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C^2 and G^2 continuous Spline Curves with Shape Parameters

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Abstract. Extensions of the classical B-spline curves by shape parameters are discussed in this paper. Using a method of linear blending, first we extend the abilities of GB-splines, originally defined by uniform shape parameter, to multiple shape parameters. Then two methods, having different types of continuity, are examined in terms of their curvature behavior.

Key Words: B-spline curve, shape parameter, C^2 continuity, G^2 continuity

MSC 2000: 68U05

1. Introduction

The classical B-spline curve is defined by its control points but otherwise it is a rigid curve. The generalizations of this curve, which allow some further shape control by shape parameters, are in the forefront of current research due to its importance in applications. Several authors attempted to extend the abilities of the B-spline curve by some shape parameter. Obviously the most known result of this attempt is the NURBS curve (c.f. [9]), but this curve has rational coefficient functions, thus alternative methods tried to incorporate shape parameters into the original, polynomial basis functions. One of the earliest methods in this way is the β-spline curve with two global parameters ([3], [1]). Further methods have been provided by direct generalization of B-spline curves as αB-splines in [8] and [11] and recently as GB-splines in [2]. Some alternative spline curves with shape parameters can be found in [5] and [6].

In this paper we examine and compare two generalizations, αB-spline curves and GB-spline curves. After defining and discussing the basic properties of these curves in Section 2 and 3, based on a linear blending approach we generalize the GB-spline curves for having multiple shape parameters. In Section 4 we focus on the continuity properties of these curves.

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We prove that, with certain restrictions, GB-spline curve with multiple shape parameters has second order geometric (G^2) continuity. By comparing the curvature plots of αB-spline and GB-spline curves we show an example where, in spite of the general supremacy of C^2 continuity, G^2 continuity yields better results in terms of smoothness, than the classical notion of continuity.

2. αB-spline curves

In [8] the *αB-spline curve* is introduced as a generalization of the B-spline curve with shape parameters. It is defined by a linear blending method, using the side of the control polygon as target curve, that is modifying the shape parameters the B-spline curve will move towards (or away from) the control polygon. The definition of the αB-spline curve with a single shape parameter is as follows (for the sake of simplicity, throughout the paper we define only a single arc of piecewise curves):

Definition 1 *Given a sequence of control points $\mathbf{p}_i$, $i = 0,\dots,3$, first consider the cubic uniform B-spline curve*

$$\mathbf{b}(t) = \sum_{i=0}^{3} N_i(t)\mathbf{p}_i\,, \quad t \in [0,1],$$

where $N_i(t)$ are the cubic B-spline basis functions. Also consider the line segment $\mathbf{p}_1\mathbf{p}_2$ parameterized by

$$\mathbf{l}(t) = (1 - s(t))\mathbf{p}_1 + s(t)\mathbf{p}_2, \quad t \in [0,1]$$

where

$$s(t) = (1 - t^3)^3. \tag{1}$$

Now the arc of the αB-spline curve $\mathbf{c}(t)$ is the linear blending of the polygon leg $\mathbf{p}_1\mathbf{p}_2$ and the original B-spline curve $\mathbf{b}(t)$ as

$$\mathbf{c}(\alpha,t) = (1-\alpha)\mathbf{b}(t) + \alpha\,\mathbf{l}(t), \quad t \in [0,1], \quad \alpha \in [0,1]$$

For αB-spline curves with *multiple* shape parameters an α_i is associated to the control points $\mathbf{p}_i$, $i = 1,2$, and the shape parameter α will be a function of t, interpolating the neighboring values α_1, α_2:

$$\alpha(t) = (1 - s(t))\alpha_1 + s(t)\alpha_2 \tag{2}$$

where $s(t) = (1 - t^3)^3$ is the same blending function that has been applied for the univariate case.

A αB-spline curve with multiple shape parameters has local tension control since α_i affect only the neighboring arcs.

3. GB-spline curves

Another generalization of the classical uniform cubic B-spline curve has been introduced in [2] and is called *GB-spline curve*. The definition of an arc of a GB-spline curve with shape parameter λ is as follows.

Definition 2 *Given a sequence of control points* $\mathbf{p}_i$, $i = 0, \ldots, 3$, *the arc of the GB-spline curve is*

$$\mathbf{c}(\lambda, t) = \sum_{i=0}^{3} B_i(\lambda, t)\mathbf{p}_i, \quad \lambda \in [0, \infty), \quad t \in [0, 1],$$

where the GB-spline basis functions are

$$\begin{aligned} B_0(\lambda, t) &= \frac{2}{12+\lambda}(1-t)^3 \\ B_1(\lambda, t) &= \frac{1}{12+\lambda}\left(2\,(3+\lambda)\,t^3 - 3\,(4+\lambda)\,t^2 + 8 + \lambda\right) \\ B_2(\lambda, t) &= \frac{1}{12+\lambda}\left(-2\,(3+\lambda)\,t^3 + 3\,(2+\lambda)\,t^2 + 6t + 2\right) \\ B_3(\lambda, t) &= \frac{2}{12+\lambda}t^3. \end{aligned}$$

At $\lambda = 0$ the GB-spline curve coincides with the original B-spline curve. Increasing the shape parameter λ the curve arc tends towards the corresponding control leg $\mathbf{p}_1\mathbf{p}_2$. In [7] it is proved, that considering a fixed point $\mathbf{c}(\lambda, t_0)$ of the curve and altering the shape parameter λ, the point will move along a straight line segment. One of the endpoints of this path is $\mathbf{c}(0, t_0)$, while the other point is

$$\lim_{\lambda\to\infty} \mathbf{c}(\lambda, t_0),$$

a point of the control leg $\mathbf{p}_1\mathbf{p}_2$. Although it is not obvious from the definition, GB-spline arc $\mathbf{c}(\lambda, t)$ can also be considered as a linear blending of the original B-spline curve $\mathbf{b}(t) = \mathbf{c}(0, t)$ and the parameterized control leg $\mathbf{p}_1\mathbf{p}_2$ (c.f. [7]), similar to the method we have seen in the case of an αB-spline curve. More precisely, one can consider the control leg as

$$\mathbf{l}(t) = s(t)\mathbf{p}_1 + (1 - s(t))\mathbf{p}_2 \tag{3}$$

where

$$s(t) = 2t^3 - 3t^2 + 1. \tag{4}$$

Applying this parametrization, the curve itself can be written in the form

$$\mathbf{c}(\lambda, t) = q(\lambda)\mathbf{c}(0, t) + (1 - q(\lambda))\,\mathbf{c}(\infty, t),$$

that is

$$\mathbf{c}(\lambda, t) = q(\lambda)\mathbf{b}(t) + (1 - q(\lambda))\,\mathbf{l}(t)$$

where

$$q(\lambda) = \frac{\lambda}{12+\lambda}.$$

4. GB-spline curves with multiple shape parameters

The linear blending description of GB-spline curves naturally yields the possibility of introducing multiple shape parameters, similarly to the case of αB-splines. Applying the basic idea we have seen in Section 2, the GB-spline curve with multiple shape parameters can also be established. Since the linear blending of the αB-spline and GB-spline curves differ only in the blending functions $s(t)$, applying function (4) instead of (1), one can interpolate the newly defined neighboring values of the shape parameters λ_1, λ_2 just as in eq. (2). That is given the shape parameters λ_1, λ_2 associated to the control points $\mathbf{p}_1, \mathbf{p}_2$, the shape parameter λ will be the function of t, interpolating the neighboring values λ_1, λ_2:

$$\lambda(t) = (1 - s(t))\lambda_1 + s(t)\lambda_2$$

where $s(t) = 2t^3 - 3t^2 + 1$. After some computation the GB-spline curve with multiple shape parameters and its basis functions can be described as follows:

$$\mathbf{g}(\lambda_1, \lambda_2, t) = \sum_{i=0}^{3} G_i(\lambda_1, \lambda_2, t)\mathbf{p}_i, \quad \lambda_i \in [0, \infty), \quad t \in [0, 1],$$

where

$$\begin{aligned}
G_0 &= \frac{2(1-t)^3}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1} \\
G_1 &= \frac{(\lambda_1 - \lambda_2)(4t^2 - 12t + 9)t^4 + 2(2\lambda_1 - \lambda_2 + 3)t^3 - 3(2\lambda_1 - \lambda_2 + 4)t^2 + 8 + \lambda_1}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1} \\
G_2 &= \frac{-(\lambda_1 - \lambda_2)(2t - 3)^2 t^4 - 2(3 + \lambda_1)t^3 + 3(2 + \lambda_1)t^2 + 6t + 2}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1} \\
G_3 &= \frac{2t^3}{12 + (\lambda_1 - \lambda_2)(2t^3 - 3t^2) + \lambda_1}.
\end{aligned}$$

5. Continuity at the joint of two arcs

So far we discussed one arc of the curves, defined by 4 control points. However, if the number of control points is $n > 4$, that is points $\mathbf{p}_i$, $i = 0, \ldots, n-1$, are given, then the cubic B-spline curve — as well as its generalizations — will contain $n - 2$ arcs, each defined by four consecutive control points. Consider two neighboring arcs of the B-spline curve, $\mathbf{b}_i(t)$ and $\mathbf{b}_{i+1}(t)$. It is a well-known fact (see, e.g., [9]), that these arcs have C^2 continuity in their point of contact, that is:

$$\frac{d^r}{dt^r}\mathbf{b}_i(1) = \frac{d^r}{dt^r}\mathbf{b}_{i+1}(0), \quad r = 0, 1, 2.$$

In case of generalizations of the B-spline curve it is computed in [8], that αB-spline arcs also possess C^2 continuity at their joint. Moreover, it is proved in [10], that generalizations by linear blending will have C^2 continuity at the point of contact of neighboring arcs if the degree of the blending polynomial $s(t)$ is at least 5. The only quintic polynomial which fulfills this criteria is (c.f. [4])

$$s(t) = (1-t)^3 \left(6t^2 + 3t + 1\right).$$

Thus the modified version of an αB-spline curve ([10]) applies this blending function instead of the original function (1), still preserving C^2-continuity.

This fact immediately yields, that the other generalization, the GB-spline curve has no C^2 continuity at the joint of the consecutive arcs $\mathbf{g}_i(t)$ and $\mathbf{g}_{i+1}(t)$, since the blending function (4) is of degree 3. However it is proved in [2] that GB-spline curve fulfills the criteria of second order *geometric continuity* (or G^2) that is, for some constants c_1 and c_2 the following equations hold:

$$\mathbf{g}_i(1) = \mathbf{g}_{i+1}(0) \tag{5}$$

$$\frac{d}{dt}\mathbf{g}_i(1) = c_1\frac{d}{dt}\mathbf{g}_{i+1}(0) \tag{6}$$

$$\frac{d^2}{dt^2}\mathbf{g}_i(1) = {c_1}^2\frac{d^2}{dt^2}\mathbf{g}_{i+1}(0) + c_2\frac{d}{dt}\mathbf{g}_{i+1}(0). \tag{7}$$

Here we prove that under some restriction the GB-spline curve with multiple shape parameters has also G^2 continuity.

Theorem 1 *Considering two consecutive arcs $\mathbf{g}_i(\lambda_i, \lambda_{i+1}, t)$ and $\mathbf{g}_{i+1}(\lambda_{i+1}, \lambda_{i+2}, t)$ of a GB-spline curve with multiple shape parameters, their joint is G^2 continuous, if $\lambda_i = \lambda_{i+2}$, while λ_{i+1} is of arbitrary value.*

Proof. By simple substitution one can see that criteria (5) is fulfilled. The first derivatives of the arcs are

$$\begin{aligned}\frac{d}{dt}\mathbf{g}_i(\lambda_i, \lambda_{i+1}, 1) &= \frac{-6}{12+\lambda_{i+1}}\mathbf{p}_i + \frac{6}{12+\lambda_{i+1}}\mathbf{p}_{i+2}\\ \frac{d}{dt}\mathbf{g}_{i+1}(\lambda_{i+1}, \lambda_{i+2}, 0) &= \frac{-6}{12+\lambda_{i+1}}\mathbf{p}_i + \frac{6}{12+\lambda_{i+1}}\mathbf{p}_{i+2},\end{aligned}$$

that is, criteria (6) also holds with $c_1 = 1$.

The second derivatives of the arcs are computed as

$$\begin{aligned}\frac{d^2}{dt^2}\mathbf{g}_i(\lambda_i, \lambda_{i+1}, 1) &= \left(\frac{12+6\lambda_{i+1}}{12+\lambda_{i+1}} - \frac{12(\lambda_i-\lambda_{i+1})}{(12+\lambda_{i+1})^2}\right)\mathbf{p}_i\\ &+ \left(\frac{-6(2\lambda_{i+1}-\lambda_i+4)}{12+\lambda_{i+1}} - \frac{6(\lambda_{i+1}+8)(\lambda_i-\lambda_{i+1})}{12+\lambda_{i+1}}\right)\mathbf{p}_{i+1}\\ &+ \left(\frac{12}{12+\lambda_{i+1}} - \frac{12(\lambda_i-\lambda_{i+1})}{(12+\lambda_{i+1})^2}\right)\mathbf{p}_{i+2}\\ \frac{d^2}{dt^2}\mathbf{g}_{i+1}(\lambda_{i+1}, \lambda_{i+2}, 0) &= \left(\frac{12}{12+\lambda_{i+1}} + \frac{12(\lambda_{i+1}-\lambda_{i+2})}{(12+\lambda_{i+1})^2}\right)\mathbf{p}_i\\ &+ \left(\frac{-12\lambda_{i+1}+6\lambda_{i+2}-24}{12+\lambda_{i+1}} + \frac{6(8+\lambda_{i+1})(\lambda_{i+1}-\lambda_{i+2})}{(12+\lambda_{i+1})^2}\right)\mathbf{p}_{i+1}\\ &+ \left(\frac{12+6\lambda_{i+1}}{12+\lambda_{i+1}} + \frac{12(\lambda_{i+1}-\lambda_{i+2})}{(12+\lambda_{i+1})^2}\right)\mathbf{p}_{i+2}.\end{aligned}$$

Since $c_1 = 1$, criteria (7) is of the form

$$\frac{d^2}{dt^2}\mathbf{g}_i(1) = \frac{d^2}{dt^2}\mathbf{g}_{i+1}(0) + c_2\frac{d}{dt}\mathbf{g}_{i+1}(0)$$

which has a unique solution as $\lambda_i = \lambda_{i+2}$ and $c_2 = \lambda_{i+1}$, and this was to be proved. □

Corollary 1 *Considering n consecutive arcs $\mathbf{g}_i(\lambda_i, \lambda_{i+1}, t)$, $i = 1, \ldots, n$, of a GB-spline curve, their joints are G^2 continuous if $\lambda_i = \lambda_{i+2}$ holds for all $i = 1, \ldots, n-1$.*

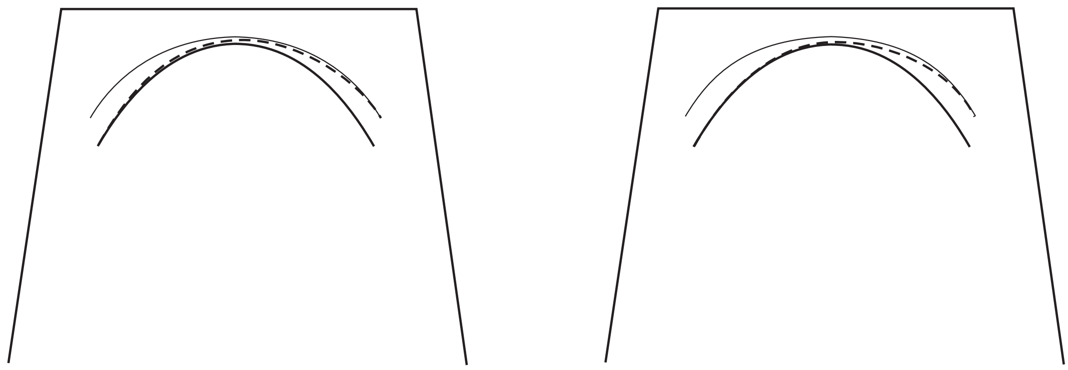

Figure 1: Two versions of spline curves with shape parameters.
Thick line: the classical B-spline curve;
Thin line: the GB-spline curve (left) and the αB-spline curve (right) with uniform shape parameter;
Dashed line: the GB-spline curve (left) and the αB-spline curve (right) with multiple shape parameters.

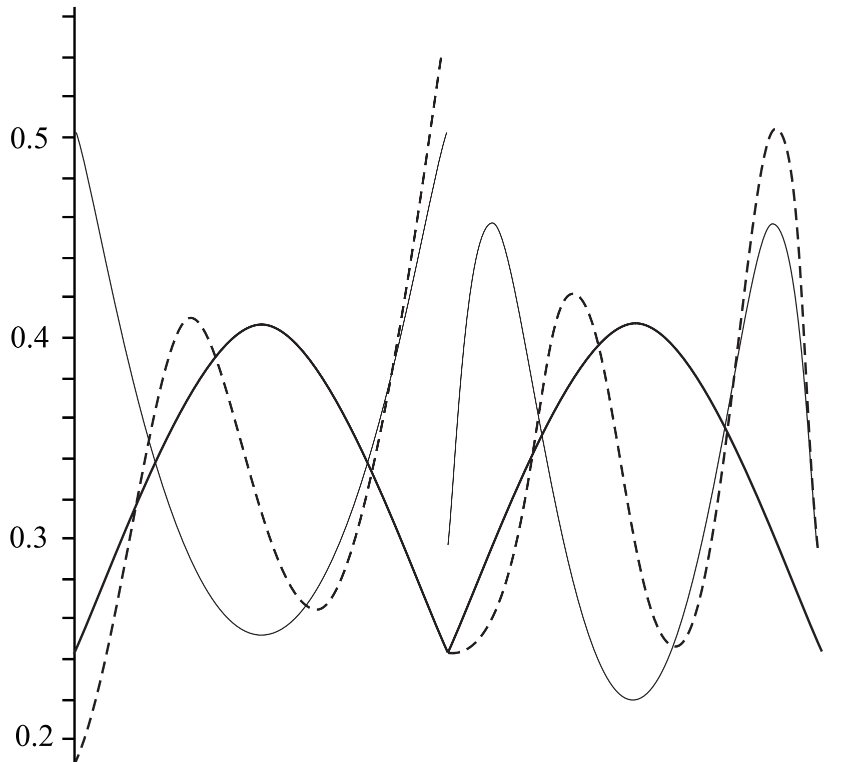

Figure 2: Curvature plot of the curves from Fig. 1. (All plots are stretched vertically and plots of the αB-spline curves are shifted towards right for better visualization)

The continuity properties of the two types of spline curves may suggest that an αB-spline curve has smoother shape due to its more rigorous continuity. This fact, however, does not necessarily hold as one can see in the following figures of a case study. Applying the GB-spline curve with multiple shape parameters, both spline curves have the same freedom in terms of design and shape control. In Fig. 1 one can see two sets of curves defined by the same control polygon. The thick curve is the classical B-spline curve, the thin ones are the generalized spline curves with uniform shape parameter (the GB-spline curve at left and the αB-spline curve at right) in a visually similar position. The dashed curves are the curves with multiple shape parameters, starting from the classical B-spline curve and ending at the modified curve. Now as one can observe in Fig. 2, although visually there is not much difference between the two curve types, the curvature plot shows, that the strong requirement of C^2 continuity yields unnecessary, heavy oscillations in curvature plot, while the GB-spline curve with G^2 continuity has a much smoother plot. This fact somehow shows that weaker conditions sometimes yield more acceptable results in practical geometric modeling.

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