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ON SETS DETERMINED BY SEQUENCES OF QUASI-CONTINUOUS FUNCTIONS

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Abstract. The aim of the paper is to characterize those sets of points at which sequence of real functions from a given class $\mathcal{F}$ converges as well as sets of points of convergence to infinity of such sequences. As $\mathcal{F}$ we consider quasi-continuous functions and some other subclasses of Baire measurable functions.

The investigation of some sets determined by sequences of functions is motivated by the well-known result due to Hahn and also Sierpiński [9] stating that a subset A of a Polish space X is of type $\mathcal{F}_{\sigma\delta}$ iff there exists a sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ of continuous functions convergent exactly at each point of A (see also [2, Theorem 23.18, p. 185]). It may be interesting to find the analogous characterization of sets of convergence points for sequences of functions from some other classes. In [10] the sequences of functions of Baire class α, derivatives and approximately continuous functions have been examined.

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Moreover, we can consider two other sets determined by a sequence of functions, i. e. sets of points of convergence to plus or minus infinity, investigated by Lipiński [4] for sequences of continuous functions. The problem we deal with in this paper is to find necessary and sufficient conditions on three pairwise disjoint sets to be the set of convergence points, the set of points of convergence to plus infinity and the set of points of convergence to minus infinity, respectively, for some sequence of functions from a given class. We investigate quasi-continuous functions (Theorem 1) as well as some other subclasses of Baire measurable functions, e.g. cliquish, pointwise discontinuous and simply continuous functions (Theorem 2 and Remark 2).

Let us establish some notations. For a subset A of a topological space X we denote by $\mathrm{int}\,(A)$, $\mathrm{cl}\,(A)$ and $\mathrm{fr}\,(A)$ the interior, closure and boundary of A, respectively. For a metric space X, $x \in X$ and $\varepsilon > 0$ let $B(x,\varepsilon)$ denote an open ball centred at x with the radius ε. Then $B(A,\varepsilon) = \bigcup_{x\in A} B(x,\varepsilon)$.

Throughout this paper the following abbreviations for some classes of subsets of a topological space X are used:

$\mathcal{SO}(X)$ — the family of all semi-open subsets of X ($\mathcal{SO}(X)$ consists of sets satisfying $A \subset \mathrm{cl}\,(\mathrm{int}\ A)$);

$\mathcal{B}aire(X)$ — the collection of subsets with the Baire property;

$\mathcal{M}(X)$ — the σ-ideal of meager (first-category) subsets of X.

For $f\colon X \to \mathbb{R}$ let $C(f)$ be the set of all continuity points of f. Then $D(f) = X \setminus C(f)$.

A function $f\colon X \to \mathbb{R}$ is said to be *quasi-continuous* iff for every $p \in X$ and open sets $U \subset X$, $W \subset \mathbb{R}$ such that $p \in U$ and $f(p) \in W$ there is a non-empty open set $G \subset U$ such that $f(G) \subset W$ (or, equivalently, $f^{-1}(V) \in \mathcal{SO}(X)$ for any open $V \subset \mathbb{R}$, see e.g. [7, Theorem 1.1]).

Let $\mathcal{QC} \subset \mathbb{R}^X$ be the class of all quasi-continuous functions. Denote also by $\mathcal{B}$ the class of Baire measurable real functions defined on X (i.e. $f \in \mathcal{B} \subset \mathbb{R}^X$ iff $f^{-1}(V)$ has the Baire property for any open $V \subset \mathbb{R}$).

For a sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ we consider the following sets:

(1) $L(\{f_n\colon n\in\mathbb{N}\}) = \{x \in X\colon (f_n(x))_n \text{ converges}\}$,
(2) $L_{+\infty}(\{f_n\colon n\in\mathbb{N}\}) = \{x\in X\colon \lim_n f_n(x) = +\infty\}$,
(3) $L_{-\infty}(\{f_n\colon n\in\mathbb{N}\}) = \{x\in X\colon \lim_n f_n(x) = -\infty\}$.

Remark 1. For any sequence $\{f_n\colon n\in\mathbb{N}\} \subset \mathbb{R}^X$

- $L(\{f_n\colon n\in\mathbb{N}\}) = \bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\bigcap_{k\in\mathbb{N}}\{x\in X\colon |f_{n+k}(x) - f_n(x)| \le 1/m\}$;
- $L_{+\infty}(\{f_n\colon n\in\mathbb{N}\}) = \bigcap_{m\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}\bigcap_{n\ge k}\{x\in X\colon f_n(x)\ge m\}$;
- $L_{-\infty}(\{f_n\colon n\in\mathbb{N}\}) = \bigcap_{m\in\mathbb{N}}\bigcup_{k\in\mathbb{N}}\bigcap_{n\ge k}\{x\in X\colon f_n(x)\le -m\}$.

By Remark 1 it is easy to see that in the case of continuous real functions $\{f_n : n \in \mathbb{N}\}$ defined on a metric space X both $L_{+\infty}(\{f_n : n \in \mathbb{N}\})$ and $L_{-\infty}(\{f_n : n \in \mathbb{N}\})$ are $\mathcal{F}_{\sigma\delta}$ sets in X. But then $L_{+\infty}(\{f_n : n \in \mathbb{N}\}) \subset \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k}\{x \in X : f_n(x) \geq 1\}$ and $L_{-\infty}(\{f_n : n \in \mathbb{N}\}) \subset \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k}\{x \in X : f_n(x) \leq -1\}$, so $L_{+\infty}(\{f_n : n \in \mathbb{N}\})$ and $L_{-\infty}(\{f_n : n \in \mathbb{N}\})$ are separated by two disjoint $\mathcal{F}_\sigma$ sets in X. The theorem proved by Lipiński (see [4]) for sequences of continuous real functions defined on $\mathbb{R}$ states that the above necessary condition is also sufficient, i.e.

> for any $\mathcal{F}_{\sigma\delta}$ sets $L_1, L_2 \subset \mathbb{R}$ separated by two disjoint $\mathcal{F}_\sigma$ sets there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^\mathbb{R}$ of continuous functions such that $L_1 = L_{+\infty}(\{f_n : n \in \mathbb{N}\})$ and $L_2 = L_{-\infty}(\{f_n : n \in \mathbb{N}\})$.

A similar characterization of triple (1), (2) and (3) has been achieved by Lunina (see [5]) for sequences of continuous real functions on a metric space.

First, we will show some necessary conditions on sets to be of the form (1), (2) or (3) for a sequence of quasi-continuous functions.

Lemma 1. *For any sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of quasi-continuous functions $L_+ = L_{+\infty}(\{f_n : n \in \mathbb{N}\}) \in \mathcal{B}aire(X)$. Moreover, if X is a Baire space and*

(4) $L_+ = (G_+ \setminus P_+) \cup Q_+$, *where G_+ is a regular open set, $P_+, Q_+ \in \mathcal{M}$, $P_+ \subset G_+$ and $Q_+ \cap G_+ = \emptyset$,*

then $B_M = \{x \in G_+ : \underline{\lim}_n f_n(x) < M\}$ is nowhere-dense for every $M > 0$.

Proof. Since $\mathcal{SO}(X) \subset \mathcal{B}aire(X)$, any quasi-continuous function has the Baire property, so the first statement of Lemma 1 is obvious by Remark 1. Let L_+ be as in (4). Fix $M > 0$, an open set $U \subset G_+$ and $x \in U \setminus P_+$. Then $x \in L_{+\infty}(\{f_n : n \in \mathbb{N}\})$, so there is a positive integer n_x such that for every $m > n_x$ we have $f_m(x) \geq M$. Consequently, since X is a Baire space, there is $n \in \mathbb{N}$ such that $A_n = \{x \in U : n_x = n\}$ is of second category. Hence we can find a non-empty open set $V \subset U$ such that A_n is dense in V. We will show that $V \cap B_M = \emptyset$. Fix $x \in V$ and $m > n$. Then $f_m(x) \geq M$, because otherwise, by the quasi-continuity of f_m at x, there is a non-empty open set $G \subset V$ such that $f_m(t) < M$ for any $t \in G$. Thus $G \cap A_n = \emptyset$, which is impossible because A_n is dense in V. Therefore $\underline{\lim}_n f_n(x) \geq M$ for any $x \in V$, which finishes the proof. □

Similar arguments apply to the next two lemmas.

Lemma 2. *For any sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of quasi-continuous functions $L_- = L_{-\infty}(\{f_n : n \in \mathbb{N}\}) \in \mathcal{B}aire(X)$. Moreover, if X is a Baire space and*

(5) $L_- = (G_- \setminus P_-) \cup Q_-$, *where G_- is a regular open set, $P_-, Q_- \in \mathcal{M}$, $P_- \subset G_-$ and $Q_- \cap G_- = \emptyset$,* *then* $D_M = \{x \in G_- \colon \overline{\lim}_n f_n(x) > M\}$ *is nowhere-dense for every* $M < 0$.

Lemma 3. *For any sequence* $\{f_n \colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ *of quasi-continuous functions* $L_0 = L(\{f_n \colon n \in \mathbb{N}\}) \in \mathcal{B}aire(X)$. *Moreover, if* X *is a Baire space and*

(**6**) $L_0 = (G_0 \setminus P_0) \cup Q_0$, *where* G_0 *is a regular open set,* $P_0, Q_0 \in \mathcal{M}$, $P_0 \subset G_0$ *and* $Q_0 \cap G_0 = \emptyset$,

then $G_0 \cap L_{+\infty}(\{f_n \colon n \in \mathbb{N}\})$ *and* $G_0 \cap L_{-\infty}(\{f_n \colon n \in \mathbb{N}\})$ *are nowhere-dense.*

Moreover, the following modification of [6, Lemma 1] will be useful.

Lemma 4. *Let* (X, d) *be a dense in itself separable metric space. If for fixed* $k \in \mathbb{N}$ *the sets* F_0, F_1, ...F_k *are nowhere dense closed subsets of* $\mathrm{cl}\,(U)$, *where* U *is an open subset of* X, *with* $F_0 \subset F_1 \subset ... \subset F_k$, *then for every* $\varepsilon > 0$ *we can choose a collection* $\{U_{k,i,j} \colon i \le k, j \le 3\}$ *of semi-open subsets of* X *such that*

1. $U_{k,i,j} \cap U_{k,l,m} = \emptyset$ *for* $(k, i, j) \ne (k, l, m)$,
2. $U_{k,i,j} \subset U \cap B(F_i, \varepsilon) \setminus F_k$ *for every* $i \le k$ *and* $j \le 3$,
3. $F_i \subset \mathrm{cl}\,(U_{k,i,j})$ *for every* $i \le k$ *and* $j \le 3$,
4. $U \setminus \bigcup_{i \le k} \bigcup_{j \le 3} U_{k,i,j}$ *is semi-open.*

Proof. We will prove the lemma for $k = 1$, just for the sake of simplicity. Denote by $\mathcal{O} = \{B_n \colon n \in \mathbb{N}\}$ an open basis of X. For $n \in \mathbb{N}$ let $W_n = B(F_1, \varepsilon/2^n)$. Then $F_1 = \bigcap_{n \in \mathbb{N}} W_n$. Let $\{G_n \colon n \in \mathbb{N}\} = \{G \in \mathcal{O} \colon G \cap F_1 \ne \emptyset\}$. For each $n \in \mathbb{N}$ we choose inductively a non-empty open set K_n with $\mathrm{cl}\,(K_n) \subset U \cap W_n \cap G_n \setminus (F_1 \cup \bigcup_{i<n} \mathrm{cl}\,(K_i))$. We have

(i) $\mathrm{cl}\,(K_n) \cap F_1 = \emptyset$ for all $n \in \mathbb{N}$ and $\mathrm{cl}\,(K_n) \cap \mathrm{cl}\,(K_m) = \emptyset$ for $n \ne m$,

(ii) for every $x \in F_1$ and its open neighbourhood V there exists an $n \in \mathbb{N}$ such that $\mathrm{cl}\,(K_n) \subset V$,

(iii) for every $x \notin F_1$ there exists an open neighbourhood V of x such that the set $\{n \in \mathbb{N} \colon \mathrm{cl}\,(K_n) \cap V \ne \emptyset\}$ has at most one element.

We see at once that (i) and (ii) hold, so we will verify (iii). Fix an $x \notin F_1$. Then there exist an $n_0 \in \mathbb{N}$ and an open V_0 with $x \in V_0$ such that $V_0 \cap W_{n_0} = \emptyset$. Clearly, $\mathrm{cl}\,(K_n) \cap V_0 = \emptyset$ for all $n \ge n_0$. Thus it is enough to take $V = V_0 \setminus \bigcup_{n<n_0} \mathrm{cl}\,(K_n)$ if $x \notin \bigcup_{n<n_0} \mathrm{cl}\,(K_n)$ or $V = V_0 \setminus \bigcup_{n<n_0, n \ne m} \mathrm{cl}\,(K_n)$ if $x \in \mathrm{cl}\,(K_m)$ for some unique $m < n_0$.

Fix an $n \in \mathbb{N}$ and choose five non-empty open sets $\{K_{n,j} \colon j = 0, 1, ..., 4\}$ such that

(iv) $\mathrm{cl}\,(K_{n,j}) \subset K_n$ and $\mathrm{cl}\,(K_{n,j}) \cap \mathrm{cl}\,(K_{n,m}) = \emptyset$ for $j \ne m$.

For each $j \le 4$ define the semi-open set $U_{1,j} = \bigcup_{n \in \mathbb{N}} \mathrm{cl}\,(K_{n,j})$. It follows that

1. $U_{1,j} \cap U_{1,m} = \emptyset$ for $j \neq m$,
2. $U_{1,j} \subset U \cap B(F_1, \varepsilon) \setminus F_1$ for every $j \leq 4$,
3. $F_1 \subset \mathrm{cl}\,(U_{1,j})$ for every $j \leq 4$,
4. $Z = U \setminus \bigcup_{j \leq 3} U_{1,j}$ is semi-open.

Of course, 1-2 hold and 3 is a consequence of (iii), so it suffices to verify 4. Fix an $x \in Z$. We will show that $x \in \mathrm{cl}\,(\mathrm{int}\ Z)$. First, take an $x \in F_1 \cap U$ and an arbitrary neighbourhood V of x. By (ii) and (iv) there is an $n \in \mathbb{N}$ such that $\mathrm{cl}\,(K_{n,4}) \subset V$. Since $\mathrm{cl}\,(K_{n,4}) \subset U_{1,4} \subset Z$, we have $V \cap \mathrm{int}\,(Z) \neq \emptyset$, so $x \in \mathrm{cl}\,(\mathrm{int}\ Z)$. By (iii) and (iv), for $x \in Z \setminus F_1$ there exists an open neighbourhood V of x such that the set $A = \{(n,j)\colon\ \mathrm{cl}\,(K_{n,j}) \cap V \neq \emptyset\}$ has at most one element. If $A = \emptyset$, then $x \in U \cap V \subset Z$, so $x \in \mathrm{cl}\,(\mathrm{int}\ Z)$. If $A = \{(n_0, j_0)\}$, we have two cases.

- If $x \in \mathrm{cl}\,(K_{n_0,j_0})$, then of course $j_0 = 4$ and $\mathrm{cl}\,(K_{n_0,j_0}) \subset Z$. Thus $U \cap V \subset Z$, so $x \in \mathrm{cl}\,(\mathrm{int}\ Z)$.
- If $x \notin \mathrm{cl}\,(K_{n_0,j_0})$, then $x \in V_0 = U \cap V \setminus \mathrm{cl}\,(K_{n_0,j_0})$. Since V_0 is a non-empty open subset of Z, we have $x \in \mathrm{cl}\,(\mathrm{int}\ Z)$.

Now, take a semi-open set $Z = U \setminus \bigcup_{j \leq 3} U_{1,j}$. For $n \in \mathbb{N}$ let $S_n = B(F_0, \varepsilon/2^n) \cap Z$. Then S_n is a non-empty semi-open subset of X. Let $\{O_n\colon n \in \mathbb{N}\} = \{O \in \mathcal{O}\colon O \cap F_0 \neq \emptyset\}$. For each $n \in \mathbb{N}$ we choose non-empty open subsets L_n of Z with $\mathrm{cl}\,(L_n) \subset S_n \cap O_n \setminus (F_1 \cup \bigcup_{i<n} \mathrm{cl}\,(L_i))$, which is possible because $S_n \cap O_n \setminus (F_1 \cup \bigcup_{i<n} \mathrm{cl}\,(L_i))$ is a non-empty semi-open set. We have

(v) $\mathrm{cl}\,(L_n) \cap F_1 = \emptyset$ for all $n \in \mathbb{N}$ and $\mathrm{cl}\,(L_n) \cap \mathrm{cl}\,(L_m) = \emptyset$ for $n \neq m$,

(vi) for every $x \in F_0$ and its open neighbourhood V there exists $n \in \mathbb{N}$ such that $\mathrm{cl}\,(L_n) \subset V$,

(vii) for every $x \notin F_1$ there exists an open neighbourhood V of x such that the set $\{n \in \mathbb{N}\colon\ \mathrm{cl}\,(L_n) \cap V \neq \emptyset\}$ has at most one element.

Obviously, (v) and (vi) hold. The proof of (vii) is similar to that of (iii).

Fix an $n \in \mathbb{N}$ and choose five non-empty open sets $\{L_{n,j}\colon j = 0, 1, ..., 4\}$ such that

(viii) $\mathrm{cl}\,(L_{n,j}) \subset L_n$ and $\mathrm{cl}\,(L_{n,j}) \cap \mathrm{cl}\,(L_{n,i}) = \emptyset$ for $j \neq i$.

For each $j \leq 4$ define a semi-open set $U_{0,j} = \bigcup_{n \in \mathbb{N}} \mathrm{cl}\,(L_{n,j})$. We apply similar arguments to those in the proof of 1-4 to obtain

5. $U_{0,j} \cap U_{0,m} = \emptyset$ for $j \neq m$ and $U_{0,j} \subset U \setminus \bigcup_{m \leq 3} U_{1,m}$ for every $j \leq 4$,
6. $U_{0,j} \subset U \cap B(F_0, \varepsilon) \setminus F_1$ for every $j \leq 4$,
7. $F_0 \subset \mathrm{cl}\,(U_{0,j})$ for every $j \leq 4$,
8. $Z_0 = U \setminus \bigcup_{i \leq 1} \bigcup_{j \leq 3} U_{i,j}$ is semi-open.

Putting $U_{1,i,j} = U_{i,j}$ for $i = 0, 1$ and $j = 0, 1, 2, 3$ we get a required collection of semi-open sets. □

Theorem 1. *Let (X, d) be a dense in itself separable metric space. Denote by X_1 the union of all first category open subsets of X. Let $X_2 = X \setminus \mathrm{cl}(X_1)$. For every pairwise disjoint sets $L_0, L_+, L_- \subset X$ the following conditions are equivalent:*

(i) *$L_+, L_-, L_0 \in \mathcal{B}aire(X)$ and for any $s \in \{0, +, -\}$ we have $L_s \cap X_2 = (G_s \setminus P_s) \cup Q_s$, where $G_s, P_s, Q_s \subset X_2$ and*
  (1) *G_s is a regular open set;*
  (2) *$P_s, Q_s \in \mathcal{M}(X)$;*
  (3) *$P_s \subset G_s$ and $Q_s \cap G_s = \emptyset$;*
  (4) *$P_+ \cap Q_-$, $P_- \cap Q_+$, $P_0 \cap Q_-$, $P_0 \cap Q_+$ are nowhere-dense;*

(ii) *there exists a sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of quasi-continuous functions such that $L_+ = L_{+\infty}(\{f_n : n \in \mathbb{N}\})$, $L_- = L_{-\infty}(\{f_n : n \in \mathbb{N}\})$ and $L_0 = L(\{f_n : n \in \mathbb{N}\})$.*

Proof. The implication (ii)⇒(i) is a consequence of Lemmas 1–3, because for any sequence $\{f_n : n \in \mathbb{N}\} \subset \mathbb{R}^X$ of quasi-continuous functions all functions $f_n|_{X_2}$ are quasi-continuous, the sets $L_s \cap X_2 \in \mathcal{B}aire(X_2)$ for $s \in \{0, +, -\}$ and, in consequence (see e.g. [8]), can be represented in the form $L_s \cap X_2 = (G_s \setminus P_s) \cup Q_s$ with G_s, P_s, Q_s satisfying (1)–(3). Since X_2 is a Baire space, G_s are pairwise disjoint sets and $P_+ \cap Q_- = G_+ \cap L_{-\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $P_- \cap Q_+ = G_- \cap L_{+\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $P_0 \cap Q_- = G_0 \cap L_{-\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $P_0 \cap Q_+ = G_0 \cap L_{+\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$ are nowhere-dense, by Lemmas 1–3.

The proof of (i)⇒(ii) consists in the construction of the required sequence. First suppose that X is a Baire space and L_+, L_-, $L_0 \in \mathcal{B}aire(X)$ are as in (i) for $X_2 = X$. Let $L_1 = X \setminus (L_0 \cup L_+ \cup L_-)$. Then $L_1 \in \mathcal{B}aire(X)$, so there are sets G_1, P_1, Q_1 such that $L_1 = (G_1 \setminus P_1) \cup Q_1$ and G_1 is a regular open set, $P_1, Q_1 \in \mathcal{M}$, $P_1 \subset G_1$ and $Q_1 \cap G_1 = \emptyset$. Let $S = \{1, 0, +, -\}$. Since X is a Baire space, $X = \bigcup_{s \in S} \mathrm{cl}(G_s)$ and G_s are pairwise disjoint sets. Fix an $s \in S$. Then $P_s \cup \mathrm{fr}(G_s) \in \mathcal{M}$, so $P_s \cup \mathrm{fr}(G_s) \subset \bigcup_{i \in \mathbb{N}} F_i^s$, where F_i^s are closed and nowhere-dense subsets of $\mathrm{cl}(G_s)$. Moreover, we can assume that $F_0^s \subset F_1^s \subset \ldots \subset F_i^s \subset F_{i+1}^s \subset \ldots$, $P_+ \cap Q_- \subset F_0^+$, $P_- \cap Q_+ \subset F_0^-$, $(P_0 \cap Q_-) \cup (P_0 \cap Q_+) \subset F_0^0$ and also $\mathrm{fr}(G_s) \subset F_0^s$ (all sets considered here are nowhere-dense by assumption).

Fix an $n \in \mathbb{N}$. By Lemma 4 there exists a collection $\{U_{n,i,j}^s : i \le n, j \le 3\}$ of semi-open subsets of X such that

1. $U_{n,i,j}^s \cap U_{n,k,l}^s = \emptyset$ for $(n, i, j) \neq (n, k, l)$,
2. $U_{n,i,j}^s \subset (G_s \cap B(F_i^s, 1/2^n)) \setminus F_n^s$ for every $i \le n$ and $j \le 3$,
3. $F_i^s \subset \mathrm{cl}(U_{n,i,j}^s)$ for every $i \le n$ and $j \le 3$,
4. $G_s \setminus \bigcup_{i \le n} \bigcup_{j \le 3} U_{n,i,j}^s$ is semi-open.

Define $\{f_n : n \in \mathbb{N}\}$ on each of G_s sets. First, take $G_+ = (G_+ \setminus P_+) \cup (P_+ \cap (Q_- \cup Q_0 \cup Q_1))$. Then $P_+ \cup \mathrm{fr}(G_+) \subset \bigcup_{i \in \mathbb{N}} F_i^+$, where F_i^+ are closed and nowhere-dense subsets of $\mathrm{cl}\,(G_+)$ with $F_0^+ \subset F_1^+ \subset \ldots \subset F_i^+ \subset F_{i+1}^+ \subset \ldots$. Moreover, $P_+ \cap Q_-$ and $\mathrm{fr}\,(G_+)$ are subsets of F_0^+. Let $E_0^+ = F_0^+$ and $E_i^+ = F_i^+ \setminus F_{i-1}^+$ for $0 < i \leq n$. Define $f_n|_{G_+}$ as follows:

$$f_n|_{G_+}(x) = \begin{cases} n & \text{if } x \in G_+ \setminus \bigcup_{i \leq n} \bigcup_{j \leq 3} (U_{n,i,j}^+ \cup E_i^+), \\ n & \text{if } x \in \bigcup_{i \leq n} \left[U_{n,i,0}^+ \cup (E_i^+ \cap (G_+ \setminus P_+)) \right], \\ -n & \text{if } x \in U_{n,0,1}^+ \cup (E_0^+ \cap P_+ \cap Q_-), \\ i & \text{if } x \in U_{n,i,2}^+ \cup (E_i^+ \cap P_+ \cap Q_0) \text{ for } i \leq n, \\ i & \text{if } x \in U_{n,i,1}^+ \qquad \text{for } 0 < i \leq n, \\ i & \text{if } x \in U_{n,i,3}^+ \cup (E_i^+ \cap P_+ \cap Q_1) \\ & \text{for } i \leq n, \text{ and } n \text{ is even}, \\ i+1 & \text{if } x \in U_{n,i,3}^+ \cup (E_i^+ \cap P_+ \cap Q_1) \\ & \text{for } i \leq n, \text{ and } n \text{ is odd}. \end{cases}$$

We see at once that $f_n|_{G_+}$ is quasi-continuous because

- it is constant on each of semi-open sets $U_{n,i,j}^+$,
- it is constant on $G_+ \setminus \bigcup_{i \leq n} \bigcup_{j \leq 3} (U_{n,i,j}^+ \cup E_i^+)$, where $G_+ \setminus \bigcup_{i \leq n} \bigcup_{j \leq 3} U_{n,i,j}^+$ is semi-open and $\bigcup_{i \leq n} E_i^+ = F_n^+$ is nowhere-dense,
- for every $i \leq n$ and $x \in E_i^+$ there is $j \leq 3$ such that $E_i^+ \subset \mathrm{cl}\,(U_{n,i,j}^+)$ and $f_n|_{G_+}(x) = f_n|_{G_+}(t)$ for any $t \in U_{n,i,j}^+$.

Let us verify the convergence. Namely,

- if $x \in P_+ \cap Q_-$ then $f_n(x) = -n$ for every n, so $\lim_n f_n(x) = -\infty$,
- if $x \in P_+ \cap Q_0$ then $x \in E_i^+$ for unique i and $f_n(x) = i$ for every $n \geq i$, which means that $\lim_n f_n(x) = i$,
- if $x \in P_+ \cap Q_1$ then $x \in E_i^+$ for unique i and for every $n \geq i$ $f_n(x) = i$ if n is even or $f_n(x) = i+1$ if n is odd , so $(f_n(x))_n$ diverges,
- if $x \in (G_+ \setminus P_+) \cap \bigcup_{i \in \mathbb{N}} E_i^+$ then $x \in E_i^+$ for unique i and $f_n(x) = n$ for every $n \geq i$, so $\lim_n f_n(x) = +\infty$,
- for $x \in (G_+ \setminus P_+) \setminus \bigcup_{i \in \mathbb{N}} E_i^+$ we have two cases.

Case 1: There is $k \in \mathbb{N}$ such that x is an element of none of $U_{n,i,j}^+$ sets for $n \geq k$. Then $\lim_n f_n(x) = +\infty$ because $f_n(x) = n$ for every $n \geq k$.

Case 2: Consider the sequence $\{U_{n_k,i_k,j_k}^+ \colon k \in \mathbb{N}\}$ of sets containing x. Then $\{i_k \colon k \in \mathbb{N}\}$ goes to infinity, because otherwise it contains a constant subsequence, which means that x belongs to the infinitely many sets of the form $U_{n_k,i_0,j_k}^+ \subset B(F_{i_0}^+, 1/2^{n_k})$. Consequently, $x \in F_{i_0}^+$, which is impossible. Since $f_n(x)$ is equal to one of the numbers n, i_k, i_k+1 for n large enough, we have $\lim_n f_n(x) = +\infty$.

For $G_- = (G_- \setminus P_-) \cup (P_- \cap (Q_+ \cup Q_0 \cup Q_1))$ the construction is similar. Let $E_0^- = F_0^-$ and $E_i^- = F_i^- \setminus F_{i-1}^-$ for $0 < i \le n$. Then

$$f_n|_{G_-}(x) = \begin{cases} -n & \text{if } x \in G_- \setminus \bigcup_{i\le n} \bigcup_{j\le 3} (U_{n,i,j}^- \cup E_i^-), \\ -n & \text{if } x \in \bigcup_{i\le n} \left[U_{n,i,1}^- \cup (E_i^- \cap (G_- \setminus P_-)) \right], \\ n & \text{if } x \in U_{n,0,0}^- \cup (E_0^- \cap P_- \cap Q_+), \\ -i & \text{if } x \in U_{n,i,2}^- \cup (E_i^- \cap P_- \cap Q_0) \text{ for } i \le n, \\ -i & \text{if } x \in U_{n,i,0}^- \qquad\qquad\qquad\quad \text{for } 0 < i \le n, \\ -i & \text{if } x \in U_{n,i,3}^- \cup (E_i^- \cap P_- \cap Q_1) \text{ for } i \le n \text{ if } n \text{ is even}, \\ -i-1 & \text{if } x \in U_{n,i,3}^- \cup (E_i^- \cap P_- \cap Q_1) \text{ for } i \le n \text{ if } n \text{ is odd}. \end{cases}$$

Then we have

- $P_- \cap Q_+ = L_{+\infty}(\{f_n|_{G_-} : n \in \mathbb{N}\})$,
- $P_- \cap Q_0 = L(\{f_n|_{G_-} : n \in \mathbb{N}\})$,
- $(f_n(x))_n$ diverges at each $x \in P_- \cap Q_1$,
- $G_- \setminus P_- = L_{-\infty}(\{f_n|_{G_-} : n \in \mathbb{N}\})$.

Now, take $G_0 = (G_0 \setminus P_0) \cup (P_0 \cap (Q_+ \cup Q_- \cup Q_1))$. Then $P_0 \cup \operatorname{fr}(G_0) \subset \bigcup_{i\in\mathbb{N}} F_i^0$, where F_i^0 are closed and nowhere-dense subsets of $\operatorname{cl}(G_0)$ with $F_0^0 \subset F_1^0 \subset \dots \subset F_i^0 \subset F_{i+1}^0 \subset \dots$ and $P_0 \cap Q_+$, $P_0 \cap Q_-$, $\operatorname{fr}(G_0)$ subsets of F_0^0. Let $E_0^0 = F_0^0$ and $E_i^0 = F_i^0 \setminus F_{i-1}^0$ for $0 < i \le n$. Define $f_n|_{G_0}$ by the formula

$$f_n|_{G_0}(x) = \begin{cases} 0 & \text{if } x \in G_0 \setminus \bigcup_{i\le n} \bigcup_{j\le 3} (U_{n,i,j}^0 \cup E_i^0), \\ 0 & \text{if } x \in \bigcup_{i\le n} \left[U_{n,i,2}^0 \cup (E_i^0 \cap (G_0 \setminus P_0)) \right], \\ n & \text{if } x \in U_{n,0,0}^0 \cup (E_0^0 \cap P_0 \cap Q_+), \\ -n & \text{if } x \in U_{n,0,1}^0 \cup (E_0^0 \cap P_0 \cap Q_-), \\ 1/(i+1) & \text{if } x \in U_{n,i,3}^0 \qquad\qquad\qquad \text{for } i \le n, \\ 1/(i+1) & \text{if } x \in U_{n,i,0}^0 \cup U_{n,i,1}^0 \qquad \text{for } 0 < i \le n, \\ 0 & \text{if } x \in \bigcup_{i\le n} (E_i^0 \cap P_0 \cap Q_1) \text{ if } n \text{ is even}, \\ 1/(i+1) & \text{if } x \in E_i^0 \cap P_0 \cap Q_1 \qquad \text{for } i \le n \text{ if } n \text{ is odd}. \end{cases}$$

Then $f_n|_{G_0}$ is quasi-continuous. In the same manner as before we can see that

- $P_0 \cap Q_+ = L_{+\infty}(\{f_n|_{G_0} : n \in \mathbb{N}\})$,
- $P_0 \cap Q_- = L_{-\infty}(\{f_n|_{G_0} : n \in \mathbb{N}\})$,
- $(f_n(x))_n$ diverges at each $x \in P_0 \cap Q_1$,
- $G_0 \setminus P_0 = \{x \in G_0 : \lim_n f_n(x) = 0\} = L(\{f_n|_{G_0} : n \in \mathbb{N}\})$.

For $G_1 = (G_1 \setminus P_1) \cup (P_1 \cap (Q_+ \cup Q_- \cup Q_0))$, where $P_1 \cup \operatorname{fr}(G_1) \subset \bigcup_{i\in\mathbb{N}} F_i^1$, F_i^1 are closed and nowhere-dense subsets of $\operatorname{cl}(G_1)$ with $F_0^1 \subset F_1^1 \subset \dots \subset F_i^1 \subset F_{i+1}^1 \subset \dots$, the construction is the following. Let $E_0^1 = F_0^1$ and $E_i^1 =$

$F_i^1 \setminus F_{i-1}^1$ for $0 < i \leq n$, $A = G_1 \setminus \bigcup_{i\leq n} \bigcup_{j\leq 3} U_{n,i,j}^1$ and $B = \bigcup_{i\leq n} \bigcup_{j\leq 3} U_{n,i,j}^1$. Then

$$f_{2n}|_A(x) = \begin{cases} (-1)^n & \text{if } x \in G_1 \setminus \bigcup_{i\leq n} \bigcup_{j\leq 3}(U_{n,i,j}^1 \cup E_i^1), \\ (-1)^n & \text{if } x \in \bigcup_{i\leq n} \left[E_i^1 \cap (G_1 \setminus P_1)\right], \\ n & \text{if } x \in \bigcup_{i\leq n}(E_i^1 \cap P_1 \cap Q_+), \\ -n & \text{if } x \in \bigcup_{i\leq n}(E_i^1 \cap P_1 \cap Q_-), \\ 1 & \text{if } x \in \bigcup_{i\leq n}(E_i^1 \cap P_1 \cap Q_0), \end{cases}$$

and

$$f_{2n}|_B(x) = \begin{cases} n & \text{if } x \in \bigcup_{i\leq n} U_{n,i,0}^1, \\ -n & \text{if } x \in \bigcup_{i\leq n} U_{n,i,1}^1, \\ 1 & \text{if } x \in \bigcup_{i\leq n} U_{n,i,2}^1, \\ -1 & \text{if } x \in \bigcup_{i\leq n} U_{n,i,3}^1. \end{cases}$$

Put also

$$f_{2n+1}|_A = f_{2n}|_A \quad \text{and} \quad f_{2n+1}|_B = -f_{2n}|_B.$$

The quasi-continuity of function defined in such a way is clear. It is also easy to verify that

- $P_1 \cap Q_+ = L_{+\infty}(\{f_n|_{G_1} : n \in \mathbb{N}\})$,
- $P_1 \cap Q_- = L_{-\infty}(\{f_n|_{G_1} : n \in \mathbb{N}\})$,
- $P_1 \cap Q_0 = \{x \in G_1 : \lim_n f_n(x) = 1\} = L(\{f_n|_{G_1} : n \in \mathbb{N}\})$.
- $(f_n(x))_n$ diverges at each $x \in G_1 \setminus P_1$.

What is left is to consider $F = \bigcup_{s\in S} \mathrm{fr}\,(G_s) \subset \bigcup_{s\in S} F_0^s$. Since G_s are pairwise disjoint regular open sets such that $X = \bigcup_{s\in S} \mathrm{cl}\,(G_s)$, we have $\mathrm{fr}\,(G_1) \subset \mathrm{fr}\,(G_+) \cup \mathrm{fr}\,(G_-) \cup \mathrm{fr}\,(G_0)$. Then $F \subset F_0^+ \cup F_0^- \cup F_0^0 \subset \bigcup_{s\in S} Q_s$. Let

$$f_n|_F(x) = \begin{cases} n & \text{if } x \in F \cap Q_+, \\ -n & \text{if } x \in F \cap Q_-, \\ 0 & \text{if } x \in F \cap Q_0, \\ n(-1)^n & \text{if } x \in F \cap Q_1. \end{cases}$$

Putting $f_n = \bigcup_{s\in S} f_n|_{G_s} \cup f_n|_F$ we get a sequence of quasi-continuous functions (notice that the quasi-continuity on F is ensured by sets $U_{n,0,j}^+$, $U_{n,0,j}^-$, $U_{n,0,j}^0$ for $j \leq 2$) satisfying all required conditions.

Now, consider an arbitrary dense in itself separable metric space X and pairwise disjoint sets L_+, L_-, $L_0 \in \mathcal{B}aire(X)$. Put $L_1 = X \setminus (L_0 \cup L_+ \cup L_-)$. Then the sets $L_+ \cap X_2$, $L_- \cap X_2$, $L_0 \cap X_2$ and $L_1 \cap X_2$ have the Baire property in X_2 and we can find (as before) a sequence $\{f_n|_{X_2} : n \in \mathbb{N}\} \subset \mathbb{R}^{X_2}$ of quasi-continuous functions with $L_+ \cap X_2 = L_{+\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $L_- \cap X_2 = L_{-\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $L_0 \cap X_2 = L(\{f_n|_{X_2} : n \in \mathbb{N}\})$.

Take X_1. By the Banach Category Theorem ([8, Theorem 16.1]), X_1 is an open set of first category in X, so $X_1 \subset \bigcup_{n\in\mathbb{N}} F_n \subset \mathrm{cl}\,(X_1)$, where F_n are nowhere-dense sets closed in X with $F_n \subset F_{n+1}$ for every $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$. Let $E_n = F_n \cup \mathrm{fr}\,(X_1)$. Then $E_n \subset \mathrm{cl}\,(X_1)$ is a nowhere-dense closed subset of X and $\mathrm{cl}\,(X_1) = \bigcup_{n\in\mathbb{N}} E_n$. By Lemma 4 there are pairwise disjoint sets $S_0^n, S_1^n, S_2^n, S_3^n \subset X_1 \setminus E_n$ such that:

(1) $S_i^n \in \mathcal{SO}(X)$ for any $i \le 3$;
(2) $X_1 \setminus E_n = \bigcup_{i\le 3} S_i^n$;
(3) $E_n \subset \mathrm{cl}\,(S_i^n)$ for any $i \le 3$.

Define $f_n|_{\mathrm{cl}\,(X_1)}\colon \mathrm{cl}\,(X_1) \to \mathbb{R}$ by

$$f_n|_{\mathrm{cl}\,(X_1)}(x) = \begin{cases} 0 & \text{if } x \in S_0^n \cup (E_n \cap L_0), \\ (-1)^n & \text{if } x \in S_1^n \cup (E_n \cap L_1), \\ n & \text{if } x \in S_2^n \cup (E_n \cap L_+), \\ -n & \text{if } x \in S_3^n \cup (E_n \cap L_-). \end{cases}$$

Then all functions $f_n|_{\mathrm{cl}\,(X_1)}$ are quasi-continuous. Moreover, $L_+ \cap \mathrm{cl}\,(X_1) = L_{+\infty}(\{f_n|_{\mathrm{cl}\,(X_1)}\colon n \in \mathbb{N}\})$, $L_- \cap \mathrm{cl}\,(X_1) = L_{-\infty}(\{f_n|_{\mathrm{cl}\,(X_1)}\colon n \in \mathbb{N}\})$ and $L_0 \cap \mathrm{cl}\,(X_1) = L(\{f_n|_{\mathrm{cl}\,(X_1)}\colon n \in \mathbb{N}\})$.

Putting $f_n = f_n|_{X_2} \cup f_n|_{\mathrm{cl}\,(X_1)}$ we get the sequence we claimed. □

It turns out that for many well-known classes of functions containing $\mathcal{QC}$ no additional assumptions are needed for three pairwise disjoint sets with the Baire property to be of the form (1), (2) or (3). This is a consequence of the next theorem, where the sequences of functions $f\colon X \to \mathbb{R}$ with dense open set of continuity points are considered. Denote the class of all such functions by $\mathcal{S}$. In other words,

$$f \in \mathcal{S} \subset \mathbb{R}^X \text{ iff } C(f) \text{ is an open dense subset of } X.$$

Theorem 2. *Let X be an arbitrary topological space. For any pairwise disjoint sets $L_+, L_-, L_0 \subset X$ the following conditions are equivalent:*

(i) $L_+, L_-, L_0 \in \mathcal{B}aire(X)$;
(ii) *there exists a sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ of Baire measurable functions such that $L_+ = L_{+\infty}(\{f_n\colon n \in \mathbb{N}\})$, $L_- = L_{-\infty}(\{f_n\colon n \in \mathbb{N}\})$ and $L_0 = L(\{f_n\colon n \in \mathbb{N}\})$;*
(iii) *there exists a sequence $\{f_n\colon n \in \mathbb{N}\} \subset \mathbb{R}^X$ of functions with the dense open set of continuity points such that $L_+ = L_{+\infty}(\{f_n\colon n \in \mathbb{N}\})$, $L_- = L_{-\infty}(\{f_n\colon n \in \mathbb{N}\})$ and $L_0 = L(\{f_n\colon n \in \mathbb{N}\})$.*

Proof. Since $\mathcal{S} \subset \mathcal{B}$, by Remark 1, the implications (iii)⇒(ii) and (ii)⇒(i) are obvious. Therefore it is enough to construct a sequence satisfying (iii) for fixed pairwise disjoint sets $L_+, L_-, L_0 \in \mathcal{B}aire(X)$.

First suppose that X is a Baire space. Let $L_1 = X \setminus (L_0 \cup L_+ \cup L_-)$. Then for every $s \in S = \{1, 0, +, -\}$ we have $L_s = (G_s \setminus P_s) \cup Q_s$, where G_s is a regular open set, $P_s, Q_s \in \mathcal{M}(X)$, $P_s \subset G_s$ and $Q_s \cap G_s = \emptyset$. Then $X = \bigcup_{s\in S} \mathrm{cl}\,(G_s)$ and G_s are pairwise disjoint sets. Moreover, $P_s \subset \bigcup_{n\in\mathbb{N}} F_n^s$, where F_n^s are closed, nowhere-dense subsets of $\mathrm{cl}\,(G_s)$ and $F_0^s \subset F_1^s \subset ....$

First we define $\{f_n \colon n \in \mathbb{N}\}$ on each of G_s sets. Fix an $n \in \mathbb{N}$ and notice that $G_+ = (G_+ \setminus P_+) \cup (P_+ \cap (Q_- \cup Q_0 \cup Q_1))$. Define $f_n|_{G_+}$ as follows:

$$f_n|_{G_+}(x) = \begin{cases} n & \text{if } x \in (G_+ \setminus F_n^+) \cup (F_n^+ \cap (G_+ \setminus P_+)), \\ -n & \text{if } x \in F_n^+ \cap P_+ \cap Q_-, \\ 0 & \text{if } x \in F_n^+ \cap P_+ \cap Q_0, \\ (-1)^n & \text{if } x \in F_n^+ \cap P_+ \cap Q_1. \end{cases}$$

Then $f_n|_{G_+}$ is constant on the open set $G_+ \setminus F_n^+$ and F_n^+ is nowhere-dense. A trivial verification shows that

- $G_+ \setminus P_+ = L_{+\infty}(\{f_n|_{G_+} \colon n \in \mathbb{N}\})$,
- $P_+ \cap Q_- = L_{-\infty}(\{f_n|_{G_+} \colon n \in \mathbb{N}\})$,
- $P_+ \cap Q_0 = L(\{f_n|_{G_+} \colon n \in \mathbb{N}\})$,
- $(f_n(x))_n$ diverges at each $x \in P_+ \cap Q_1$.

Similarly, we define

$$f_n|_{G_-}(x) = \begin{cases} -n & \text{if } x \in (G_- \setminus F_n^-) \cup (F_n^- \cap (G_- \setminus P_-)), \\ n & \text{if } x \in F_n^- \cap P_- \cap Q_+, \\ 0 & \text{if } x \in F_n^- \cap P_- \cap Q_0, \\ (-1)^n & \text{if } x \in F_n^- \cap P_- \cap Q_1, \end{cases}$$

$$f_n|_{G_0}(x) = \begin{cases} 0 & \text{if } x \in (G_0 \setminus F_n^0) \cup (F_n^0 \cap (G_0 \setminus P_0)), \\ n & \text{if } x \in F_n^0 \cap P_0 \cap Q_+, \\ -n & \text{if } x \in F_n^0 \cap P_0 \cap Q_-, \\ (-1)^n & \text{if } x \in F_n^0 \cap P_0 \cap Q_1, \end{cases}$$

$$f_n|_{G_1}(x) = \begin{cases} (-1)^n & \text{if } x \in (G_1 \setminus F_n^1) \cup (F_n^1 \cap (G_1 \setminus P_1)), \\ n & \text{if } x \in F_n^1 \cap P_1 \cap Q_+, \\ -n & \text{if } x \in F_n^1 \cap P_1 \cap Q_-, \\ 0 & \text{if } x \in F_n^1 \cap P_1 \cap Q_0. \end{cases}$$

Finally, let us consider a nowhere-dense set $F = \bigcup_{s\in S} \mathrm{fr}\,(G_s) \subset \bigcup_{s\in S} Q_s$. Put

$$f_n|_F(x) = \begin{cases} n & \text{if } x \in F \cap Q_+, \\ -n & \text{if } x \in F \cap Q_-, \\ 0 & \text{if } x \in F \cap Q_0, \\ (-1)^n & \text{if } x \in F \cap Q_1. \end{cases}$$

Writing $f_n = \bigcup_{s\in S} f_n|_{G_s} \cup f_n|_F$ we get $L_+ = L_{+\infty}(\{f_n : n \in \mathbb{N}\})$, $L_- = L_{-\infty}(\{f_n : n \in \mathbb{N}\})$, $L_0 = L(\{f_n : n \in \mathbb{N}\})$. Moreover, $D(f_n) \subset \bigcup_{s\in S} F_n^s \cup F$ for any $n \in \mathbb{N}$, so it is nowhere-dense. Consequently, $C(f)$ is a dense subset of X. Since f_n has a finite range, $C(f)$ is open.

Now, consider an arbitrary topological space X. Let X_1 be the union of all first category open subsets of X. Then $X_2 = X \setminus \mathrm{cl}\,(X_1)$ is an open Baire subspace of X and the sets $L_+ \cap X_2$, $L_- \cap X_2$, $L_0 \cap X_2$ and $L_1 = X_2 \setminus (L_0 \cup L_+ \cup L_-)$ have the Baire property in X_2, so we can find (as before) a sequence $\{f_n|_{X_2} : n \in \mathbb{N}\} \subset \mathbb{R}^{X_2}$ of functions with finite ranges such that $L_+ \cap X_2 = L_{+\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $L_- \cap X_2 = L_{-\infty}(\{f_n|_{X_2} : n \in \mathbb{N}\})$, $L_0 \cap X_2 = L(\{f_n|_{X_2} : n \in \mathbb{N}\})$ and $D(f_n|_{X_2})$ is closed and nowhere-dense in X_2 for any $n \in \mathbb{N}$.

Take X_1. By the Banach Category Theorem, X_1 is an open set of the first category in X, so $X_1 \subset \bigcup_{n\in\mathbb{N}} F_n \subset \mathrm{cl}\,(X_1)$, where F_n are nowhere-dense sets closed in X with $F_n \subset F_{n+1}$ for every $n \in \mathbb{N}$. Fix an $n \in \mathbb{N}$. Let $E_n = F_n \cup \mathrm{fr}\,(X_1)$. Then E_n is a nowhere-dense closed subset of X and $\mathrm{cl}\,(X) = \bigcup_{n\in\mathbb{N}} E_n$. Define $f_n|_{\mathrm{cl}\,(X_1)} \colon \mathrm{cl}\,(X_1) \to \mathbb{R}$ by

$$f_n|_{\mathrm{cl}\,(X_1)}(x) = \begin{cases} n & \text{if } x \in E_n \cap L_+, \\ -n & \text{if } x \in E_n \cap L_-, \\ 0 & \text{if } x \in E_n \cap L_0, \\ (-1)^n & \text{otherwise.} \end{cases}$$

Note that $f_n|_{\mathrm{cl}\,(X_1)}$ is constant on the open set $X_1 \setminus E_n$. Moreover, $L_+ \cap \mathrm{cl}\,(X_1) = L_{+\infty}(\{f_n|_{\mathrm{cl}\,(X_1)} : n \in \mathbb{N}\})$, $L_- \cap \mathrm{cl}\,(X_1) = L_{-\infty}(\{f_n|_{\mathrm{cl}\,(X_1)} : n \in \mathbb{N}\})$ and $L_0 \cap \mathrm{cl}\,(X_1) = L(\{f_n|_{\mathrm{cl}\,(X_1)} : n \in \mathbb{N}\})$. Put $f_n = f_n|_{\mathrm{cl}\,(X_1)} \cup f_n|_{X_2}$. Then $C(f_n)$ is open for any $n \in \mathbb{N}$ and $D(f_n)$ is a subset of $D(f_n|_{X_2}) \cup E_n \cup \mathrm{fr}\,(X_1)$ which is nowhere-dense in X. Consequently, $\{f_n : n \in \mathbb{N}\}$ has all required properties. □

Remark 2. Theorem 2 holds for any class $\mathcal{F} \subset \mathbb{R}^X$ between $\mathcal{S}$ and $\mathcal{B}$, such as *pointwise discontinuous* functions (see [3], p. 74), *simply continuous* functions (cf. [1, Lemma 1]), or *cliquish* functions (see [7]). Notice also that these last two lie in-between $\mathcal{QC}$ and $\mathcal{B}$ (see [7]). As a corollary from Theorems 1 and 2 we obtain an analogue of Hahn - Sierpiński theorem for some subclasses of Baire measurable functions. For a family of functions $\mathcal{F} \subset \mathbb{R}^X$ define

$$\mathcal{L}(\mathcal{F}) = \{L(\{f_n : n \in \mathbb{N}\}) : \{f_n : n \in \mathbb{N}\} \text{ is a sequence of functions from } \mathcal{F}\}\,.$$

Then for any class $\mathcal{F} \subset \mathbb{R}^X$ such that $\mathcal{S} \subseteq \mathcal{F} \subseteq \mathcal{B}$ we have $\mathcal{L}(\mathcal{F}) = \mathcal{B}aire(X)$. Moreover, if $\mathcal{QC} \subset \mathbb{R}^X$, where X is a dense in itself separable metric space, then $\mathcal{L}(\mathcal{QC}) = \mathcal{B}aire(X)$.

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