous (see (7.1)), there exists $\varepsilon_1 \in (0, \varepsilon_0/2)$ such that

$$D_f(\widehat{z}, \xi) < \varepsilon_0/8 \text{ for all } \xi \in K \text{ satisfying } ||\xi - T_\gamma^N x|| \leq \varepsilon_1. \tag{8.6}$$

It follows from the continuity of T_γ that there exist a neighborhood U of T_γ in $\mathcal{M}_c$ and a number $\delta \in (0,1)$ such that for each $S \in U$ and each $y \in K$ satisfying $||y - x|| \leq \delta$,

$$||S^N y - T_\gamma^N x|| \leq \varepsilon_1 \tag{8.7}$$

(see Lemma 6.5).

Assume that

$$S \in U, \; y \in K, \text{ and } ||y - x|| \leq \delta.$$

By the definition of U and δ, the inequality (8.7) is valid. By (8.7) and (8.6), $D_f(\widehat{z}, S^N y) < \varepsilon_0/8$. This implies that $D_f(\widehat{z}, S^n y) < \varepsilon_0/8$ for all integers $n \geq N$. When combined with (8.3) this implies that $||\widehat{z} - S^n y|| \leq \varepsilon$ for all integers $n \geq N$. Lemma 8.1 is proved. $\square$

Proof of Theorem 7.1. Let $x \in K$. By Lemma 8.1, for each $T \in \mathcal{M}_c$, each $\gamma \in (0,1)$ and each integer $i \geq 1$, there exist an open neighborhood $\mathcal{U}(T,\gamma,i)$ of T_γ in $\mathcal{M}_c$, a natural number $N(T,\gamma,i)$, a point $z(T,\gamma,i) \in F$ and a number $\delta(T,\gamma,i) > 0$ such that the following property holds:

P(v) For each $S \in \mathcal{U}(T,\gamma,i)$, each $y \in K$ satisfying $||x-y|| \leq \delta(T,\gamma,i)$ and each integer $n \geq N(T,\gamma,i)$,

$$||S^n y - z(T,\gamma,i)|| \leq 2^{-i}.$$

Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \{\mathcal{U}(T,\gamma,q) :\ T \in \mathcal{M}_c,\ \gamma \in (0,1)\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c$.

Let $B \in \mathcal{F}$ and $\varepsilon > 0$. There exists an integer $q \geq 1$ such that

$$2^{-q} < 4^{-1}\varepsilon. \tag{8.8}$$

There also exist $T \in \mathcal{M}_c$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(T,\gamma,q). \tag{8.9}$$

It follows from property P(v) and (8.8) that the following property also holds:

P(vi) For each $S \in \mathcal{U}(T,\gamma,q)$, each $y \in K$ satisfying $||y-x|| \leq \delta(T,\gamma,q)$ and each integer $n \geq N(T,\gamma,q)$,

$$||S^n y - z(T,\gamma,q)|| \leq 4^{-1}\varepsilon. \tag{8.10}$$

Since ε is an arbitrary positive number, we conclude that $\{B^n x\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\lim_{n\to\infty} B^n x$. The inequality (8.10) implies that

$$|| \lim_{n\to\infty} B^n x - z(T,\gamma,q)|| \leq 4^{-1}\varepsilon.$$

Since ε is an arbitrary positive number, we conclude that $\lim_{n\to\infty} B^n x$ belongs to F. It follows from this inequality and property P(vi) that for each $S \in \mathcal{U}(T,\gamma,q)$, each $y \in K$ satisfying $||y-x|| \leq \delta(T,\gamma,q)$, and each integer $n \geq N(T,\gamma,q)$,

$$||S^n y - \lim_{i\to\infty} B^i x|| \leq 2^{-1}\varepsilon.$$

Theorem 7.1 is proved. □

Proof of Theorem 7.2. By Lemma 8.1, for each $(x,T) \in K \times \mathcal{M}_c$, each $\gamma \in (0,1)$, and each integer $i \geq 1$, there exist an open neighborhood $\mathcal{U}(x,T,\gamma,i)$ of (x,T_γ) in $K \times \mathcal{M}_c$, a natural number $N(x,T,\gamma,i)$ and a point $z(x,T,\gamma,i) \in F$ such that the following property holds:

P(vii) For each $(y, S) \in \mathcal{U}(x, T, \gamma, i)$ and each integer $n \geq N(x, T, \gamma, i)$,

$$||S^n y - z(x, T, \gamma, i)|| \leq 2^{-i}.$$

Define

$$\mathcal{F} = \bigcap_{q=1}^{\infty} \bigcup \{\mathcal{U}(x, T, \gamma, q) : \ (x, T) \in K \times \mathcal{M}_c, \ \gamma \in (0, 1)\}.$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $K \times \mathcal{M}_c$.

Let $(z, B) \in \mathcal{F}$ and $\varepsilon > 0$. There exists an integer $q \geq 1$ such that

$$2^{-q} < 4^{-1}\varepsilon. \tag{8.11}$$

There exist $x \in K$, $T \in \mathcal{M}_c$, and $\gamma \in (0, 1)$ such that

$$(z, B) \in \mathcal{U}(x, T, \gamma, q). \tag{8.12}$$

By (8.11) and property P(vii), the following property also holds:

P(viii) For each $(y, S) \in \mathcal{U}(x, T, \gamma, q)$ and each integer $n \geq N(x, T, \gamma, q)$,

$$||S^n y - z(x, T, \gamma, q)|| \leq 4^{-1}\varepsilon. \tag{8.13}$$

Since ε is an arbitrary positive number we conclude that $\{B^n z\}_{n=1}^{\infty}$ is a Cauchy sequence and there exists $\lim_{n\to\infty} B^n z$. Property P(viii) and (8.12) now imply that

$$|| \lim_{n\to\infty} B^n z - z(x, T, \gamma, q)|| \leq 4^{-1}\varepsilon. \tag{8.14}$$

Since ε is an arbitrary positive number, we conclude that $\lim_{n\to\infty} B^n z \in F$. It follows from (8.14) and property P(viii) that for each $(y, S) \in \mathcal{U}(x, T, \gamma, q)$ and each integer $n \geq N(x, T, \gamma, q)$,

$$||S^n y - \lim_{i\to\infty} B^i z|| \leq 2^{-1}\varepsilon.$$

This completes the proof of Theorem 7.2. □

Proof of Theorem 7.3. Assume that K_0 is a nonempty closed separable subset of K. Let $\{x_j\}_{j=1}^{\infty} \subset K_0$ be a sequence such that K_0 is the closure of $\{x_j\}_{j=1}^{\infty}$. For each integer $p \geq 1$, there exists by Theorem 7.1 a set $\mathcal{F}_p \subset \mathcal{M}_c$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c$ such that for each $T \in \mathcal{F}_p$ the following properties hold:

C(i) There exists $\lim_{n\to\infty} T^n x_p \in F$.

C(ii) For each $\varepsilon > 0$, there exist a neighborhood U of T in $\mathcal{M}_c$, a number $\delta > 0$ and a natural number N such that for each $S \in U$, each $y \in K$ satisfying $||y - x_p|| \leq \delta$ and each integer $m \geq N$,

$$||S^m y - \lim_{n\to\infty} T^n x_p|| \leq \varepsilon.$$

Set

$$\mathcal{F} = \bigcap_{p=1}^{\infty} \mathcal{F}_p. \tag{8.15}$$

Clearly $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{M}_c$.

Assume that $T \in \mathcal{F}$. Then for each $p \geq 1$ there exists $\lim_{n\to\infty} T^n x_p \in F$.

Now we will construct the set $\mathcal{K}_T \subset K_0$. By property C(ii), for each pair of natural numbers q, i there exist a neighborhood $\mathcal{U}(q,i)$ of T in $\mathcal{M}_c$, a number $\delta(q,i) > 0$ and a natural number $N(q,i)$ such that the following property holds:

C(iii) For each $S \in \mathcal{U}(q,i)$, each $y \in K$ satisfying $||y - x_q|| \leq \delta(q,i)$, and each integer $m \geq N(q,i)$,

$$||S^m y - \lim_{n\to\infty} T^n x_q|| \leq 2^{-i}.$$

Define

$$\mathcal{K}_T = \bigcap_{n=1}^{\infty} \bigcup \{\{y \in K_0 : \ ||y - x_q|| < \delta(q,i)\} : \ q \geq 1, \ i \geq n\}. \tag{8.16}$$

Clearly $\mathcal{K}_T$ is a countable intersection of open everywhere dense subsets of K_0.

Assume that $x \in \mathcal{K}_T$ and $\varepsilon > 0$. There exists an integer $n \geq 1$ such that

$$2^{-n} < 4^{-1}\varepsilon. \tag{8.17}$$

By (8.16) there exist a natural number q and an integer $i \geq n$ such that

$$||x - x_q|| < \delta(q,i). \tag{8.18}$$

It follows from (8.17) and C(iii) that the following property also holds:

C(iv) For each $S \in \mathcal{U}(q,i)$, each $y \in K$ satisfying $||y - x_q|| \leq \delta(q,i)$, and each integer $m \geq N(q,i)$,

$$||S^m y - \lim_{j\to\infty} T^j x_q|| \leq 4^{-1}\varepsilon.$$

By property C(iv) and (8.18),

$$||T^m x - \lim_{j\to\infty} T^j x_q|| \leq 4^{-1}\varepsilon$$

for all integers $m \geq N(q,i)$. Since ε is an arbitrary positive number, we conclude that $\{T^m x\}_{m=1}^{\infty}$ is a Cauchy sequence and there exists $\lim_{m\to\infty} T^m x$. We also have

$$||\lim_{m\to\infty} T^m x - \lim_{m\to\infty} T^m x_q|| \leq 4^{-1}\varepsilon. \tag{8.19}$$

Since $\lim_{m\to\infty} T^m x_q \in F$, we conclude that $\lim_{m\to\infty} T^m x$ also belongs to F. By (8.19) and property C(iv), for each $S \in \mathcal{U}(q,i)$, each $y \in K$ satisfying $||y - x|| < \delta(q,i) - ||x - x_q||$, and each integer $m \geq N(q,i)$, we have

$$||S^m y - \lim_{j\to\infty} T^j x|| \leq 2^{-1}\varepsilon.$$

Theorem 7.3 is proved. □

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