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Creating Ruled Surfaces Using the Base Curve of the Frenet Trihedron

Serhiy Pylypaka[1], Andrii Nesvidomin[1], Tetiana Volina[1], Oleksandra Trokhaniak[1], Iryna Taras[2]

[1] *University of Life and Environmental Sciences of Ukraine, Kyiv, Ukraine*
ps55@ukr.net, a.nesvidomin@gmail.com, t.n.zaharova@ukr.net, klendii_o@ukr.net

[2] *Ivano-Frankivsk National Technical University of Oil and Gas, Ivano-Frankivsk, Ukraine*
iryna.taras@nung.edu.ua

Abstract. A ruled surface is formed by moving a straight line (as a generator) along a base curve. If the straight line's direction remains constant in a fixed coordinate system, the resulting surface will be cylindrical. In the general case, the direction of the generator can be described by a unit vector in the projections on the coordinate axis and will vary depending on the point on the base curve. This surface can be defined by a grid of both rectangular and oblique coordinate lines. Additionally, a ruled surface can be either developable or non-developable. In the general case, the surface is associated with an oblique grid of coordinate lines and is non-developable. In order to obtain specific cases, it is necessary to impose restrictions on the direction of the unit vector of the straight line.

The article describes a method for creating ruled surfaces using the Frenet trihedron of the base curve. It explores cases where a straight line rotates within the accompanying trihedron of the base curve or remains fixed in place. The article also establishes conditions for non-developable surfaces when the straight line rotates. It provides the parametrization of the ruled surfaces and their first quadratic forms based on the Frenet trihedron. Additionally, the article presents examples of ruled surfaces and provides visual representations.

Key Words: Frenet formulas, curvature, torsion, vector surface equation, first quadratic form

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1 Introduction

Ruled surfaces are widely used because they can be easily manufactured and can create various forms, especially when constructing screw surfaces for transporting materials [4, 5, 7, 12]. In

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[6], their connection to science, nature, and architectural structures is explored. Various methods of creating them and studying their properties are discussed in existing literature [1, 2, 8–10, 13]. The conditions under which the ruled surface will be non-developable are also examined [3]. Additionally, projective geometry methods are used in their comprehensive study [14]. A related paper to our research is [11], where the surface properties are considered from the perspective of inner geometry.

A ruled surface is created by moving a straight line (as a generator) along a directional curve with the direction of the straight line depending on the point on the curve. The parametrization of the ruled surface formed in this manner takes the following form:

$$\overline{R} = \overline{r}(v) + u\overline{w}(v), \tag{1}$$

where $\overline{r}(v)$ denotes the parametrization of the directional curve specified as a function of the independent variable v;

$\overline{w}(v)$ represents a single direction vector that specifies the direction of the rectilinear surface;

u stands for the second variable of the surface, serving as the length of the straight line, starting the count from the point on the base curve.

We will use a spatial curve on a cylinder with a unit radius as our base curve. The parametrization of this curve in terms of the arc length s was obtained in [10]. In order to simplify the parametrization, we will change the variable from s to a new variable v by introducing the substitution $v = \operatorname{Arcsinh} s$. After making this substitution, the equation of the curve can be expressed as:

$$x = \cos v; \qquad y = \sin v; \qquad z = \cosh v. \tag{2}$$

For further research, we need to find the first, second, and third derivatives of the equations of the curve (2):

$$x' = -\sin v; \qquad y' = \cos v; \qquad z' = \sinh v; \tag{3}$$

$$x'' = -\cos v; \qquad y'' = -\sin v; \qquad z'' = \cosh v; \tag{4}$$

$$x''' = \sin v; \qquad y''' = -\cos v; \qquad z''' = \sinh v. \tag{5}$$

Using the given derivatives, we can calculate the curvature k and torsion σ of curve (2):

$$k = \frac{\sqrt{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}^2 + \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^2 + \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^2}}{(x'^2 + y'^2 + z'^2)^{3/2}} = \frac{\sqrt{2}}{\cosh v}, \tag{6}$$

$$\sigma = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}^2 + \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}^2 + \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^2} = \frac{\sinh v}{\cosh^2 v}. \tag{7}$$

In the future, we will construct a ruled surface using the Frenet trihedron of the base curve (2). To do this, it is necessary to determine the orientation of the trihedron in the Cartesian coordinate system $OXYZ$. This is accomplished using directional cosines. We denote the angles formed by the tangent of the trihedron with the axes OX, OY and OZ axes as α_τ, β_τ, γ_τ respectively. Similarly, we use α_n, β_n, γ_n for the principal normal $\overline{n}$ and α_b, β_b, γ_b for the binormal $\overline{b}$. To find the direction cosines of these angles, we introduce the following notation:

$$A = y'z'' - y''z'; \quad B = z'x'' - z''x'; \quad C = x'y'' - x''y'. \tag{8}$$

Let's find the direction cosines of the curve (2):

$$\begin{aligned} \cos\alpha_\tau &= \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} = -\frac{\sin v}{\cosh v}; \\ \cos\beta_\tau &= \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} = \frac{\cos v}{\cosh v}; \\ \cos\gamma_\tau &= \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} = \tanh v. \end{aligned} \tag{9}$$

$$\begin{aligned} \cos\alpha_n &= \frac{Bz' - Cy'}{\sqrt{(x'^2 + y'^2 + z'^2)(A^2 + B^2 + C^2)}} = \frac{\sin v \tanh v - \cos v}{\sqrt{2}}; \\ \cos\beta_n &= \frac{Cx' - Az'}{\sqrt{(x'^2 + y'^2 + z'^2)(A^2 + B^2 + C^2)}} = -\frac{\cos v \tanh v + \sin v}{\sqrt{2}}; \\ \cos\gamma_n &= \frac{Ay' - Bx'}{\sqrt{(x'^2 + y'^2 + z'^2)(A^2 + B^2 + C^2)}} = \frac{1}{\sqrt{2}\cosh v}. \end{aligned} \tag{10}$$

$$\begin{aligned} \cos\alpha_b &= \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \frac{\sin v \tanh v + \cos v}{\sqrt{2}}; \\ \cos\beta_b &= \frac{B}{\sqrt{A^2 + B^2 + C^2}} = \frac{-\cos v \tanh v + \sin v}{\sqrt{2}}; \\ \cos\gamma_b &= \frac{C}{\sqrt{A^2 + B^2 + C^2}} = \frac{1}{\sqrt{2}\cosh v}. \end{aligned} \tag{11}$$

If in the trihedron system with unit vectors denoting the projections of the unit vector as $\overline{\tau}$, $\overline{n}$, $\overline{b}$, then its projections in the $OXYZ$ system will be written through direction cosines as follows:

$$\begin{aligned} w_x &= w_\tau \cos\alpha_\tau + w_n \cos\alpha_n + w_b \cos\alpha_b; \\ w_y &= w_\tau \cos\beta_\tau + w_n \cos\beta_n + w_b \cos\beta_b; \\ w_z &= w_\tau \cos\gamma_\tau + w_n \cos\gamma_n + w_b \cos\gamma_b. \end{aligned} \tag{12}$$

2 Generator in the Direction Plane of the Frenet Trihedron

In Fig. 1 the accompanying trihedron $\overline{\tau}$, $\overline{n}$, $\overline{b}$ at point A of the base curve is depicted. A vector $\overline{w}$ is located in the directional plane $\overline{\tau}$, $\overline{b}$, and its projections on the corresponding coordinates are marked. The position of the vector $\overline{w}$ can also be specified by the angle γ, which starts from the orthogonal point $\overline{\tau}$. The vector $\overline{w}$ determines the position of the straight line in the system of the trihedron, passing through its vertex, point A. As the trihedron moves along the base curve, the angle γ can be constant or change, depending on the position of point A on the curve. In this way, it is possible to form different ruled surfaces, which are united

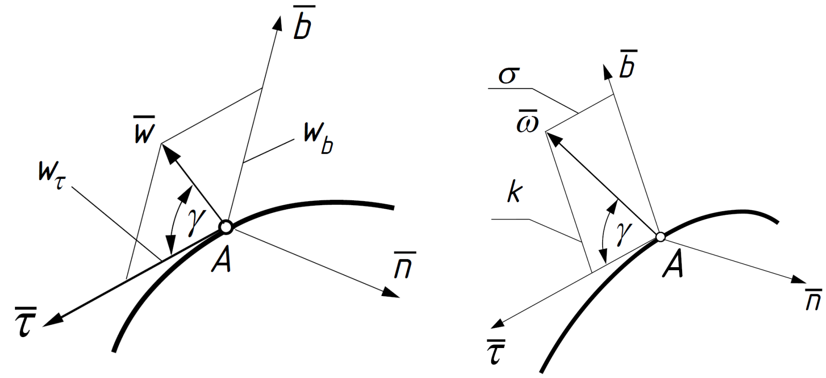

Figure 1: Frenet trihedron of a directional curve with a vector in the directional plane: a) a unit vector that specifies the direction of the rectilinear surface; b) Darboux vector of the instantaneous axis of rotation of the trihedron

by one common property. This property is that the directional plane will be tangent to the surface, as it contains a straight line and is tangent to the directional curve. This means that the normal to the surface coincides with the principal normal of the curve, that is, the unit vector $\overline{n}$. This condition is sufficient for the base curve to be a geodesic for a ruled surface. In the case where $\gamma = 0$, the tangent (unit vector $\overline{\tau}$) and the derivative coincide. The tangent plane becomes undefined, and the surface itself will be warped, with the base curve being the edge of the return.

The projections of the unit vector $\overline{w}$ in the trihedron system (Fig. 1) are expressed as follows:

$$w_\tau = \cos\gamma; \quad w_n = 0; \quad w_b = \sin\gamma. \tag{13}$$

To find the direction cosines of the rectilinear derivative in the Cartesian coordinate system $OXYZ$, it is essential to substitute Equation (13) into Equation (12). Then the parametrization (1) of the ruled surface in projections on the axes of this system can be written:

$$\begin{aligned}
X &= \cos v + u\Big(-\cos\gamma\frac{\sin v}{\cosh v} + \sin\gamma\tfrac{\sin v\tanh v+\cos v}{\sqrt{2}}\Big);\\
Y &= \sin v + u\Big(\cos\gamma\frac{\cos v}{\cosh v} - \sin\gamma\frac{\cos v\tanh v - \sin v}{\sqrt{2}}\Big);\\
Z &= \cosh v + u\Big(\cos\gamma\tanh v + \frac{\sin\gamma}{\sqrt{2}\cosh v}\Big).
\end{aligned} \tag{14}$$

Example

Based on Equations 14 in Fig. 2, ruled surfaces are created at a constant angle γ. The limits for the changing parameters v and u are as follows: $v = -\pi/2\ldots\pi/2$; $u = -1\ldots1$. When $\gamma = 0$, the surface is non-developable, with the base curve being the turning edge (Fig. 2). When $\gamma = \pi/2$, the ruled surface forms a surface of binormals (Fig. 2).

Fig. 3 shows constructed surfaces with variable angle γ. The angle dependence $\gamma = \gamma(v)$ can be chosen arbitrarily or under a certain condition, such as when the surface is non-developable. In Fig. 3, a surface is constructed based on the dependence $\gamma = 0.5\pi + 0.25v$, and in Fig. 3 the surface is constructed under the condition that it is non-developable. It is evident from the figure that all straight lines intersect at a common point, indicating that the

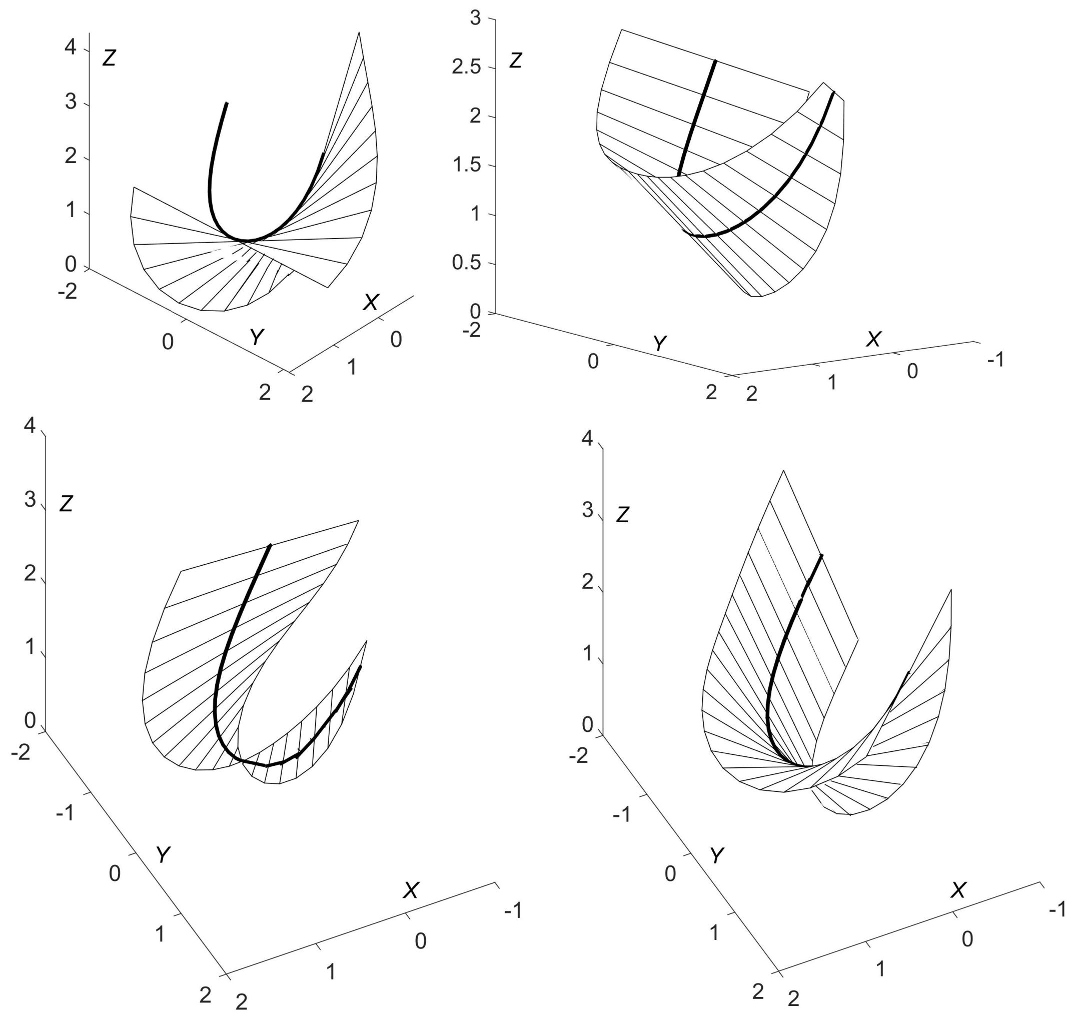

Figure 2: Ruled surfaces in which straight lines are located in the reference plane and intersect the reference curve at a constant angle γ: a) $\gamma = 0$; b) $\gamma = \pi/2$; c) $\gamma = \pi/4$; d) $\gamma = -\pi/4$.

resulting surface is conical. The next subsection will consider the condition for determining the dependence $\gamma = \gamma(v)$ for which the ruled surface will be developable.

3 Construction of the Non-Developable Surface With a Base Curve that is a Geodesic Line

A ruled surface is expansive if the mixed product of vectors equals zero:

$$\overline{r}'\overline{w}\,\overline{w}' = 0, \tag{15}$$

where $\overline{r}'$ is the tangent vector to the base curve; $\overline{w}$, $\overline{w}'$ is the normal vector to the surface and its derivative vector.

In order to derive a generalized result taking into account the curvature and torsion of the directional curve, it is necessary to calculate the mixed product in the Frenet trihedron system. We have a vector $\overline{r}' = \overline{\tau}$, and its projections in the trihedron system can be represented as: $\{1, 0, 0\}$. The projections of the vector $\overline{w}$ are determined by the Fig. 1 and can be written as:

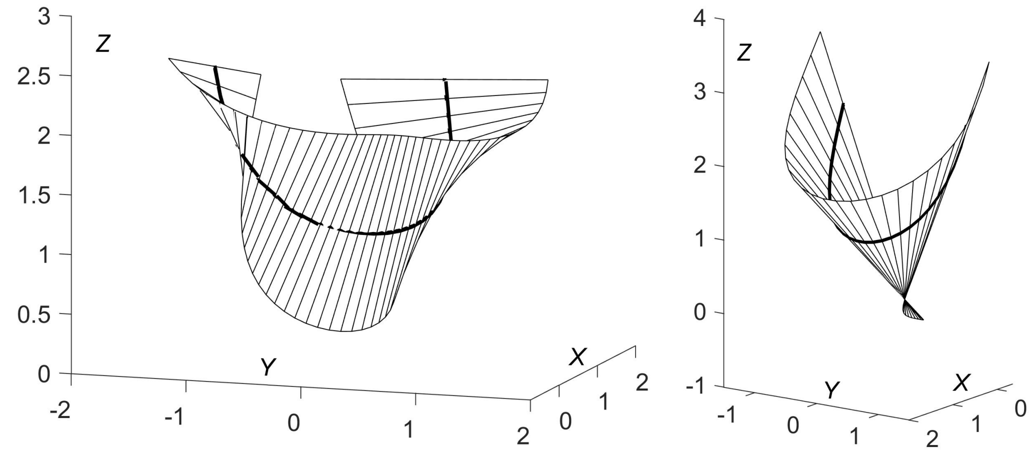

Figure 3: A ruled surface is defined by straight lines located in the direction plane and crossing the direction curve according based on the given dependency $\gamma = \gamma(v)$: a) $\gamma = 0.5\pi + 0.25v$; b) the dependence $\gamma = \gamma(v)$ is defined under the condition that the ruled surface is non-developable.

$\{\cos\gamma, 0, \sin\gamma\}$. Let's express this in vector form and then proceed with differentiation:

$$\overline{w} = \overline{\tau}\cos\gamma + \overline{b}\sin\gamma, \tag{16}$$

$$\overline{w}' = \overline{\tau}'\cos\gamma - \overline{\tau}\gamma'\sin\gamma + \overline{b}'\sin\gamma + \overline{b}\gamma'\cos\gamma. \tag{17}$$

Next, it is essential to use Frenet's formulas:

$$\overline{\tau}' = \overline{n}k; \quad \overline{n}' = -\overline{\tau}k + \overline{b}\sigma; \quad \overline{b}' = -\overline{n}\sigma. \tag{18}$$

Substitute formulas (18) into (17), group the expressions according to the directions of the unit vectors and finally obtain:

$$\overline{w}' = -\overline{\tau}\gamma'\sin\gamma + \overline{n}(k\cos\gamma - \sigma\sin\gamma) + \overline{b}\gamma'\cos\gamma. \tag{19}$$

The mixed product (15) will be written:

$$\begin{vmatrix} 1 & 0 & 0 \\ \cos\gamma & 0 & \sin\gamma \\ -\gamma'\sin\gamma & k\cos\gamma - \sigma\sin\gamma & \gamma'\cos\gamma \end{vmatrix} = \sin\gamma(\sigma\sin\gamma - k\cos\gamma) = 0. \tag{20}$$

Based on the result of the mixed product (20), there are two possible cases:

Case 1: If $\sin\gamma = 0$, in this scenario, the straight lines align with the unit vector $\overline{\tau}$, therefore, for the non-developable surface, the base curve will be the edge of the return;

Case 2:

$$\sin\gamma = \frac{k}{\sqrt{k^2 + \sigma^2}}; \quad \cos\gamma = \frac{\sigma}{\sqrt{k^2 + \sigma^2}}. \tag{21}$$

In this case, the projections of the unit vector of the straight line are determined by the curvature and torsion of the directional curve (Fig. 1). This vector is called the Darboux vector. The vector $\overline{\omega}$ represents the axis of rotation of the trihedron as it moves along the base curve. It's worth noting that in Frenet's formulas (18), the independent variable is the length s of the directional curve, which will need to be considered when necessary.

4 First Quadratic Form of the Non-Developable Surface when the Base Curve is a Geodesic Line

In order for Equations (14) be applicable to the non-developable surface, it is necessary to use the derived dependencies (21). It is necessary to substitute the expressions for curvature (6) and torsion (7) into dependence (21):

$$\sin\gamma = \frac{2}{\sqrt{3+\cosh 2v}}; \quad \cos\gamma = \frac{\sinh v}{\sqrt{2+\sinh^2 v}}. \tag{22}$$

After substituting expressions (22) into equation (14) and solving for the parametrization of the non-developable surface:

$$\begin{aligned} X &= \cos v + u\frac{\sqrt{2}\cos v}{\sqrt{3+\cosh 2v}}; \\ Y &= \sin v + u\frac{\sqrt{2}\sin v}{\sqrt{3+\cosh 2v}}; \\ Z &= \cosh v + u\frac{\sqrt{2}\cosh v}{\sqrt{3+\cosh 2v}}. \end{aligned} \tag{23}$$

The surface described by Equations (23) is visualized in Fig. 3.

We will transition from the independent variable v to the natural parameter s which represents the length of the arc of the base curve in Equations (23). To achieve this, we will substitute v with s in Equations (23): $v = \operatorname{Arcsinh} s$. Next, we will calculate the partial derivatives and the coefficients of the first quadratic form using the symbolic mathematics package in the software "Mathematica". The resulting expressions are complex, so we will only provide the final outcome:

$$\begin{aligned} E &= \left(\tfrac{\partial X}{\partial u}\right)^2 + \left(\tfrac{\partial Y}{\partial u}\right)^2 + \left(\tfrac{\partial Z}{\partial u}\right)^2 = 1; \\ F &= \tfrac{\partial X}{\partial u}\cdot\tfrac{\partial X}{\partial s} + \tfrac{\partial Y}{\partial u}\cdot\tfrac{\partial Y}{\partial s} + \tfrac{\partial Z}{\partial u}\cdot\tfrac{\partial Z}{\partial s} = \tfrac{s}{\sqrt{2+s^2}}; \\ G &= \left(\tfrac{\partial X}{\partial s}\right)^2 + \left(\tfrac{\partial Y}{\partial s}\right)^2 + \left(\tfrac{\partial Z}{\partial s}\right)^2 = 1 + \tfrac{2u(2\sqrt{2+s^2}+u)}{(2+s^2)^2}. \end{aligned} \tag{24}$$

Let's determine the coefficients of the first quadratic form in the Frenet trihedron system. The parametrization (1) for a ruled surface, where the straight lines are located in the reference plane and intersect the base curve at an angle $\gamma = \gamma(s)$ (Fig. 1) in the system of the trihedron, is given by:

$$\overline{R} = \overline{r}(s) + u(\overline{\tau}\cos\gamma + \overline{b}\sin\gamma). \tag{25}$$

We find the partial derivative surfaces (25). When finding the derivative with respect to the variable s we use Frenet's formulas (18) and then group by unit vector. Here is the finished result:

$$\begin{aligned} \tfrac{\partial\overline{R}}{\partial u} &= \overline{\tau}\cos\gamma + \overline{b}\sin\gamma; \\ \tfrac{\partial\overline{R}}{\partial s} &= \overline{\tau}(1 - u\gamma'\sin\gamma) + \overline{n}u(k\cos\gamma - \sigma\sin\gamma) + \overline{b}u\gamma'\cos\gamma. \end{aligned} \tag{26}$$

We determine the coefficients of the first quadratic form:

$$\begin{aligned} E &= \left(\tfrac{\partial \overline{R}}{\partial u}\right)^2 = \cos^2\gamma + \sin^2\gamma = 1; \\ F &= \tfrac{\partial \overline{R}}{\partial u} \cdot \tfrac{\partial \overline{R}}{\partial s} = \cos\gamma(1 - u\gamma' \sin\gamma) + u\gamma' \cos\gamma \sin\gamma = \cos\gamma; \\ G &= (\tfrac{\partial \overline{R}}{\partial s})^2 = (1 - u\gamma' \sin\gamma)^2 + u^2(k\cos\gamma - \sigma\sin\gamma)^2 \\ &\quad + u^2\gamma'^2\cos^2\gamma = 1 - 2u\gamma'\sin\gamma + u^2\gamma'^2 + u^2(k\cos\gamma - \sigma\sin\gamma)^2. \end{aligned} \tag{27}$$

Let's now consider the coefficient $F = \cos\gamma$ (27) in the expression that depends on the variable s. To do this, we substitute $v = \text{Arcsinh}\, s$ in expressions (22) and after simplifications, we obtain:

$$\sin\gamma = \frac{\sqrt{2}}{\sqrt{2+s^2}}; \quad \cos\gamma = \frac{s}{\sqrt{2+s^2}}. \tag{28}$$

The second expression (28) indicates that the coefficients F are identical in (24) and (27). We will also demonstrate a comprehensive similarity of the coefficient G in expressions (24) and (27). To accomplish this, we provide the required expressions in terms of the length of the arc s:

$$k = \frac{\sqrt{2}}{1+s^2}; \quad \sigma = \frac{s}{1+s^2}; \quad \gamma' = -\frac{\sqrt{2}}{2+s^2}. \tag{29}$$

After substituting expressions (28) and (29) into the expression of the coefficient G (27) and simplifying, we obtain an expression similar to the one for the coefficient G (24). This demonstrates the consistency of the equations for the same surface in the Cartesian coordinate system $OXYZ$ and the system of the accompanying trihedron of the base curve.

5 Generator in the Normal Plane of the Frenet Trihedron

According to Fig. 4 parametrization of the ruled surface will be written as follows:

$$\overline{R} = \overline{r}(s) + u(\overline{n}\cos\varepsilon + \overline{b}\sin\varepsilon). \tag{30}$$

We find the partial derivatives:

$$\begin{aligned} \tfrac{\partial \overline{R}}{\partial u} &= \overline{n}\cos\varepsilon + \overline{b}\sin\varepsilon; \\ \tfrac{\partial \overline{R}}{\partial s} &= \overline{\tau}(1 - uk\cos\varepsilon) - \overline{n}u(\varepsilon' + \sigma)\sin\varepsilon + \overline{b}u(\varepsilon' + \sigma)\cos\varepsilon. \end{aligned} \tag{31}$$

We find the coefficients of the first quadratic form:

$$E = 1; \quad F = 0; \quad G = (1 - uk\cos\varepsilon)^2 + u^2(\varepsilon' + \sigma)^2. \tag{32}$$

The coefficient F is zero, so the surface refers to a rectangular grid of coordinate lines.

In order to construct the surface, it is necessary to convert the parametrization (30) from the trihedron system to the Cartesian system $OXYZ$. By using equations (12), the unit vector of the straight line in the $OXYZ$ system can be determined. Keep in mind that in the trihedron system, it has coordinates $\{0; \cos\varepsilon; \sin\varepsilon\}$. Then, one has an opportunity to express the parametrization of the surface:

$$\begin{aligned} X &= \cos v + \tfrac{u}{\sqrt{2}}[(\sin\varepsilon - \cos\varepsilon)\cos v + (\sin\varepsilon + \cos\varepsilon)\sin v \tanh v]; \\ Y &= \sin v + \tfrac{u}{\sqrt{2}}[(\sin\varepsilon - \cos\varepsilon)\sin v - (\sin\varepsilon + \cos\varepsilon)\cos v \tanh v]; \\ Z &= \cosh v + u\,\frac{\sin\varepsilon + \cos\varepsilon}{\sqrt{2}\cosh v}. \end{aligned} \tag{33}$$

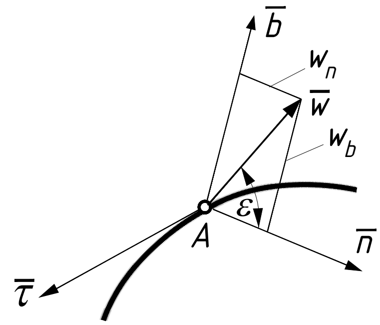

Figure 4: Diagram showing the location of the unit vector, specifying the direction of the straight line in the normal plane of the trihedron

Let's determine the condition under which the ruled surface (33) will not be developable. The mixed product (15) in the trihedron system will be expressed as follows:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ -k\cos\varepsilon & -(\varepsilon'+\sigma)\sin\varepsilon & (\varepsilon'+\sigma)\cos\varepsilon \end{vmatrix} = \varepsilon' + \sigma = 0. \tag{34}$$

Using Equation (34), the rate at which the angle ε changes can be determined:

$$\varepsilon = -\int \sigma\,\mathrm{d}s + \varepsilon_0, \tag{35}$$

where ε_0 is a constant of integration (initial reference angle from the unit vector $\overline{n}$).

Taking into account the fact that $\mathrm{d}s = \cosh\alpha\,\mathrm{d}\alpha$ and given the expression for the torsion σ in (7), we can analyze the regularity of the angle change ε with respect to the variable v using (35):

$$\varepsilon = -\int \frac{\sinh v}{\cosh^2 v}\cosh v\,\mathrm{d}v = \varepsilon_0 - \ln(\cosh v). \tag{36}$$

Equations (33) make it possible to create ruled surfaces linked to a rectangular grid of coordinate lines, both at a constant and variable angle ε. When $\varepsilon = 0$, we obtain the surface of principal normals, and when $\varepsilon = \pi/2$ we get the surface of binormals. This is depicted in Fig. 2 based on Equations (14), as they become completely identical when $\gamma = \pi/2$ and Equation (33) when $\varepsilon = \pi/2$.

Example

Equations (33) allow us to create ruled surfaces for a rectangular grid of coordinate lines, both at a constant (Fig. 5) and variable (Fig. 6) angle ε.

The line of curvature for non-developable surfaces is a direction curve. Straight lines for non-developable surfaces and all curves perpendicular to them are also lines of curvature.

6 Generator in the Tangent Plane of the Frenet Trihedron

The parametrization of the ruled surface based on Fig. 7 is as follows:

$$\overline{R} = \overline{r}(s) + u(\overline{\tau}\cos\gamma + \overline{n}\sin\gamma g). \tag{37}$$

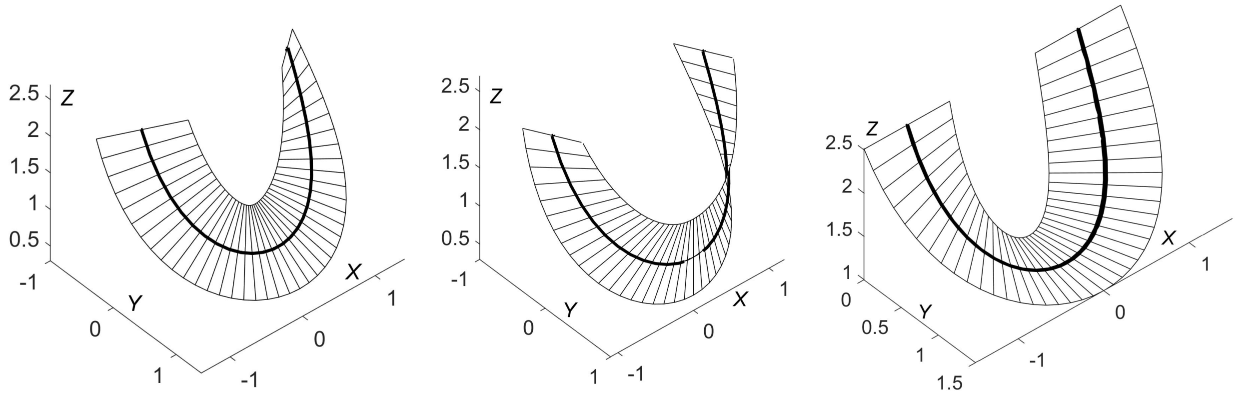

Figure 5: Ruled surfaces in which straight lines are located in a normal plane with a constant angle of rotation ε relative to the principal normal: a) $\varepsilon = 0$; b) $\varepsilon = \pi/4$; c) $\varepsilon = -\pi/4$.

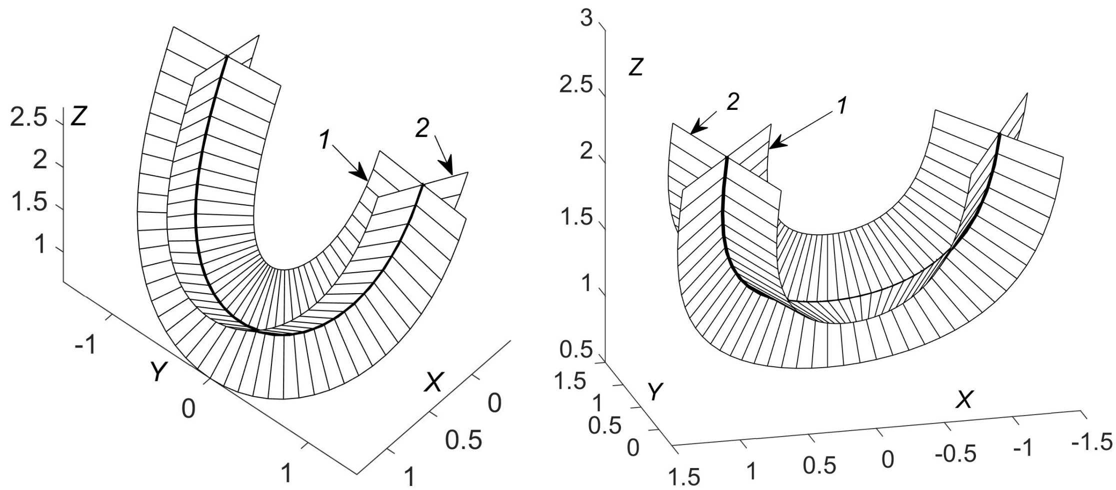

Figure 6: Non-developable surfaces in which straight lines are located in a normal plane with different angles ε_o: a) 1 corresponds to $\varepsilon_o = 0$; 2 corresponds to $\varepsilon_o = \pi/2$; b) 1 corresponds to $\varepsilon_o = \pi/4$; 2 corresponds to $\varepsilon_o = -\pi/4$

The partial derivatives will be written:

$$\begin{aligned} \frac{\partial \overline{R}}{\partial u} &= \overline{\tau}\cos\gamma + \overline{n}\sin\gamma; \\ \frac{\partial \overline{R}}{\partial s} &= \overline{\tau}[1 - u(\gamma' + k)\sin\gamma] + \overline{n}u(\gamma' + k)\cos\gamma + \overline{b}u\sigma\sin\gamma. \end{aligned} \tag{38}$$

The coefficients of the first quadratic form:

$$E = 1; \quad F = \cos\gamma; \quad G = 1 - 2u(\gamma' + k)\sin\gamma + u^2(\gamma' + k)^2 + u^2\sigma^2\sin^2\gamma. \tag{39}$$

Similarly, as in the previous cases, we determine the parametrization of the surface:

$$\begin{aligned} X &= \cos v + u\Big[\tfrac{\sin\gamma}{\sqrt{2}}(\sin v\tanh v - \cos v) - \cos\gamma\tfrac{\sin v}{\cosh v}\Big]; \\ Y &= \sin v - u\Big[\tfrac{\sin\gamma}{\sqrt{2}}(\cos v\tanh v + \sin v) - \cos\gamma\tfrac{\cos v}{\cosh v}\Big]; \\ Z &= \cosh v + u\Big(\tfrac{\sin\gamma}{\sqrt{2}\cosh v} + \cos\gamma\tanh v\Big). \end{aligned} \tag{40}$$

The mixed product (15) in the Frenet trihedron system indicates that a non-developable surface is possible when $\gamma = 0$, meaning the base curve will be the edge of the return for it.

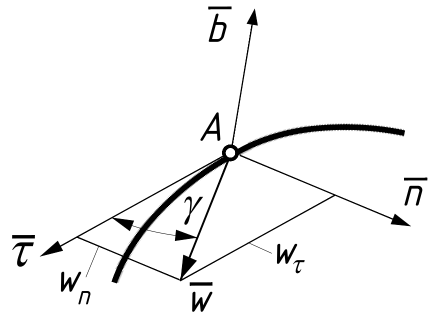

Figure 7: The scheme of the location of the unit vector, which specifies the direction of the straight line in the tangent plane of the trihedron.

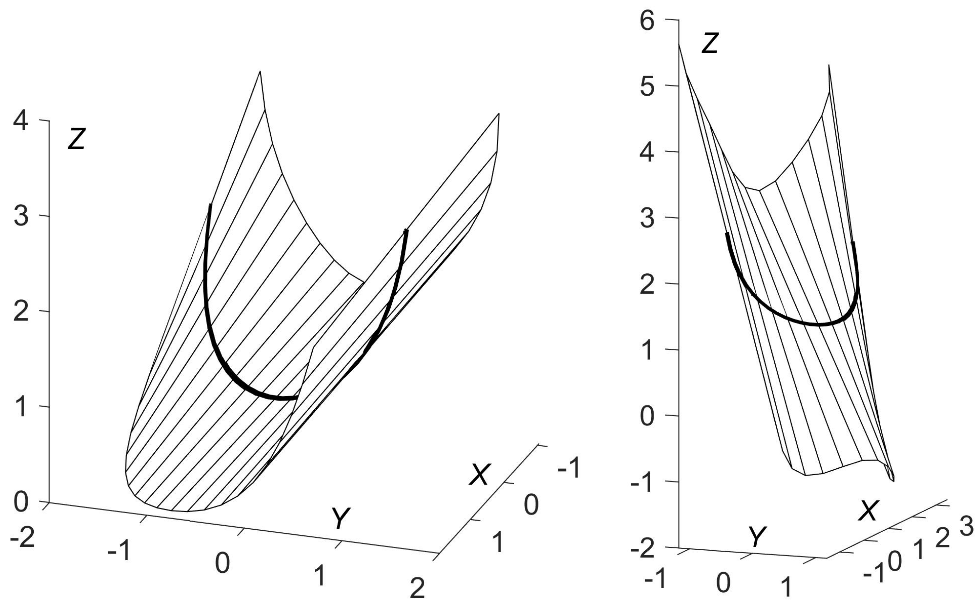

Figure 8: Ruled surfaces in which the straight lines are located in the tangent plane of the Frenet trihedron: a) $\gamma = \pi/2$; b) $\gamma_0 = \pi/2$ (directive curve – striction line of the surface)

It is easy to see that the coefficient G (39) will be significantly simplified if $\gamma' + k = 0$. Under this condition, the base curve will be a striction line for a ruled surface, and the regularity of the angle γ change is determined from the expression:

$$\gamma = -\int k\,\mathrm{d}s + \gamma_0 = -\int \tfrac{\sqrt{2}}{\cosh v}\cosh v\,\mathrm{d}v = \gamma_0 - 2\sqrt{2}A\tan\Bigl(\tanh\frac{v}{2}\Bigr). \tag{41}$$

Example

In Fig. 8, ruled surfaces are constructed based on equations (40) at a constant angle γ (Fig. 8) and at a variable angle γ (Fig. 8), which is determined from dependence (41).

In Fig. 8, all straight lines intersect the base curve at a right angle. In Fig. 8, all straight lines approach each other at the shortest distance along the base curve.

7 Conclusion

Ruled surfaces formed by the movement of a Frenet trihedron of a base curve with a straight line (as a generator) in one of its faces have their own characteristics. A straight line can be stationary in a face, or rotate around its vertex according to a given law. When forming a surface using the base plane of a trihedron, the base curve will be a geodesic line for such surfaces. If the straight line rotates in the reference plane so that its direction all the time coincides with the direction of the Darboux vector, then it will be non-developable. When creating a surface using the normal plane of a trihedron, it will be assigned to a rectangular grid of coordinate lines, regardless of whether the straight line on the face of the trihedron rotates or not. The relationship between the rotation and the non-developable nature of the surface has been identified. In this scenario, the base curve will be a curve line. When creating a surface using the tangent plane of a trihedron, a non-developable surface is possible if the direction of the straight line coincides with the unit vector of the tangent of the trihedron. In this case, the base curve will be the edge of the surface reversal. Additionally, it is possible to create a surface for which the base curve will be a striction line.

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