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FUNCTIONS OF TWO VARIABLES WHOSE VERTICAL SECTIONS ARE EQUIDERIVATIVES

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Abstract. We examine functions of two variables whose all vertical sections are equiderivatives. In particular we show that a bounded function whose horizontal sections are strongly measurable and vertical sections are equiderivatives, is strongly measurable. The theorems we prove are generalizations of the results of Z. Grande [3].

Let $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. Let $(X, \mathcal{M})$ be a measurable space and let $\mathcal{I} \subset \mathcal{M}$ be a proper σ-ideal of subsets of X. Assume that Z is a Banach space.

Let $h\colon X \to Z$. Recall that h is *measurable*, if $h^{-1}(U) \in \mathcal{M}$ for every open set $U \subset Z$. In [2], I introduced the following two kinds of measurability of a function. We say that h is *nearly simple*, if there exist elements $\alpha_1, \alpha_2, \ldots \in Z$ and a sequence of pairwise disjoint sets $A_1, A_2, \ldots \in \mathcal{M}$ such that $h = \alpha_n$ on A_n for each n, and $X = \bigcup_{n=1}^{\infty} A_n$. We say that h is *strongly measurable* with respect to $(\mathcal{M}, \mathcal{I})$, if there exists a sequence of nearly simple functions (h_n) and a set $A \in \mathcal{I}$ such that $h_n \to h$ on $X \setminus A$.

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Remark 1. Clearly Bochner integrable functions are strongly measurable, because they are pointwise limits of sequences of simple functions. Moreover strongly measurable functions are measurable. Easy examples show that the converse implications do not hold.

Remark 2. Usually in functional analysis (see, e.g., [6]) one considers a different kind of strong measurability, namely one assumes that there exists a sequence of simple functions (h_n) (i.e., measurable functions with finite range) and a set $A \in \mathcal{I}$ such that $h_n \to h$ on $X \setminus A$. If we can find a σ-finite measure on $(X, \mathcal{M})$, then these two notions coincide.

We will need the following property of strongly measurable functions.

Proposition 1. *Let $h\colon X \to Z$ be strongly measurable with respect to $(\mathcal{M}, \mathcal{I})$. Then*

(1) *for each $\varepsilon > 0$ and each $A \in \mathcal{M} \setminus \mathcal{I}$ there is a set $B \in \mathcal{M} \setminus \mathcal{I}$ with $B \subset A$ such that $\operatorname{osc}_B h < \varepsilon$,*

where $\operatorname{osc}_B h = \sup\{\|h(x) - h(y)\|\colon x, y \in B\}$.

Proof. Let $\varepsilon > 0$ and $A \in \mathcal{M} \setminus \mathcal{I}$. By [2, Corollary 3], there is a set $S \in \mathcal{I}$ such that $h(X \setminus S)$ is a separable subspace of Z. So, there are $z_1, z_2, \ldots \in Z$ such that

$$h(X \setminus S) \subset \bigcup_{n=1}^{\infty} K_n,$$

where $K_n = \{z \in Z\colon \|z - z_n\| < \varepsilon/3\}$. Thus

$$A = (A \cap S) \cup \bigcup_{n=1}^{\infty} \big(A \cap h^{-1}(K_n)\big).$$

Since $A \notin \mathcal{I}$, there is an $n \in \mathbb{N}$ such that $B = A \cap h^{-1}(K_n) \notin \mathcal{I}$. Clearly $B \in \mathcal{M}$ and $\operatorname{osc}_B h \le 2\varepsilon/3 < \varepsilon$. □

Remark 3. Recall that if condition (ccc) is fulfilled in $(X, \mathcal{M}, \mathcal{I})$ (i.e., if each family of pairwise disjoint elements of $\mathcal{M} \setminus \mathcal{I}$ is at most countable), then condition (1) implies the strong measurability of h. (See [2, Proposition 6].)

Through the article we fix a triple $(X, \mathcal{M}, \mathcal{I})$ as above and a Banach space Z. We assume that condition (ccc) is fulfilled in $(X, \mathcal{M}, \mathcal{I})$.

Let $(Y, \mathcal{N}, \nu)$ be a measure space. We will consider the product σ-ideal $\widetilde{\mathcal{I}}$ (the family of all subsets of the sets of the form $A \times Y$, where $A \in \mathcal{I}$) and the product σ-field $\mathcal{M} \otimes \mathcal{N}$ (the smallest σ-field containing the σ-ideal $\widetilde{\mathcal{I}}$ and the family of all sets of the form $M \times N$, where $M \in \mathcal{M}$ and $N \in \mathcal{N}$).

We assume that we have a net structure $\mathcal{J}$ in $(Y, \mathcal{N}, \nu)$ (see, e.g., [1]). Recall that a *net* in Y is an at most countable cover of Y consisting of pairwise disjoint measurable sets of positive finite measure. The individual sets in the net are called *cells*. The family $\mathcal{J} = \bigcup_{n=1}^{\infty} \mathcal{J}_n$, where each $\mathcal{J}_n$ is a net, is called a *net structure*. Observe that for each $y \in Y$ and $n \in \mathbb{N}$, there is a unique cell from the net $\mathcal{J}_n$ which contains y. We will denote this cell by $J_n(y)$. Several examples of net structures can be found, e.g., in [4] or [5].

A function $g\colon Y \to Z$ is called a *derivative* (with respect to the net structure $\mathcal{J}$), if g is Bochner integrable over each $J \in \mathcal{J}$ and for each $y \in Y$,

$$\lim_{n\to\infty} \frac{1}{\nu\big(J_n(y)\big)} \int_{J_n(y)} g = g(y).$$

Let $\mathcal{F} \subset Z^Y$ be a family of Bochner integrable functions. We say that the functions in $\mathcal{F}$ are *equiderivatives at a point* $y \in Y$ (with respect to the net structure $\mathcal{J}$) [3], if for each $\varepsilon > 0$ we can find an $N \in \mathbb{N}$ such that for all $n > N$ and $f \in \mathcal{F}$,

$$\Big\| \frac{1}{\nu\big(J_n(y)\big)} \int_{J_n(y)} f - f(y) \Big\| < \varepsilon.$$

We will say that a family $\mathcal{A} \subset \mathcal{J}$ has *property* (V) *in a set* J, if for each $y \in J$ and each $n \in \mathbb{N}$, there is an $m > n$ with $J_m(y) \in \mathcal{A}$. We define the following property, pertaining to a net structure $\mathcal{J}$:

(2) for each $J \in \mathcal{J}$, each family $\mathcal{A} \subset \mathcal{J}$ which possesses property (V) in J, and each $\varepsilon > 0$, we can choose pairwise disjoint sets $A_1, \dots, A_k \in \mathcal{A}$ such that $\nu\big(J \triangle \bigcup_{i=1}^{k} A_i\big) < \varepsilon$.

This property is a generalization of the Vitali property for intervals in $\mathbb{R}^m$.

Example 1. Let $\mathcal{J} = \bigcup_{n\in\mathbb{N}} \mathcal{J}_n$, where $\mathcal{J}_n$ denotes the family of all intervals of the form

$$\Big[\frac{i_1 - 1}{2^n}, \frac{i_1}{2^n}\Big) \times \cdots \times \Big[\frac{i_m - 1}{2^n}, \frac{i_m}{2^n}\Big),$$

where $i_1, \dots, i_m$ are integers. Then $\mathcal{J}$ possesses property (2).

Theorem 2. *Assume that $\mathcal{J}$ possesses property* (2)*. Let $g\colon X \times Y \to Z$ be a bounded function, whose all horizontal sections g^y are strongly measurable with respect to $(\mathcal{M}, \mathcal{I})$, while all vertical sections g_x are equiderivatives at each $y \in Y$. Then g is strongly measurable with respect to $\big(\mathcal{M} \otimes \mathcal{N}, \widetilde{\mathcal{I}}\big)$.*

Proof. Let $M \in \mathbb{R}$ be such that $\|g(x, y)\| < M$ for each $(x, y) \in X \times Y$. Let $J \in \mathcal{J}$. Define a function $h_J\colon X \to Z$ by $h_J(x) = \int_J g_x$. We will use Remark 3 to show that h_J is strongly measurable with respect to $(\mathcal{M}, \mathcal{I})$.

Fix $\varepsilon > 0$ and $A \in \mathcal{M} \setminus \mathcal{I}$. Since the vertical sections of g are equiderivatives, for each $y \in Y$ there is an $N(y) \in \mathbb{N}$ such that for all $n > N(y)$ and $x \in X$,

$$\Big\| \frac{1}{\nu\big(J_n(y)\big)} \int_{J_n(y)} g_x - g_x(y) \Big\| < \frac{\varepsilon}{16\nu(J)}.$$

Clearly the family $\mathcal{A} = \big\{J_n(y) \colon y \in J, n > N(y)\big\}$ possesses the property (V) in J. By assumption (2), we can choose $y_1, \dots, y_k \in J$ and $n_1, \dots, n_k \in \mathbb{N}$ such that the cells $J_{n_1}(y_1), \dots, J_{n_k}(y_k)$ are pairwise disjoint, $n_i > N(y_i)$ for each i, and

$$\nu(K \cup L) < \min\Big\{\frac{\varepsilon}{4M}, \nu(J)\Big\},$$

where

$$K = J \setminus \bigcup_{i=1}^k J_{n_i}(y_i), \qquad L = \bigcup_{i=1}^k J_{n_i}(y_i) \setminus J.$$

Recall that all sections g^{y_i} are strongly measurable. So, using k times Proposition 1 we can find a set $B \in \mathcal{M} \setminus \mathcal{I}$ with $B \subset A$ such that for all $v, w \in B$ and $i \in \{1, \dots, k\}$,

$$\|g^{y_i}(v) - g^{y_i}(w)\| < \frac{\varepsilon}{8\nu(J)}.$$

Consequently, for all $v, w \in B$,

$$\begin{aligned}
&\|h_J(v) - h_J(w)\| = \Big\| \int_J g_v - \int_J g_w \Big\| = \Big\| \int_J (g_v - g_w) \Big\| \\
&= \Big\| \sum_{i=1}^k \int_{J_{n_i}(y_i)} (g_v - g_w) + \int_K (g_v - g_w) - \int_L (g_v - g_w) \Big\| \\
&\le \sum_{i=1}^k \Big\| \int_{J_{n_i}(y_i)} (g_v - g_w) \Big\| + \int_{K \cup L} \|g_v - g_w\| \\
&\le \sum_{i=1}^k \Big\| \int_{J_{n_i}(y_i)} \big(g_v - g(v, y_i)\big) \Big\| + \sum_{i=1}^k \Big\| \int_{J_{n_i}(y_i)} \big(g(v, y_i) - g(w, y_i)\big) \Big\| \\
&\qquad + \sum_{i=1}^k \Big\| \int_{J_{n_i}(y_i)} \big(g(w, y_i) - g_w\big) \Big\| + 2M \frac{\varepsilon}{4M}
\end{aligned}$$

$$\leq \sum_{i=1}^{k} \nu\big(J_{n_i}(y_i)\big) \Big\| \frac{1}{\nu\big(J_{n_i}(y_i)\big)} \int_{J_{n_i}(y_i)} g_v - g_v(y_i) \Big\|$$
$$+ \sum_{i=1}^{k} \int_{J_{n_i}(y_i)} \big\| g(v, y_i) - g(w, y_i) \big\|$$
$$+ \sum_{i=1}^{k} \nu\big(J_{n_i}(y_i)\big) \Big\| \frac{1}{\nu\big(J_{n_i}(y_i)\big)} \int_{J_{n_i}(y_i)} g_w - g_w(y_i) \Big\| + \frac{\varepsilon}{2}$$
$$\leq \sum_{i=1}^{k} \nu\big(J_{n_i}(y_i)\big) \Big(\frac{\varepsilon}{16\nu(J)} + \frac{\varepsilon}{8\nu(J)} + \frac{\varepsilon}{16\nu(J)} \Big) + \frac{\varepsilon}{2}$$
$$= \nu\big(\textstyle\bigcup_{i=1}^{k} J_{n_i}(y_i)\big) \dfrac{\varepsilon}{4\nu(J)} + \dfrac{\varepsilon}{2}$$
$$\leq \nu(J \cup L) \frac{\varepsilon}{4\nu(J)} + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{4} + \nu(L) \frac{\varepsilon}{4\nu(J)} + \frac{\varepsilon}{2} < \varepsilon.$$

We have proved that h_J fulfills condition (1). By Remark 3, h_J is strongly measurable with respect to $(\mathcal{M}, \mathcal{I})$. Now repeating the last part of the proof of [2, Theorem 7] we can show that g is strongly measurable as well. $\square$

Corollary 3. *Assume that $\mathcal{J}$ possesses property (2). Let $f\colon X \times Y \to Z$ be a bounded function, whose all horizontal sections f^y are strongly measurable with respect to $(\mathcal{M}, \mathcal{I})$. Assume moreover that there is a set $S \in \mathcal{I}$ such that the vertical sections f_x, $x \in X \setminus S$, are equiderivatives at each $y \in Y$. Then f is strongly measurable with respect to $\big(\mathcal{M} \otimes \mathcal{N}, \widetilde{\mathcal{I}}\big)$.*

Proof. Define

$$g(x, y) = \begin{cases} f(x, y) & \text{if } x \in X \setminus S, \\ 0 & \text{if } x \in S. \end{cases}$$

One can easily see that g fulfills the assumptions of Theorem 2, so g is strongly measurable with respect to $\big(\mathcal{M} \otimes \mathcal{N}, \widetilde{\mathcal{I}}\big)$. Since the equality $f = g$ holds $\widetilde{\mathcal{I}}$-almost everywhere, f is strongly measurable as well. $\square$

Now assume that $\mathcal{I} = \big\{A \in \mathcal{M}\colon \mu(A) = 0\big\}$, where $\mu\colon \mathcal{M} \to [0, \infty]$ is a nontrivial measure, and that $\mathcal{K} = \bigcup_{m=1}^{\infty} \mathcal{K}_m$ is a net structure in X. For each $x \in X$ and $m \in \mathbb{N}$ we denote by $K_m(x)$ the unique cell from the net $\mathcal{K}_m$ which contains x.

Theorem 4. *Assume that $\mathcal{J}$ possesses property (2). Let $f\colon X \times Y \to Z$ be a bounded function, whose all horizontal sections f^y are derivatives with* *respect to $\mathcal{K}$, while all vertical sections f_x are equiderivatives at each $y \in Y$. Then f is a derivative in the strong sense, i.e., for each $(x,y) \in X \times Y$,*

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \frac{1}{\mu\big(K_m(x)\big)\nu\big(J_n(y)\big)} \int_{K_m(x)\times J_n(y)} f(u,v)\, d(\mu\times\nu)(u,v) = f(x,y).$$

Proof. Fix $(x,y) \in X \times Y$ and $\varepsilon > 0$. Since all vertical sections f_u are equiderivatives at y, there is an $N \in \mathbb{N}$ such that for all $n > N$ and $u \in X$,

$$\Big\|\frac{1}{\nu\big(J_n(y)\big)} \int_{J_n(y)} f_u(v)\, d\nu(v) - f_u(y)\Big\| < \frac{\varepsilon}{2}.$$

Since f^y is a derivative at x, there is an $M \in \mathbb{N}$ such that for each $m > M$,

$$\Big\|\frac{1}{\mu\big(K_m(x)\big)} \int_{K_m(x)} f^y(u)\, d\mu(u) - f^y(x)\Big\| < \frac{\varepsilon}{2}.$$

By Theorem 2, function f is strongly measurable. So, it is Bochner integrable on $K_m(x)\times J_n(y)$. (See, e.g., [6, p. 133].) Using the Fubini Theorem, for all $n > N$ and $m > M$ we obtain

$$\begin{aligned}
&\Big\|\frac{1}{\mu\big(K_m(x)\big)\nu\big(J_n(y)\big)} \int_{K_m(x)\times J_n(y)} f(u,v)\, d(\mu\times\nu)(u,v) - f(x,y)\Big\| \\
&= \Big\|\frac{1}{\mu\big(K_m(x)\big)\nu\big(J_n(y)\big)} \int_{K_m(x)} \Big(\int_{J_n(y)} f_u(v)\, d\nu(v)\Big)\, d\mu(u) - f(x,y)\Big\| \\
&\le \Big\|\frac{1}{\mu\big(K_m(x)\big)} \int_{K_m(x)} \Big(\frac{1}{\nu\big(J_n(y)\big)} \int_{J_n(y)} f_u(v)\, d\nu(v) - f_u(y)\Big)\, d\mu(u)\Big\| \\
&\qquad + \Big\|\frac{1}{\mu\big(K_m(x)\big)} \int_{K_m(x)} f_u(y)\, d\mu(u) - f(x,y)\Big\| \\
&\le \frac{1}{\mu\big(K_m(x)\big)} \int_{K_m(x)} \Big\|\frac{1}{\nu\big(J_n(y)\big)} \int_{J_n(y)} f_u(v)\, d\nu(v) - f_u(y)\Big\|\, d\mu(u) \\
&\qquad + \Big\|\frac{1}{\mu\big(K_m(x)\big)} \int_{K_m(x)} f^y(u)\, d\mu(u) - f^y(x)\Big\| \\
&\le \frac{1}{\mu\big(K_m(x)\big)} \frac{\varepsilon}{2} \mu\big(K_m(x)\big) + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

Remark 4. If $\mathcal{J} = \mathcal{K}$ is the net structure from Example 1 (for $m = 1$), $\mathcal{M}$ is the σ-field of Lebesgue measurable sets, $\mathcal{I}$ is the σ-ideal of null sets, and $\mu = \nu$ is the Lebesgue measure, then Corollary 3 and Theorem 4 are precisely the main results of [3].

References

[1] Bruckner, A. M., *Differentiation of integrals*, p. II, Amer. Math. Monthly **78**(9) (1971).
[2] Chmielewska, K., *On the strong product measurability*, Real Anal. Exchange **26**(1) (2000–01), 437–443.
[3] Grande, Z., *On equi-derivatives*, Real Anal. Exchange **21**(2) (1995–96), 637–647.
[4] Mišík, L., *Über den Mittelwertsatz für additive Zellenfunktionen*, Mat.-Fyz. Časopis. Sloven. Acad. Vied **15**(4) (1963), 260–274.
[5] Pauc, C., *Dérivées et Intégrales*: *Fonctions de cellule*, Conférence faite au Centre Mathématique International de Varenna, 15–25 Aug. 1954, pub. Mathematical Institute, Rome.
[6] Yosida, K., *Functional Analysis*, Springer–Verlag, Berlin-Heidelberg-New York, 1980.

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